THE MATHEMATICAL ASSOCIATION OF AMERICA

LIST OF OFFICERS AND CHARTER MEMBERS

DECEMBER, 1916

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1916

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INTRODUCTORY STATEMENT.

The MATHEMATICAL ASSOCIATION OF AMERICA was organized at Columbus, Ohio, on Thursday and Friday, December thirtieth and thirty-first, 1915, as the definite outcome of a wide demand for an organization which should have for its field that of collegiate mathematics. The call for this meeting was authorized by four hundred and fifty persons representing every state in the Union, the District of Columbia, and Canada.

The report of the organization meeting and the constitution as there adopted, were given in the January, 1916, number of THE AMERICAN MATHEMATICAL MONTHLY, which had been made the official journal of the ASSOCIATION. In accordance with a provision of the constitution, those who should be admitted to membership before April first, 1916, were to constitute the list of Charter Members. This list of 1,045 individual members and 52 institutional members is presented herewith. As fully as the members furnished the data asked for, there is given (1) the name in full; (2) the leading degree, with the institution by which it was given; (3) the official position, with the institution represented (if any); (4) the mailing address, where it is not simply the institution given in the preceding. The data have been brought up to December 1, 1916, so far as was possible, in order that this list may be the more useful to the members of the ASSOCIATION.

To the Charter Membership list is appended the names of individual and institutional members elected at the meeting of the Council at Cambridge, Massachusetts, September first, 1916.

It is believed that the data as here assembled contain no serious inaccuracies, as they have been carefully checked on the basis of the returns to inquiries sent to the members by the Secretary. At the same time, the Secretary will be glad to have his attention called to any errors which may still remain in the list.

W. D. CAIRNS,
Secretary-Treasurer.

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To serve until January 1, 1918.

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M. B. PORTER, University of Texas.
J. W. YOUNG, Dartmouth College.

To serve until January 1, 1919.

B. F. FINKEL, Drury College.
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ALEXANDER ZIWET, University of Michigan.

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WESTFALL, WILHELMUS DAVID ALLEN, Ph.D. (Göttingen). Associate Professor of Mathematics, University of Missouri, Columbia, Mo. *309 Hicks Avenue.*

WHEELER, ALBERT HARRY, B.S. (Worcester Polytechnic). Teacher of Mathematics, High School of Commerce, Worcester, Mass. *8 Shawmut Street.*

WHEELER, CHARLES AUGUSTUS, A.M. (Yale). Professor of Mathematics, Connecticut Agricultural College, Storrs, Conn.

WHEELER, J. J. Assistant Professor of Mathematics, University of Kansas, Lawrence, Kan. *1024 Alabama Street.*

WHITE, A. E., M.S. (Purdue). Associate Professor of Mathematics, Kansas State Agricultural College, Manhattan, Kan.

WHITE, CHARLES EDGAR, A.M. (Indiana). Professor of Mathematics, West Virginia Wesleyan College, Buckhannon, W. Va.

WHITE, MARION BALLANTYNE, Ph.D. (Chicago). Associate Professor of Mathematics and Dean of Women, Michigan State Normal College, Ypsilanti, Mich. *314 Forest Avenue.*

WHITFORD, ALFRED EDWARD, A.M. (Wisconsin). Professor of Mathematics and Physics, Milton College, Milton, Wis.

WHITFORD, EDWARD EVERETT, Ph.D. (Columbia). Assistant Professor of Mathematics, College of the City of New York, New York, N. Y. *180 Claremont Avenue.*

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WHITON, EMMA KIRTLAND, A.B. (Colorado College). Graduate Student, University of California, Berkeley, Cal. *Hotel Shattuck, Berkeley.*

WHITTED, JAMES ALBERT, A.M. (Southwestern College). Professor of Mathematics, Hedding College, Abingdon, Ill. *208 West Monmouth Street.*

WIGHT, AMELIA C. Head of the Department of Mathematics, Philadelphia High School for Girls, Philadelphia, Pa. *2009 Mount Vernon Street.*

WILCZYNSKI, ERNEST JULIUS, Ph.D. (Berlin). Professor of Mathematics, University of Chicago, Chicago, Ill.

WILDER, CHARLES EDWARD, Ph.D. (Harvard). Instructor in Mathematics, Northwestern University, Evanston, Ill. *102 Hinman House.*

WILDER, GEORGE F., A.M. (Columbia). Teacher of Mathematics, Erasmus Hall High School, Brooklyn, N. Y.

WILEY, FORBES BAGLEY, Ph.D. (Chicago). Professor and Head of the Department of Mathematics, Denison University, Granville, Ohio.

WILLETT, HUGH CAREY, A.M. (University of Southern California). Associate Professor of Mathematics, University of Southern California, Los Angeles, Cal.

WILLIAMS, ARTHUR ROBINSON, Ph.D. (California). Lecturer in Mathematics, University of California, Berkeley, Cal. *2415 Durant Avenue.*

WILLIAMS, CLARKE BENEDICT, A.M. (Princeton). Dean and Olney Professor of Mathematics, Kalamazoo College, Kalamazoo, Mich. *214 Stuart Avenue.*

WILLIAMS, FRANK BLAIR, Ph.D. (Clark). Professor of Mathematics, Collegiate Department, and Instructor in Mathematics, Graduate Department, Clark University, Worcester, Mass.

WILLIAMS, JOHN EDWARD, Ph.D. (Virginia). Professor of Mathematics, Virginia Polytechnic Institute, Blacksburg, Va.

WILLIAMS, KENNETH POWERS, Ph.D. (Princeton). Assistant Professor of Mathematics, Indiana University, Bloomington, Ind. *405 West Indiana Avenue.*

WILLIAMS, WILLIAM HILL, A.M. (Williams). Head of the Department of Mathematics, State Normal School, Platteville, Wis.

WILLIS, RUBY, A.B. (Wellesley). Instructor in Mathematics, Wells College, Aurora, N. Y.

WILLSON, FREDERICK NEWTON, C.E. (Rensselaer), A.M. (Princeton). Professor of Descriptive Geometry, Pure and Applied, Princeton University, Princeton, N. J. *Box 63.*

WILSON, ALBERT HARRIS, Ph.D. (Chicago). Associate Professor of Mathematics, Haverford College, Haverford, Pa.

WILSON, DELONZA TATE, Ph.D. (Chicago). Associate Professor of Mathematics and Astronomy, Case School of Applied Science, Cleveland, Ohio.

WILSON, EDWIN BIDWELL, Ph.D. (Yale). Professor of Mathematics, Massachusetts Institute of Technology, Cambridge, Mass.

WILSON, ROBERT EDWARD, Ph.M. (Northwestern). Assistant Professor of Mathematics, Northwestern University, Evanston, Ill. *2015 Sherman Avenue.*

WILSON, WALLACE ALVIN, Ph.D. (Yale). Assistant Professor of Mathematics, Yale College, New Haven, Conn. *228 Park Street.*

WINBIGLER, ALICE, A.M. (Monmouth). Professor of Mathematics, Monmouth College, Monmouth, Ill.

WINTER, OLICE. Instructor in Mathematics, Harrison Technical High School, Chicago, Ill. *3534 Walnut Street, Garfield Park Station, Chicago.*

WOLFE, CLYDE LYNNE EARLE, M.S. (Occidental), A.M. (Harvard). Associate Professor of Mathematics, Occidental College, Los Angeles, Cal.

WOLFE, HAROLD EICHHOLTZ, A.M. (Indiana). Fellow in Mathematics, Indiana University, Bloomington, Ind. *North Manchester, Ind.*

WOOD, FREDRICK, A.B. (Wisconsin). Instructor in Mathematics, University of Wisconsin, Madison, Wis. *710 West Dayton Street.*

WOOD, ROSE BELL, A.B. (Barnard College, Columbia). Teacher of Mathematics, Hardin College, Mexico, Mo.

WOODARD, DUDLEY WELDON, M.S. (Chicago). Professor of Mathematics, Wilberforce University, Wilberforce, Ohio.

WOODMANSEE, WILSON R., A.M. (Wisconsin). Professor of Mathematics and Registrar, Ripon College, Ripon, Wis.

WOODS, BALDWIN MUNGER, E.E. (Texas), Ph.D. (California). Assistant Professor of Theoretical Mechanics, University of California, Berkeley, Cal. *Mechanics Building, University of California.*

WOODS, FREDERICK SHENSTONE, Ph.D. (Göttingen). Professor of Mathematics, Massachusetts Institute of Technology, Cambridge, Mass.

WOODWARD, ROBERT SIMPSON, C.E., Ph.D. (Michigan), Sc.D. (Columbia, Pennsylvania), LL.D. (Wisconsin, Johns Hopkins). President, Carnegie Institution of Washington, Washington, D. C.

WOODYARD, ELLA, A.B. B.Pd. (Baker). Instructor in Mathematics, High School, Kansas City, Kan. *702 Oakland Street.*

WORTHINGTON, EUPHEMIA RICHARDSON, Ph.D. (Yale). Instructor in Mathematics, Wellesley College, Wellesley, Mass. *10 Waban Street.*

WRIGHT, HARRY NOBLE, Ph.D. (California). Instructor in Mathematics, University of California, Berkeley, Cal. *2409 Dwight Way.*

WRIGHT, WALTER CHANNING. Consulting Actuary and Accountant, Medford, Mass. *204 Forest Street.*

WRIGHT, WALTER LIVINGSTON, A.M. (Princeton). Professor of Mathematics, Lincoln University, Lincoln University, Pa.

WUNDER, CHARLES NEWMAN, Ph.D. (Virginia). Professor of Mathematics, Southwestern University, Georgetown, Tex.

YANNEY, BENJAMIN FRANKLIN, A.M. (Mount Union). Johnson Professor of Mathematics, College of Wooster, Wooster, Ohio. *666 North Bever Street.*

YEATON, CHESTER HENRY, Ph.D. (Chicago). Assistant Professor of Mathematics, Northwestern University, Evanston, Ill.

YOUNG, ANNA IRWIN, A.M. (Columbia). Professor of Mathematics, Agnes Scott College, Decatur, Ga.

YOUNG, JOHN WESLEY, Ph.D. (Cornell). B. P. Cheney Professor and Head of the Department of Mathematics, Dartmouth College, Hanover, N. H.

YOUNG, JACOB WILLIAM ALBERT, Ph.D. (Clark). Associate Professor of the Pedagogy of Mathematics, University of Chicago, Chicago, Ill. *5422 Blackstone Avenue.*

YOUNG, MABEL M., Ph.D. (Johns Hopkins). Instructor in Mathematics, Wellesley College, Wellesley, Mass. *6 Norfolk Terrace.*

YOWELL, EVERETT IRVING, C.E., Ph.D. (Cincinnati). First Astronomer, Cincinnati Observatory, University of Cincinnati, Cincinnati, Ohio. *Griest and Corbett Avenues.*

ZEHRING, WILLIAM ARTHUR, A.B. (Harvard), A.M. (Otterbein). Assistant Professor of Mathematics, Purdue University, LaFayette, Ind. *303 Russell Street.*

ZEIGEL, WILLIAM HENRY, A.M. (Missouri). Head of the Department of Mathematics, First District Normal School, Kirksville, Mo. *502 South Stanford Street.*

ZIMMERMAN, JOHN, A.M. (Hope College, Michigan). Head of the Department of Mathematics, Dubuque German College and Seminary, Dubuque, Ia. *75 North Glen Oak Avenue.*

ZIWET, ALEXANDER, C.E. (Karlsruhe). Professor of Mathematics, University of Michigan, Ann Arbor, Mich. *644 South Ingalls Street.*

INSTITUTIONAL CHARTER MEMBERS

UNIVERSITY OF ST. FRANCIS XAVIER, Antigonish, N. S., Can.
COLORADO COLLEGE, Colorado Springs, Col.
COLORADO SCHOOL OF MINES, Golden, Col.
WESLEYAN UNIVERSITY, Middletown, Conn.
UNIVERSITY OF GEORGIA, Athens, Ga.
CARTHAGE COLLEGE, Carthage, Ill.
ARMOUR INSTITUTE OF TECHNOLOGY, Chicago, Ill.
UNIVERSITY OF CHICAGO, Chicago, Ill.
NORTHWESTERN UNIVERSITY, Evanston, Ill.
ROCKFORD COLLEGE, Rockford, Ill.
UNIVERSITY OF ILLINOIS, Urbana, Ill.
PURDUE UNIVERSITY, LaFayette, Ind.
IOWA STATE COLLEGE, Ames, Ia.
STATE UNIVERSITY OF IOWA, Iowa City, Ia.
UNIVERSITY OF KANSAS, Lawrence, Kan.
BETHEL COLLEGE, Newton, Kan.
UNIVERSITY OF LOUISVILLE, Louisville, Ky.
UNIVERSITY OF MAINE, Orono, Me.
AMHERST COLLEGE, Amherst, Mass.
UNIVERSITY OF MICHIGAN, Ann Arbor, Mich.
KALAMAZOO COLLEGE, Kalamazoo, Mich.
UNIVERSITY OF MINNESOTA, Minneapolis, Minn.
COLLEGE OF ST. CATHERINE, St. Paul, Minn.
BLUE MOUNTAIN COLLEGE, Blue Mountain, Miss.
UNIVERSITY OF MISSOURI, Columbia, Mo.
CENTRAL COLLEGE, Fayette, Mo.
WASHINGTON UNIVERSITY, St. Louis, Mo.
UNIVERSITY OF MONTANA, Missoula, Mont.
CREIGHTON UNIVERSITY, Omaha, Neb.
RUTGERS COLLEGE, New Brunswick, N. J.
NEW YORK STATE COLLEGE FOR TEACHERS, Albany, N. Y.
UNIVERSITY OF BUFFALO, Buffalo, N. Y.
THE COLLEGE OF THE CITY OF NEW YORK, New York, N. Y.
THE COOPER UNION, New York, N. Y.
CASE SCHOOL OF APPLIED SCIENCE, Cleveland, Ohio.
WESTERN RESERVE UNIVERSITY, Cleveland, Ohio.
OHIO STATE UNIVERSITY, Columbus, Ohio.
OHIO WESLEYAN UNIVERSITY, Delaware, Ohio.
KENYON COLLEGE, Gambier, Ohio.
OBERLIN COLLEGE, Oberlin, Ohio.
LAKE ERIE COLLEGE, Painesville, Ohio.
ALBRIGHT COLLEGE, Myerstown, Pa.
DREXEL INSTITUTE, Philadelphia, Pa.
CARNEGIE INSTITUTE OF TECHNOLOGY, Pittsburgh, Pa.
SWARTHMORE COLLEGE, Swarthmore, Pa.
VANDERBILT UNIVERSITY, Nashville, Tenn.
UNIVERSITY OF TEXAS, Austin, Tex.

Southern Methodist University, Dallas, Tex.
Baylor University, Waco, Tex.
Middlebury College, Middlebury, Vt.
University of Virginia, University, Va.
University of Washington, Seattle, Wash.

INDIVIDUAL MEMBERS ELECTED SEPTEMBER 1, 1916.

Berry, Grace Ella, A.M. (Mount Holyoke). Dean of Women and Assistant Professor of Mathematics, Pomona College, Claremont, Cal.

Boren, Wetz E., A.B. (Indiana). Head of the Department of Mathematics, College Course, Milwaukee State Normal School, Milwaukee, Wis.

Clark, Lewis H. Head of the Department of Mathematics, State Normal School, River Falls, Wis.

Fisher, Willard J., Ph.D. (Cornell). Honorary Fellow in Physics, Clark University, Worcester, Mass. *28 Baker Street.*

Garabedian, Carl Arshag, M.S. (Tufts). Instructor in Mathematics, New Hampshire College, Durham, N. H.

Hacker, Jonathan A., B.S. (Valparaiso). Professor of Mathematics, Sioux Falls College, Sioux Falls, S. D. *1411 Norton Avenue.*

Hill, Prescott William, A.M. (Brown). Instructor in Mathematics, Wabash College, Crawfordsville, Ind. *603 Washington Street.*

Langellotti, Frank. Assistant, Nautical Almanac Office, Washington, D. C.

McMillan, Mary Bell, A.M. (Wisconsin). Instructor in Mathematics, State Normal School, River Falls, Wis.

Moore, Wesley Adolphus, A.B. (Southern University). 110 Fourth Street, Montgomery, Ala.

Perry, Winona Merle, A.M. (Brown). Head of the Department of Mathematics, Judson College, Marion, Ala.

Roeser, Harry M., B.S. (Oklahoma Agricultural and Mechanical College). Bureau of Standards, Washington, D. C.

Wiley, W. O., A.M. (Columbia). Secretary, John Wiley and Sons, Inc., New York, N. Y. *432 Fourth Avenue.*

INSTITUTIONAL MEMBERS ELECTED SEPTEMBER 1, 1916.

George Washington University, Washington, D. C.
Shurtleff College, Alton, Ill.
Drake University, Des Moines, Ia.
Wellesley College, Wellesley, Mass.
Dartmouth College, Hanover, N. H.
Elmira College, Elmira, N. Y.
New York University, New York, N. Y.
La Crosse State Normal School, La Crosse, Wis.

GEOGRAPHICAL DISTRIBUTION OF INDIVIDUAL MEMBERS.

ALABAMA. (10)

Carmichael. Eagles. Perry. Sayre.
Chapman. Luckie. Moore. Shi.
Crenshaw. Messick.

ARIZONA. (4)

Butler. Cresse. Lampland. Leonard.

ARKANSAS. (8)

Armitage. Bragg. Harding. Miser.
Bigbee. Droke. McAlister. Steirnagle.

CALIFORNIA. (42)

Allen. Eggen. Moreno. Tuttle.
Arnold. Green. Morris. Van Buskirk.
Ballaseyus. Hackley. Noble. van Huystee.
Bernstein. Haskell. Ponzer. Wapple.
Berry. Higdon. Posey. Whiton.
Blichfeldt. Hoskins. Putnam. Willett.
Brackett. Irwin. Russell. Williams.
Collier. Lehmer. See. Wolfe.
Cook. McCarty. Stager. Woods.
Crofts. McEwen. Titus. Wright.
Dick. Mathews.

CANADA. (15)

Anning. Draxten. Ling. Rosebrugh.
Baker. Findlay. Matheson. Sheldon.
Buchanan. Gummer. Murray. Warren.
Campbell. Kingston. Pounder.

COLORADO. (20)

Bateman. DeLong. Kendall. Morgan.
B. J. Brown. Denis. Lester. Odell.
E. L. Brown. Finley. Light. Robertson.
Burger. Hill. Loud. Showman.
Cajori. Howe. Macdonald. Sperry.

CONNECTICUT. (16)

Barrow. Kovarik. Reynolds. Tracey.
Brown. Leib. Robbins. Uhler.
Dimick. Longley. Rosenbaum. Wheeler.
Flynn. Miles. Smith. Wilson.

DELAWARE. (4)

Harter. Hebb. Rawlins. Short.

District of Columbia. (17)

Adams.
Bixby.
Cromwell.
Davis.
English.
Hamilton.
Harris.
Hodgkins.
Lambert.
Director, Department of Terrestrial Magnetism, Washington.
Landry.
Langellotti.
Mangold.
Martin.
Roeser.
Van Orstrand.
Woodward.

Florida. (4)

Allen.
Keppel.
Palmer.
Sanders.

Georgia. (14)

Bingley.
Brindle.
Burton.
Field.
Hemke.
Morton.
Peed.
Pond.
Rumble.
Skiles.
Smith.
Stamy.
Stephens.
Young.

Idaho. (2)

Conwell.
Snow.

Illinois. (97)

Abrams.
Allen.
Anderson.
Austin.
Barnett.
Barnhart.
Baudin.
Bliss.
Cahill.
Campbell.
Card.
Carmichael.
Carus.
Cederberg.
Challacombe.
Chittenden.
Cobb.
Comstock.
Coultrap.
Crathorne.
Curtiss.
Dappert.
Davis.
Denton.
Dickson.
Dotterer.
Emch.
Foberg.
Ginnings.
Green.
Hassler.
Hebblethwaite.
Hess.
Hilton.
Holgate.
Holmes.
Ingels.
Jacobs.
Jarrett.
Jones.
Kempner.
Kerr.
Kinney.
Krathwohl.
Lamson.
Laves.
LeStourgeon.
Lunn.
Lytle.
Macdonald.
MacMillan.
McNeill.
Miller, Bessie I.
Miller, G. A.
Moore.
Morrison.
Moulton, E. J.
Moulton, F. R.
Musselman.
Myers.
Olson.
Ott.
Palmer.
Pettersen.
Rietz.
Risley.
Roman.
Schottenfels.
Schweitzer.
Sellew.
Shaw.
Shively.
Simpson.
Sisam.
Skarstedt.
Slaught.
Smith.
Soderholm.
Spunar.
Stahl.
Suffa.
Taylor, E. H.
Taylor, W. H.
Thielbar.
Townsend.
Wahlin.
Walton.
Weida.
Weld.
Whitted.
Wilczynski.
Wilder.
Wilson.
Winbigler.
Winter.
Yeaton.
Young.

Indiana. (30)

Albert.
Bates.
Beckett.
Caparo.
Clarke.
Cox.
Cragwall.
Davisson.
Graves.
Hadley.
Hanna.
Hazard.
Hennel.
Hill.
Hodge.
Johnson.
Kenyon.
Knisely.
Lawrence.
Leckrone.
Lehman.
Mendenhall.
Motts.
Robbins.
Rothrock.
Spitzer.
Tyner.
Williams.
Wolfe.
Zehring.

Iowa. (32)

Baker.	Emmons.	Madson.	†Smith.
Beck.	Farnum.	Neff.	Stanton.
Chaney.	Gouwens.	Norton.	Stein.
Coffin.	Hadley.	Pattengill.	Thomas.
Colpitts.	Herr.	Reilly.	Thompson.
Condit.	Kreth.	Roberts.	Van Horne.
Corey.	McClenon.	Rusk.	Wester.
Daniells.	McGaw.	Simonson.	Zimmerman.

Kansas. (34)

Andrews.	Garrett.	Mergendahl.	Stahl.
Ashton.	Harnly.	Mitchell.	Steimley.
Bell.	Harshbarger.	Newson.	Stouffer.
Bouse.	Hoare.	Olson.	Stratton.
Bowerman.	Hyde.	Porter.	Van der Vries.
Caris.	Jordan.	Remick.	Wheeler.
Dougherty.	Larsen.	Richert.	White.
Dueker.	Lefschetz.	Shirk.	Woodyard.
Flagg.	Lindquist.		

Kentucky. (6)

Boyd.	Lloyd.	Rees.	Rhoton.
Davis.	Phalen.		

Louisiana. (6)

Bogard.	Dinwiddie.	Nicholson.	Spencer.
Cater.	Hedges.		

Maine. (14)

Adams.	Goodrich.	Milne.	Reed.
Ashcraft.	Hart.	Moody.	Robinson.
Carter.	† Holmes.	Ramsdell.	Trefethen.
Colson.	Lord.		

Maryland. (26)

Bacon.	Converse.	Johnson.	Robert.
Bateman.	Eppes.	Lewis.	Root.
Brown.	Galloway.	Morley.	Shenton.
Bullard.	Hall.	Morrow.	Smith.
Capron.	Harry.	Norris.	Tanzola.
Coble.	Hobbs.	Norwood.	Tyler.
Cohen.	Hulburt.		

Massachusetts. (68)

Agard.	Clark.	Fisher.	Libby.
Bailey.	† Conant.	Gaylord.	Lipka.
Barney.	Cook.	Green.	Lupien.
Benedict.	Coolidge.	Haigler.	Martin.
Birkhoff.	Copeland.	Hardy.	Merrill.
Bouton.	Davis.	Hobbs.	Moore.
Bradley.	Duncan.	Huntington.	Moriarty.
Brigham.	Esty, T. C.	Jackson.	Morley.
Bruce.	† Esty, W. C.	Kenison.	Munroe.
Bryant.	Evans.	Kennelly.	Olds.
Butterfield.	Fash.	King.	Osgood.
Chandler.	Ferry.	Kircher.	Pell, Anna J.

† Deceased.

MASSACHUSETTS (*continued*).

Pell, Alexander.
Phillips.
Preston.
Ransom.
Rutledge.
Sheffer.
Smith, Clara.
Smith, Sarah.
Tyler.
Vivian.
Wagar.
Washburne.
Wentworth.
Wheeler.
Williams.
Wilson.
Woods.
Worthington.
Wright.
Young.

MICHIGAN. (35)

Baldwin.
Beman.
Bradshaw.
Butts.
Emmons.
Erickson.
Everett.
Field.
Ford.
Garretson.
Glover.
Grant.
Herron.
Hildebrandt.
Hussey.
Karpinski.
Küstermann.
Lyman.
Markley.
Mirick.
Nelson.
Notestein.
Plant.
Reece.
Rinck.
Running.
Sleight.
Speeker.
Spooner.
Thome.
Tripp.
Welton.
White.
Williams.
Ziwet.

MINNESOTA. (19)

Barton.
Bauer.
Beal.
Berger.
Brooke.
Bussey.
Clevenger.
Dalaker.
Hansen.
Hartwell.
Holzinger.
Lunn.
Newkirk.
Peterson.
Reeve.
Royuley.
Shumway.
Slobin.
Underhill.

MISSISSIPPI. (3)

Hume.
Sharp.
Walker.

MISSOURI. (54)

Ames.
Ammerman.
Andrews.
Borgmeyer.
Brennan.
Calman.
Campbell.
Cosby.
Dunkel.
Epperson.
Escott.
Ferguson.
Finkel.
Fleet.
Forsman.
Gibson.
Hedrick.
Hinrichs.
Horne.
Huff.
Hughes.
Huntington.
Ingold, Byron.
Ingold, Louis.
Jackson.
James.
Jamison.
Jenison.
Johnson, B. F.
Johnson, Stella.
Kellogg.
Kent.
Levy.
Long.
Luby.
Nauer.
Patton.
† Rayworth.
Rider.
Roever.
Rowe.
Scarborough.
Schwartz.
Shannon.
Stellwagen.
Touton.
Urban.
Waldo.
Waldron.
Weeks.
Wells.
Westfall.
Wood.
Zeigel.

MONTANA. (3)

Calderwood.
Carey.
Lennes.

NEBRASKA. (16)

Babbitt.
Bennett.
Blumberg.
Brenke.
Chatburn.
Counts.
Davis.
Feemster.
Fitzpatrick.
Frankish.
Howie.
Logsdon.
Pratt.
Reeves.
Schmiedel.
Walker.

† Deceased.

NEVADA. (1)

Haseman.

NEW HAMPSHIRE. (12)

Beetle.	Forsyth.	Mathewson.	Steck.
Bill.	Garabedian.	Moore.	Sweet.
Dines.	Hopkins.	Morgan.	Young.

NEW JERSEY. (28)

Adams.	Fine.	Morris.	Stanwick.
Sister Blanche Marie.	Gunther.	Nelson.	Sullivan.
Caster.	Kent.	Nyberg.	Thompson.
Colliton.	Kline.	Seymour.	Titsworth.
Dederick.	Lord.	Smith, H. L.	Veblen.
Durell.	Martin.	Smith, I. C.	Webb.
Eisenhart.	† McClintock.	Somers.	Willson.

NEW MEXICO. (3)

Edington.	Pearson.	Rodgers.

NEW YORK. (110)

Allen.	Durfee.	Kasner.	Roe.
Auerbach.	Echols.	Kelly.	Saurel.
Beatley.	Eckersley.	Koch.	Schmall.
Belcher.	Edmondson.	Kristal.	Schuyler.
Benway.	Eshleman.	Langman.	Siceloff.
Bergstresser.	Fischer.	Lanigan.	Silverman.
Berry.	Fiske.	Latham.	Simons.
Betz, Herman.	Fitch.	Lindsey.	Smith, A. W.
Betz, William.	Fite.	Linehan.	Smith, D. E.
Birchenough.	† Frankland.	Locke.	Smith, R. R.
Bowden.	Gaba.	McKelvey.	Spearing.
Breckenridge.	Gale.	Maclay.	Taylor, J. M.
Brewster.	Gillespie.	MacNeish.	Taylor, W. E.
Brown.	Goertz.	McMahon.	Turner.
Bullard.	Grossbaum.	Merriman.	Van Anda.
Callan.	Grove.	Mitchell.	Van Benschoten.
Campbell.	Hanson.	Molina.	Vedder.
Carruth.	Hartmann.	Norton.	Walker.
Carver.	Hawkes.	Owens.	Walsh.
Chamberlain.	Hayes.	Paaswell.	Webster.
Conwell.	Henderson.	Pedersen.	Wechsler.
Cook.	Himwich.	Philip.	Wells.
Cowley.	Hirsch.	Pi Mu Epsilon.	Westfall.
Cox.	Hodgdon.	Ranum.	Whitford.
Day.	Howe.	Reddick.	Wilder.
Decker.	Hurwitz.	Requa.	Wiley.
Dougherty.	Joffe.	Reynolds.	Willis.
Douglas.	Johnson.		

NORTH CAROLINA. (11)

Amick.	Cain.	Mendenhall.	Rankin.
Barton.	Douglas.	Patterson.	Richardson.
Brinton.	Lasley.	Ragsdale.	

† Deceased.

North Dakota. (1)

Hitchcock.

Ohio. (76)

Allen.
Anderegg.
Anderson.
Armstrong.
Arnold.
Austin.
Bareis.
Beckwith.
Börger.
Bohannan.
Brand.
Brandeberry.
Burnell.
Cain.
Cairns.
Caris.
Carr.
Carscallen.
Coddington.
Deming.
Dustheimer.
Faught.
Focke.
Glazier.
Graber.
Haldeman.
Hancock.
Hirschler.
Hofmann.
Hoover.
Horn.
Hornung.
Hutchinson.
Jenkins.
Johnson, R. A.
Johnson, W. W.
Kindle.
Konantz.
Kuhn.
McCain.
McCoard.
† Maglott.
Mayer.
Metcalf.
Miller.
Moore.
Morgan.
Morningstar.
Morris.
Overman.
Owen.
Palmié.
Pfahl.
Pitcher.
Preston, Amy F.
Preston, J. B.
Rasor.
Rickard.
Sinclair.
Singer.
Slocum.
Smith.
Spinks.
Stetson.
Swartzel.
Thomas, C. F.
Thomas, R. P.
Urner.
Wallace.
West.
Whiting.
Wiley.
Wilson.
Woodard.
Yanney.
Yowell.

Oklahoma. (8)

Altshiller.
Duval.
Gossard.
Gundersen.
Livingston.
Reaves.
Riley.
Short.

Oregon. (7)

De Cou.
Griffin.
Ludlow.
Merriss.
Palmer.
Thayer.
West.

Pennsylvania. (65)

Akers.
Apple.
Atchison.
Baird.
Bauman.
Bert.
Bishop.
Bland.
Burley.
Chambers.
Charles.
Clark.
Clarke.
Clawson.
Crawley.
Davis.
Dill.
Durham.
Evans.
Fisher.
Foraker.
Franklin.
Glenn.
Granville.
Gravatt.
Gummere.
Haines.
Hall.
Hazlett.
Holder.
Kiess.
Lambert.
Landis.
Linton.
MacNutt.
Marriott.
Miller.
Moore.
Nolan.
O'Shaughnessy.
Partridge.
Quinn.
Ramsey.
Rau.
Reusswig.
Reynolds.
Riggs.
Rowe.
Safford.
Scott.
Sensenig.
Simpson.
Smith, E. R.
Smith, W. M.
Snyder.
Sousley.
Taber.
Thompson.
Thornburg.
Travis.
Weaver.
Webber.
Wight.
Wilson.
Wright.

Rhode Island. (11)

Archibald.
Brown.
Burgess.
Currier.
Davis.
French.
Manning.
Peaslee.
Richardson.
Swain.
Tyler.

Deceased.

SOUTH CAROLINA. (4)

Coleman.	Earle.	Hightower.	Thomas.

SOUTH DAKOTA. (10)

Boyce.	Field.	McLaury.	Myers.
Brown.	Hacker.	Mills.	Scott.
Canaday.	McKinney.		

TENNESSEE. (7)

Barton.	Cox.	Jones.	Sherrer.
Buchanan.	Hooper.	Knapp.	

TEXAS. (23)

Alexander.	Decherd.	Jones.	Porter, M. B.
Allen.	de la Garza.	Lovett.	Porter, P. C.
Benedict.	Dodd.	Mahoney.	Roach.
Bennett.	Ettlinger.	Nelson.	Underwood.
Bond.	Hagelstein.	Nichols.	Wunder.
Chandler.	Harrell.	Noel.	

UTAH. (4)

Gibson.	Pehrson.	Saxer.	Sweazey.

VERMONT. (4)

Donahue	Mabrey	Perkins.	Swift.

VIRGINIA. (17)

Carpenter.	Gaines.	Miller.	Smith, L. W.
Colaw.	Larew.	Morenus.	Stone.
Dickinson.	Lord.	Oglesby.	Watts.
Duke.	Luck.	Smith, J. B.	Williams.
Echols.			

WASHINGTON. (13)

Bell.	Eells.	Hanawalt.	Neikirk.
Boothroyd.	Gavett.	Hix.	Raynor.
Bratton.	Guy.	Moritz.	Wear.
Carpenter.			

WEST VIRGINIA. (8)

Baker.	Eiesland.	Githens.	Turner.
Balch.	Flanagan.	Hodgson.	White.

WISCONSIN. (21)

Adkins.	Dowling.	Haynes.	Warner.
Arnold.	Dresden.	McMillan.	Whitford.
Boren.	Ericson.	Simpson.	Williams.
Burgess.	Frumveller.	Slichter.	Wood.
Clark.	Hamilton.	Smith.	Woodmansee.
Sister Mariola Dobbin.			

WYOMING. (3)

Fitterer.	Ridgaway.	Stromquist.

CHINA. (6)

Chang Shen Fu. Kuo Chiu Liu. Patten. Sanford.
Heinz. Leavens.

ENGLAND. (2)

Jackson. Richmond.

INDIA. (1)

Pandya.

ITALY. (1)

Prompt.

SOUTH AFRICA. (1)

Muir.

TURKEY. (1)

Mourad.

RECAPITULATION OF MEMBERSHIP.

Individual charter members	1,045	
Institutional charter members	52	
Total charter members		1,097
Individual members elected September, 1916	13	
Institutional members elected September, 1916	8	
Total members elected September, 1916		21
Total individual members December 1, 1916	1,058	
Total institutional members December 1, 1916	60	
Grand total December 1, 1916		1,118

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.
2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.
2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secrecary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.
2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appcinted by the Council.
3. The Council shall transact the official business of the Association and shall report its actions at the annual meetıng of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Associaticn may make known to the Council their individual views.
4. The Council shall have authority to fill vacancies *ad interim.*

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.
2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.
2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

VOLUME XXIII

JANUARY–DECEMBER, 1916

PUBLISHED BY THE ASSOCIATION

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1912. FROM 1912 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the MANAGING EDITOR, H. E. SLAUGHT 5548 Kenwood Avenue, Chicago, Ill.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER, of the ASSOCIATION, W. D. CAIRNS, 55 East Lorain Street, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME XXIII JANUARY, 1916 NUMBER 1

THE MATHEMATICAL ASSOCIATION OF AMERICA.

In accordance with the call for an organization meeting of a new national mathematical association, signed by four hundred and fifty persons representing every state in the Union, the District of Columbia, and Canada, such a meeting was held at Columbus, Ohio, on Thursday, December 30, 1915, in room 101 of Page Hall, on the Campus of Ohio State University. The meeting extended through two sessions, the latter being held on Friday morning, December 31, at which time the constitution was finally adopted.

The following one hundred and four persons were in attendance, a large number of whom took active part in the proceedings:

C. E. Albright, North High School, Columbus; R. B. Allen, Kenyon College; Frederick Anderegg, Oberlin College; W. E. Anderson, Wittenberg College; G. N. Armstrong, Ohio Wesleyan University; C. L. Arnold, Ohio State University; R. P. Baker, Iowa State University; Grace M. Bareis, Ohio State University; W. H. Bates, Purdue University; H. M. Beatty, High School, Newark, Ohio; R. D. Bohannan, Ohio State University; J. C. Boldt, High School, Dayton, Ohio; P. P. Boyd, University of Kentucky; J. B. Brandeberry, Toledo University; S. W. Brown, Ohio State University; Daniel Buchanan, Queen's University; H. T. Burgess, University of Wisconsin; W. H. Cahill, Loyola University; W. D. Cairns, Oberlin College; A. G. Caris, Defiance College; R. D. Carmichael, University of Illinois; G. E. Carscallen, Hiram College; H. E. Cobb, Lewis Institute; E. T. Coddington, Ohio State University; Elizabeth B. Cowley, Vassar College; L. C. Cox, Purdue University; D. R. Curtiss, Northwestern University; S. C. Davisson, Indiana University; W. W. Denton, University of Illinois; L. E. Dickson, University of Chicago; O. L. Dustheimer, Baldwin-Wallace College; Peter Field, Jr., University of Michigan; B. F. Finkel, Drury College; T. W. Focke, Case School of Applied Science; W. S. Franklin, New York City; C. E. Githens, Superintendent of Schools, Wheeling, W. Va.;

Harriet E. Glazier, Western College for Women; M. E. Graber, Heidelberg University; S. M. Hadley, Penn College; W. A. Hamilton, Beloit College; Harris Hancock, University of Cincinnati; C. T. Hazard, Purdue University; E. R. Hedrick, University of Missouri; Cora B. Hennel, Indiana University; F. J. Holder, University of Pittsburgh; L. A. Hopkins, University of Michigan; Marvel C. Horn, Muskingum College; C. A. Hutchinson, Wittenberg College; L. C. Karpinski, University of Michigan; A. M. Kenyon, Purdue University; J. H. Kindle, University of Cincinnati; Emma L. Konantz, Ohio Wesleyan University; H. W. Kuhn, Ohio State University; T. E. Mason, Purdue University; Gertrude McCain, Oxford College for Women; J. V. McKelvey, Cornell University; W. O. Mendenhall, Earlham College; F. E. Miller, Otterbein University; G. A. Miller, University of Illinois; J. A. Miller, Swarthmore College; U. G. Mitchell, University of Kansas; C. N. Moore, University of Cincinnati; Charlotte Morningstar, Ohio State University; C. C. Morris, Ohio State University; F. R. Moulton, University of Chicago; J. R. Musselman, Johns Hopkins University; J. R. Overman, Ohio State Normal College; C. P. Parkhurst, care of Ginn and Company, Columbus, Ohio; A. D. Pitcher, Western Reserve University; V. C. Poor, University of Michigan; J. B. Preston, Ohio State University; S. E. Rasor, Ohio State University; H. W. Reddick, Cooper Union, New York; Hortense Rickard, Ohio State University; H. L. Rietz, University of Illinois; W. J. Risley, James Millikin University; W. H. Roever, Washington University; R. E. Root, United States Naval Academy; D. A. Rothrock, Indiana University; F. H. Safford, University of Pennsylvania; Ida M. Schottenfels, Chicago, Ill.; A. R. Schweitzer, Chicago, Ill.; H. M. Sheffer, St. Louis, Mo.; S. A. Singer, Capital University, Ohio; H. E. Slaught, University of Chicago; G. W. Smith, University of Illinois; W. M. Steirnagle, High School, Fowler, Indiana; K. D. Swartzel, Ohio State University; E. H. Taylor, Eastern Illinois State Normal School; C. E. Van Orstrand, U. S. Geological Survey; C. A. Waldo, Washington University; C. P. Weinland, North High School, Columbus, Ohio; W. P. Webber, University of Pittsburgh; P. M. Weida, St. Alban's School, Knoxville, Ill.; C. J. West, Ohio State University; H. S. White, Vassar College; E. J. Wilczynski, University of Chicago; F. B. Wiley, Denison University; C. B. Williams, Kalamazoo College; D. T. Wilson, Case School of Applied Science; B. F. Yanney, College of Wooster; A. E. Young, Miami University; W. H. Zeigel, Kirksville Normal School, Missouri; Alexander Ziwet, University of Michigan.

When the meeting was called to order, E. R. Hedrick, University of Missouri, was elected temporary Chairman, and W. D. Cairns, Oberlin College, temporary Secretary. Upon the request of the Chairman, some introductory remarks were made by H. E. Slaught, as the representative of the Board of Editors of the American Mathematical Monthly who had been responsible for proposing the call for the meeting. Referring to the history of this movement, as outlined in the October, 1915, issue of the Monthly, he emphasized the fact that this journal had stood consistently, since its reorganization, for advancing the interests of mathematics in the collegiate and advanced secondary fields, and expressed the

hope that the new organization might carry forward these aims with still greater effectiveness, coöperating, on the one hand, with the various well-organized secondary associations, and, on the other hand, with the American Mathematical Society in its chosen field of scientific research, but being careful to encroach upon neither of these fields.

The meeting was then resolved into a committee of the whole for the consideration, section by section, of a constitution and by-laws, tentative drafts of which had been prepared in advance. After three hours of patient and painstaking deliberation, all mooted questions were settled except the name of the new organization. This was left to a committee to choose from the eighteen different variations which had been proposed and to report the following morning. A committee was also delegated to assist the temporary secretary in smoothing out any verbal inconsistencies or inaccuracies in the constitution and by-laws. The name finally chosen by the committee was adopted without a dissenting vote, as embodying more favorable points and fewer objections than any other that had been suggested.

CONSTITUTION.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

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1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

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1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

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1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

The initiation fee shall be waived in case of those who join the Association before April 1, 1916, and this clause shall be dropped after its provisions have been fulfilled.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

Twelve members of the Council shall be elected at the first meeting of the Association, and the secretary shall draw lots to determine which four of those elected shall serve for one, for two, and for three years respectively. (This clause shall be dropped after its provisions have been fulfilled.)

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

This shall not be construed to affect existing contracts with any subscribers or news agencies for the year 1916, who may decline to readjust on the new basis. (This clause shall be dropped after its provisions have been fulfilled.)

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

As the provisions of the By-laws with respect to the nomination and election of officers could not be fulfilled in the first instance, a special resolution at the meeting on Thursday provided for a nominating committee, consisting of L. E. DICKSON, University of Chicago; D. R. CURTISS, Northwestern University; H. L. RIETZ, University of Illinois; S. E. RASOR, Ohio State University; and R. E. ROOT, United States Naval Academy. This committee reported immediately after the final adoption of the Constitution and By-laws on Friday morning. Their report, after full opportunity for further nominations from the floor, was adopted and the following officers were elected:

For President, E. R. HEDRICK, University of Missouri;

For Vice-Presidents, E. V. HUNTINGTON, Harvard University, and
G. A. MILLER, University of Illinois;

For Secretary-Treasurer, W. D. CAIRNS, Oberlin College;

For additional members of the Executive Council:

To serve for one year.

D. N. LEHMER, University of California,
R. E. MORITZ, University of Washington,
K. D. SWARTZEL, Ohio State University,
OSWALD VEBLEN, Princeton University.

To serve for two years.

R. C. ARCHIBALD, Brown University,
FLORIAN CAJORI, Colorado College,
M. B. PORTER, University of Texas,
J. W. YOUNG, Dartmouth College.

To serve for three years.

B. F. FINKEL, Drury College,
E. H. MOORE, University of Chicago,
J. N. VAN DER VRIES, University of Kansas,
ALEXANDER ZIWET, University of Michigan.

The Council met on Friday afternoon and immediately completed its quota by the appointment of the Committee on Publications consisting of H. E. SLAUGHT, of the University of Chicago, managing editor, together with R. D. CARMICHAEL, University of Illinois, and W. H. BUSSEY, University of Minnesota.

The Council appointed a committee consisting of ALEXANDER ZIWET and K. D. SWARTZEL to act with power in negotiating with the owners of the AMERICAN MATHEMATICAL MONTHLY with a view to securing this as the official journal of the ASSOCIATION. The result of these negotiations was that the MONTHLY was formally transferred to the ASSOCIATION and thus, under these new auspices, it will begin its twenty-third year of continuous service.

The Council empowered the Committee on Publications to fill out the Editorial Board by the appointment of associate editors. This has been done as follows:

R. P. Baker, University of Iowa; W. C. Brenke, University of Nebraska; A. Cohen, Johns Hopkins University; B. F. Finkel, Drury College; L. C. Karpinski, University of Michigan; G. H. Ling, University of Saskatchewan; Helen A. Merrill, Wellesley College; U. G. Mitchell, University of Kansas; W. H. Roever, Washington University; D. A. Rothrock, Indiana University; C. H. Slichter, University of Wisconsin; and D. E. Smith, Columbia University.

The Editorial Board will be further organized by the appointment of committees in the various departments.

The Council also appointed a Standing Committee on Finance consisting of the President, the Secretary and the Managing Editor; also a special committee to secure a bond for the Secretary-Treasurer in the amount of $2,000.

The Secretary was instructed to draw lots to determine the terms of service of the twelve members of the Council just elected, the result being as indicated above.

The Council received formal applications from duly authorized representatives of three states requesting authority for organizing Sections of the Association; namely, from Kansas, Missouri and Ohio. The Kansas meeting was held early in the Autumn, the Missouri meeting at Thanksgiving time, and the Ohio meeting on Thursday afternoon at Columbus, the latter having some thirty-five delegates present. The Council appointed a committee consisting of E. R. Hedrick, Alexander Ziwet, and K. D. Swartzel, to formulate the terms under which such petitions may be granted, as provided by the Constitution, and to act with power on these and other similar petitions which may be received before the next meeting.

Finally, the Council authorized the President, Secretary, and Managing Editor to act with power in respect to all names proposed for membership during the formation period.

Although no program had been originally planned, aside from the arduous business of organizing the new Association, yet the temporary committee on arrangements was most fortunate in securing Professor L. C. Karpinski, of the University of Michigan, to give his illustrated address on "The Story of Algebra," which he did on Friday morning immediately following the business session. It is not too much to say that for the space of an hour he both charmed and edified an enthusiastic audience of approximately one hundred persons.

Hearty thanks are due to the local committee of arrangements, under the direction of Professor S. E. Rasor, for the accommodations provided for these meetings, including especially the luncheon on Friday given by the Department of Mathematics of Ohio State University to all mathematicians attending the Columbus meetings. Thanks are also due to the members of the American Mathematical Society who were in attendance at Columbus for the very hearty cordiality and coöperation extended by them to the new Association.

For further notes on the Columbus meeting see the last pages of this issue.

W. D. Cairns, *Secretary.*

DIGITAL RECKONING AMONG THE ANCIENTS.

By LEON J. RICHARDSON, University of California.

Primitive man developed his notions of number to no small extent by aid of the fingers. Hence his habit of counting in systems of five and ten. As social relations grew, the needs of communication led to certain uniformities of practice. These signs and arithmetical processes must have differed among tribes; at the same time the fixed elements in the case, namely the ten fingers, made for similarities. Such devices for reckoning, moreover, once hit upon, especially if marked by superior convenience, tended to spread from tribe to tribe and from region to region, just as cleverly devised modes of writing numbers passed with little change from the Egyptians to the Phœnicians and thence to Palmyra and the Syrians.

I. Use of the Fingers in Representing Numbers.

The practice of indicating numbers on the fingers was common among ancient Egyptians, Babylonians, Greeks, and Romans. Moreover, the system was eventually developed to such an extent that all numbers from one to 10,000, and sometimes even larger numbers, could be so expressed. References to the subject are often met with in Greek and Latin literature, authors usually taking it for granted that their readers will understand details. Pliny the Elder (*Natural History*, 34, 7, 33) says that "king Numa dedicated a statue of two-faced Janus . . . the fingers being put in a position to show 365 . . . and thus to represent him as the god of time and duration." Macrobius (1, 9, 10), referring to copies of this statue, says the number 300 was "held" in the right hand and 65 in the left. A statue of the philosopher Chrysippus, who devoted much attention to mathematics, had "the fingers drawn together so as to indicate numbers" (Sidonius: *Epist.*, 9, 9, 14). The ring-finger, for example, when bent, indicated the number six (Macrobius, 7, 13, 10). The sign for 500 was made *flexo pollice* (Quintilian, 11, 3, 117). St. Jerome (*Adversus Iovinianum*, 1, 3) informs us that "Thirty is associated with marriage. For the very union of the fingers, as if embracing, . . . depict the husband and wife. Sixty indeed is associated with widows, for the reason that they are placed in straits and misery. . . . The number one hundred however (pray, reader, attend carefully) passes over from the left hand to the right . . . and forming a circle bodies forth the crown of virginity." Cassiodorus—historian, statesman and monk of the sixth century—in commenting on the sixtieth Psalm remarks: "The numbering of this Psalm moreover is not barren of interest. For the number sixty belongs fitly to celibates and widows, being represented by a bound position of the fingers."

The practise of representing numbers on the fingers gave rise to the English word *digit* as a name for each of the numerals below ten. The earliest known instance of this word in our literature, namely in the works of John de Trevisa under date of 1398, takes the form *digitus*, the Latin word having developed that meaning in the late middle ages, notably in books on algorism.

Fortunately we have not been left in the dark as to how the individual numbers were represented, for the complete system is set forth by the Venerable Bede and by Nicolaus Rhabda of Smyrna, both of the eighth century, men dwelling in widely separated parts of the world, one writing in Latin and the other in Greek. Bede's account, which antedates all others that have come down to us, is given in his book *De loquela per gestum digitorum* and Rhabda's is incorporated in the work of Nicolaus Caussinus *De eloquentia sacra et humana* (Paris, 1636). There exist still other accounts, notably one contained in the Persian and Arabic lexicon of Ghiyás and translated into English by E. H. Palmer. These writers set down what had been traditional, doubtless, from early times, for they agree substantially among themselves and their matter harmonizes in the main with the scattered and casual observations of classical authors.

From the foregoing sources we learn that numbers were expressed by the following signs. The hand, unless otherwise stated, was held upward, the palm flat, the fingers together, except the thumb, which did not touch the second finger.

1—5th finger of the left hand bent at the middle joint.
2—4th and 5th fingers bent at the middle joint.
3—3d, 4th and 5th fingers bent at the middle joint.
4—3d and 4th fingers bent at the middle joint.
5—3d finger bent at the middle joint.
6—4th finger bent at the middle joint.
7—5th finger closed on the palm.
8—4th and 5th fingers closed on the palm.
9—3d, 4th and 5th fingers closed on the palm.
10—Tip of the 2d (or index) finger touched the middle joint of the thumb.
11—The signs for 10 and 1 were made coincidentally. The same method applied to numbers from 12 to 19.
20—The thumb was placed between the 2d and 3d fingers in such a way that the thumb nail touched the middle joint of the 2d finger.
30—The thumb and the 2d finger formed a circle.
40—The thumb and 2d finger stood erect and close together.
50—The thumb, bent at both joints, rested on the palm.
60—The 2d finger was bent forward over the thumb, which remained in the position just described.
70—The first joint of the 2d finger rested upon the first joint of the thumb, which was held nearly straight.
80—The tip of the 2d finger rested upon the first joint of the thumb.
90—The thumb was bent over the first joint of the 2d finger.

The signs so far described were all made with the left hand. We now pass to the part played by the other hand. The sign for 100 did not differ from the sign for 10, except that it was made with the right hand. Similarly related were the signs for 200 and 20, 300 and 30 and so on through 900 and 90.

The sign for 1,000 did not differ from the sign for 1, except that it was made with the right hand. Similarly related were the signs for 2,000 and 2, 3,000 and 3 and so on through 9,000 and 9.

The sign for 10,000 was made by laying the left hand on the chest. 20,000, 30,000 and so on through 90,000 were made by touching various parts of the body

with the same hand. (See Bede, 692.) Similarly the signs for 100,000–900,000 were made by touching corresponding parts of the body with the right hand. The sign for 1,000,000 was the hands clasped, the fingers being interlocked.

II. Use of the Fingers in Counting and Reckoning.

Let us now turn to the other phase of the subject, the use of the fingers in counting and reckoning. The Homeric verb πεμπάζειν means "to count on the five fingers," "to count by fives." Herodotus (6, 63) employs the expression ἐπὶ δακτύλων συμβάλλεσθαι meaning "to reckon on the fingers." Significant also in this connection is a passage in Aristophanes (*Vespæ,* 655–657): "Hear then . . . and first of all do an easy sum—not with counters, but with your fingers—the tribute collectively which accrues to us from the cities."

There is a verse in Plautus (*Miles Gloriosus,* 204) which runs: "He reckons the pros and cons on the fingers of the right hand" (*dextera digitis rationem conputat*). To a Roman familiar with current mathematical usages the expression might convey something more than appears in our literal translation. The clue to the subtler meaning may be found in several ancient writings, but is also to be gathered, as it chances, from Sir Thomas Browne's *Pseudodoxia Epidemica* (iv, iv, 186): "On the left [hand] they accounted their digits and articulate numbers unto an hundred, on the right hand hundreds and thousands." Accordingly the "pros and cons" aforesaid, being reckoned on the right hand, are by implication many in number—jocosely represented as a hundred or more. A similar point is made by the poet Juvenal (*Saturnae,* x, 246): "If one has any faith in great Homer, [Nestor] was an instance of life inferior in duration only to the crow's. Happy was he indeed who put off the hour of his death so long and at last begins to count his years on his right hand" (*suos iam dextra conputat annos*).

An important phrase for our purpose is found in a letter that Cicero once wrote to his friend Atticus, the capitalist (*Ad Atticum,* 5, 21, 12–13): "Everybody present exclaimed that nothing was more shameless than Scaptius, who was not satisfied with 12 per cent., compound interest. . . . Lately a decree of the senate has been passed . . . on the subject of creditors fixing the rate at 12 per cent., simple interest. What difference this makes, if I know your skill at reckoning, you have certainly computed." The phrase "if I know your skill at reckoning" is literally "if I know your fingers" (*si tuos digitos novi*).

Ovid (*Fasti,* 3, 123) has the expression: "the fingers by the aid of which we are wont to count" (*digiti per quos numerare solemus*). But one of the most illuminating passages bearing on our subject is found in Quintilian (1, 10, 35): "As to geometry, people admit that attention to it is of advantage in tender years; for they allow that by this study the thinking powers are excited, the intellect sharpened and quickness of perception produced; but they fancy that it is not, like other sciences, profitable after it has been acquired, but only while it is being studied. Such is the common opinion respecting it. Not without reason, however, have the greatest men devoted much attention to this science;

for while geometry comprises numbers and forms, a knowledge of numbers assuredly is necessary not only to a speaker, but to any one taking even the first steps along the path of learning. For pleading cases in court it is very often in request. On these occasions, to say nothing of becoming confused about sums, if a speaker, by any uncertain or awkward movement of the fingers, differs from the accepted mode of calculation, he is thought to be poorly trained."

Pliny the Younger (2, 20, 3), speaking of a man occupied with thoughts about becoming the heir of a rich woman, says "He moves his lips, keeps his fingers going, reckons" (*movet labra, agitat digitos, computat*). Again Suetonius (*Claudius*, 21) makes this observation on the Roman emperor Claudius: "He gave many gladiatorial shows. . . . There was no form of entertainment at which he was more familiar and free, even thrusting out his left hand, like the commons, and counting aloud on his fingers the gold pieces which were paid to the victors" (*aureos . . . voce digitisque numeraret*).

Apuleius (*Apologia*, 89) protests that a certain lady's age has been represented as sixty, when she was really thirty: "If instead of ten you had said thirty years, you might seemingly have erred in the manual sign of the calculation, [in the former case, the tip of the forefinger touching the middle joint of the thumb,] in the latter those fingers forming a circle. Forty, however, is shown by the flat palm—the simplest of manual signs—and you increase that number by half. An error in the manual sign is out of the question, unless, supposing Pudentilla to be thirty, you counted both consuls with each year."

When we come down to the fourth and fifth centuries A. D., we find St. Jerome (*Adversus Iovinianum*, 1, 46) saying: "He shows that wives are wont to be selected more on the basis of wealth than of character and that many are guided, not by their eyes, but by their fingers, in marrying" (*multos non oculis sed digitis uxores ducere*). In the same period St. Augustine (*City of God*, 18, 53) writes: "He puts aside the fingers of the computers (*calculantium digitos*) and orders silence, who says 'It is not for you to know the times, which the Father hath put in His own power.'"

III. The Four Operations Represented by Use of the Fingers.

Enough has been presented to show how common among the ancients was the use of the fingers in dealing with numbers. The illustrations have been drawn for the most part from the Romans, to be sure, but similar material could be found among the Greeks and other peoples. It is worth remarking in this connection that no trace of finger computation of the sort we have described has been found among the Hindus, to whom ultimately we owe our own system of numerals. It is now time to consider how the fingers were employed in mathematical processes. This is not easy to answer in detail. The case of addition, however, is fairly well understood. When a series of numbers was to be added, the fingers at the outset were made to indicate the first number. The second number was then added mentally and the fingers put in a position to indicate the sum. The third number in its turn was added mentally and the fingers

changed to show the new sum. In this way a person proceeded until he arrived at the final sum. (See Marquardt, 1: 99.) The sight of the fingers was an aid in performing the successive steps of the addition. In subtraction the fingers seem to have been similarly used, according to evidence that has reached us. The same may be said of multiplication and division, for the former may be performed by a series of additions and the latter by a series of subtractions.

Concerning the matter of multiplication, some curious evidence has come to light. Dacia, which in ancient geography corresponds in the main to modern Roumania and Transylvania, became a Roman province under Trajan. It was in fact governed from Rome from about 101 to 256 A. D. During that period four military roads were constructed and several forts were built to protect the inhabitants from the incursions of surrounding barbarians. Numerous colonists from Italy settled in the country. The Dacians on their part adopted the religion and language of the conquerors. The Roman occupation, though comparatively short, may still be traced in Greek and Latin literature, in monoliths, inscriptions and coins, as well as in the language and customs of the Roumanians. Interestingly enough, the Walachian peasants, who dwell in southern Roumania, have preserved an old method of multiplication on the fingers. How old the custom is, or whence derived, no one can tell. It can hardly be of local origin, for it is not absolutely unique. It may hark back to oriental or Greek sources, for an algorism manuscript of about 1200 A. D., now preserved at Heidelberg, contains something similar. (See Cantor, 1: 780.) Again it may be of Roman origin. The last theory is supported by the fact that similar usages, probably of Roman origin, have been noted among French peasants. (See Cantor, 1: 447; also Sittl, p. 262.) Moreover, there were current in the middle ages similar mathematical usages which almost surely arose from Roman sources. The method of multiplication just mentioned is as follows:

TABLE I.

1st cycle $(6\cdots10)$ formula: $10(e + e') + cc'$.
2d cycle $(11\cdots15)$ formula: $15(e + e') + cc' + 75$.
3d cycle $(16\cdots20)$ formula: $20(e + e') + cc' + 200$.
4th cycle $(21\cdots25)$ formula: $25(e + e') + cc' + 375$.
5th cycle $(26\cdots30)$ formula: $30(e + e') + cc' + 600$.
. . . .
nth cycle $[5n + 1\cdots5(n + 1)]$ formula: $5(n + 1)(e + e') + cc' + 5^2(n^2 - 1)$.

ABBREVIATIONS:

e = extended fingers of the right hand.
e' = extended fingers of the left hand.
c = closed fingers of the right hand.
c' = closed fingers of the left hand.

To illustrate the foregoing formulæ, take a problem falling within the first cycle. How many are 7×7? Hold up each hand clenched. Extend the right thumb and index finger (= 7). Extend the left thumb and index finger (= 7) [Note that in this cycle the fingers beginning with the thumb have values respectively of 6, 7, 8, 9, 10.] How many fingers are extended? [The thumbs count as fingers.] Four. $4 \times 10 = 40$. How many fingers are closed? Three on the

right hand and three on the left. $3 \times 3 = 9$. Therefore $7 \times 7 = 49$. This process, it will be observed, presupposes a knowledge of the multiplication table in the modern sense through the fives.

Again, how many are 6×8? Hold up each hand clenched. Extend the right thumb (= 6). Extend the left thumb, index finger and middle finger (= 8). How many fingers are extended? Four. $4 \times 10 = 40$. How many fingers are closed? Four on the right hand and two on the left. $4 \times 2 = 8$. Therefore $6 \times 8 = 48$.

Second cycle problem: How many are 12×13? Hold up each hand clenched. Extend the right thumb and index finger (= 12). Extend the left thumb, index finger and middle finger (= 13). [In this cycle the fingers have the values of 11, 12, 13, 14, 15.] How many fingers are extended? Five. $5 \times 15 = 75$. How many are closed? Three on the right hand and two on the left. $3 \times 2 = 6$. $75 + 6 + 75 = 156$. Therefore $12 \times 13 = 156$.

In order to multiply two numbers belonging to different cycles, *e. g.*, 9×13, the larger number is so divided as to bring the problem within a single cycle. *E. g.*, $9 \times 13 = (9 \times 6) + (9 \times 7)$.

In actual practice the Walachians do not go beyond the first cycle. The other cycles as here set forth have been made out inferentially.

The formulæ given above may assume a slightly different form:

TABLE II.

1st cycle	$(6 \cdots 10)$	formula: $10(e + e') + cc'$.
2d cycle	$(11 \cdots 15)$	formula: $10(e + e') + ee' + 100$.
3d cycle	$(16 \cdots 20)$	formula: $20(e + e') + cc' + 200$.
4th cycle	$(21 \cdots 25)$	formula: $20(e + e') + ee' + 400$.
5th cycle	$(26 \cdots 30)$	formula: $30(e + e') + cc' + 600$.

As regards the nth cycle: when n is an odd number, the general formula is the same as that given in Table I; when, however, n is an even number, the general formula is:

$$n\text{th cycle } [5n + 1 \cdots 5(n + 1)] \text{ formula: } 5n(e + e') + ee' + 5^2n^2.$$

By way of illustration, take the problem: How many are 12×13? Hold up each hand clenched. Extend the right thumb and index finger (= 12). Extend the left thumb, index finger and middle finger (= 13). How many fingers are extended? Five. $5 \times 10 = 50$. How many are extended on each hand? Two on the right hand and three on the left. $2 \times 3 = 6$. $50 + 6 + 100 = 156$.

The modulus 5 is natural under the influence of the decimal system. However, it is readily possible to substitute 4, 6, 10, or even other numbers. The modulus 6 is illustrated in

TABLE III.

1st cycle	$(7 \cdots 12)$	formula: $12(e + e') + cc'$.
2d cycle	$(13 \cdots 18)$	formula: $18(e + e') + cc' + 108$.
3d cycle	$(19 \cdots 24)$	formula: $24(e + e') + cc' + 288$.
4th cycle	$(25 \cdots 30)$	formula: $30(e + e') + cc' + 540$.
5th cycle	$(31 \cdots 36)$	formula: $36(e + e') + cc' + 864$.
. . . . .	. . . . .	. . . . .
nth cycle	$[6n + 1 \cdots 6(n + 1)]$	formula: $6(n + 1)(e + e') + cc' + 6^2(n^2 - 1)$.

First cycle problem: How many are 8×9? Hold up each hand clenched. Extend the right thumb and index finger ($= 8$). Extend the left thumb, index finger and middle finger ($= 9$). [In order to use this table, one must imagine that he has two little fingers on each hand. The fingers then beginning with the thumb have values respectively of 7, 8, 9, 10, 11, 12.] How many fingers are extended? Five. $5 \times 12 = 60$. How many are closed? Four on the right hand and three on the left. $4 \times 3 = 12$. Since $60 + 12 = 72$, $8 \times 9 = 72$.

The foregoing material may be cast in a form analogous to Table II as follows:

TABLE IV:

1st cycle $(7 \cdots 12)$ formula: $12(e + e') + cc'$.
2d cycle $(13 \cdots 18)$ formula: $12(e + e') + ee' + 144$.
3d cycle $(19 \cdots 24)$ formula: $24(e + e') + cc' + 288$.
4th cycle $(25 \cdots 30)$ formula: $24(e + e') + ee' + 576$.
5th cycle $(31 \cdots 36)$ formula: $36(e + e') + cc' + 864$.

The general formula as given in Table III is here valid, when n is an odd number; when, however, n is an even number, the formula is:

$$n\text{th cycle } [6n + 1 \cdots 6(n + 1)] \text{ formula: } 6n(e + e') + ee' + 6^2n^2.$$

Latin passages similar to those discussed above are as follows:

Plautus: *Stich.*, 706. Ovid: *Pont.*, 2, 3, 117. Seneca: *Epist.*, 88, 10. Pliny: *Natural History*, 2, 87. Irenæus: 5, 30, 1. Tertullian: *Apologeticus*, 19. Martianus Capella: 2, 102; 7, 729 and 746. Firmicus: *Mathesis*, 1, 4, 13. Ambrose: *Tob.*, 7, 25. Pacianus: *Epist. ad Sym. tertia*, 3, 25. Macrobius: *Saturnalia*, 1, 1, 6; *Somnium*, 2, 11, 17. Augustine: *Sermo.* (ed. A. Mai; nova patrum bibliotheca, I), 158, 14; 270, 7. Cassianus: *Conl.*, 24, 26, 7. Boethius: *In Porphy. Comm.*, Sec. 1, 2, p. 138, 19 (Migne, 64).

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A SIMPLE PROOF OF HART'S THEOREM.

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The theorem which forms the subject of the present note was discovered by Sir Andrew Hart[1] and may be stated as follows:

If a triangle be formed by the arcs of three circles, the inscribed and the three escribed circles are all tangent to a new circle or line.

This theorem is to be found in numerous works on modern elementary geometry and has been proved also in countless shorter articles. All of the proofs with which I am familiar, at least all of those which are elementary in character, are essentially alike and depend upon Casey's criterion for four circles tangent to a fifth. If the circles be c_1, c_2, c_3, c_4, a necessary and sufficient condition that they be tangent to a circle or line is that there should exist among their common tangents a relation

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0,$$

where t_{ij} means the length of a common tangent to c_i and c_j.[2] Hart's theorem comes very quickly by four successive applications of this relation. The trouble is that the amount of confidence that one is willing to repose in this criterion varies inversely as the amount of care with which one regards it. The difficulties are as follows:

(*a*) The proof that this equation affords a sufficient condition that four circles should be tangent to a fifth is rather intricate.[3]

(*b*) The condition, as a theorem in elementary geometry, is not necessary. We may have four circles tangent to a fifth where one of the four surrounds the other three. Here, three of the common tangents simply do not exist in the universe of discourse.

(*c*) Some of the common tangents may be direct and some transverse, and exact statement covering all cases correctly is cumbersome.

(*d*) It is a tedious task to look after all of the $+$ and $-$ signs. The usual method is to write them all $+$ and trust in Providence.

It is for these reasons that it seems to me worth while to give another proof based upon different considerations. Our method consists simply in throwing Hart's theorem back upon the theorem of Feuerbach, and assuming that the reader is familiar with one, at least, of the many ordinary proofs of the latter.[4] Feuerbach's theorem states that the inscribed and escribed circles of a triangle are tangent to a fourth circle, namely, the nine-point circle.

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[1] "On the Extension of Terquem's Theorem," *Quarterly Journal*, Vol. IV, 1860.

[2] Conf. Casey's *Sequel to Euclid*, second ed., London, 1881, p. 101.

[3] Conf. Lachlan, *Modern Pure Geometry*, London, 1893, p. 244. Casey avoids the difficulty by assuming that a necessary condition must be sufficient.

[4] Certain mathematicians seem to consider that the proving of Feuerbach's theorem constitutes a separate branch of mathematics. Conf. Sawayama, "Nouvelles démonstrations d'un théorème relatif au cercle de neuf points," *L'Enseignement mathématique*, Vol. XIII, 1911. This paper contains nine new proofs.

Proof. The circles constituting the triangle shall be $c_1c_2c_3$, the inscribed circle c, while the escribed circles need not be named. Let c' be the circle tangent to c_1, c_2, c_3 which couples with c, *i. e.*, has either exactly the same contacts with all three, or exactly the opposite contacts. If c' reduce to a point, we may at once invert c_1, c_2, c_3 into three lines, and we have reached Feuerbach's theorem. If c' be not a point or a straight line (a case which we may avoid by a preliminary inversion) let N be a point on the axis of this circle at a radius's distance from the center. We take a sphere of inversion with N as center, passing through c'. The plane will invert into that sphere which has N for north pole and c' for equator, and our problem consists in proving Hart's theorem for the sphere, where the given circles touch the equator. We next make the simple transformation which consists in replacing each great circle by one of its poles, say that which lies in the northern hemisphere. This transformation carries a circle into a circle, and tangent circles into tangent circles. Hence we have merely to prove Hart's theorem on a sphere where the three given circles pass through the north pole. But if we repeat our previous inversion, the three circles through the north pole become three straight lines in the plane, and we are carried back to Feuerbach's theorem once more. Our proof of the dependence of the one theorem upon the other is, thus, complete.

A TRIBUTE TO JOHN HOWARD VAN AMRINGE.

In the death of Professor John Howard Van Amringe September 10, 1915, in his eighty-first year, there passed away from Columbia College one of the greatest men, and without doubt the most beloved man, ever connected with that venerable institution.

Born of Dutch parentage in Philadelphia in 1836 and prepared for college mainly by his father, he entered Yale in 1854. At the end of his sophomore year he transferred to Columbia College and was graduated from this institution with the degree of A.B. in 1860, at the age of twenty-four. So brilliant and many-sided were his native powers and his attainments that even before graduation he had been tendered an instructorship in no fewer than five widely diverse departments: Greek, Latin, history, chemistry, mathematics. He might with equal propriety have been invited into the department of English, had there been such a department at that time, for his extant writings, including many published addresses, show that he had a remarkable control over the resources of English speech.

He chose mathematics and this subject he taught till his resignation from Columbia University, October, 1909, after fifty years of service as teacher and many years of service as dean of the college. Why he chose this science I do not know, but, as President Butler has felicitously said, "There was something curiously appropriate in his choice of mathematics as the agency of his academic influence, for there was in it that rigor of demonstration and that accuracy of statement which marched so well with his sturdy uprightness, his straight

thinking, and his unbending integrity; and there was also in it that beauty of expression, that perfection of form, that appeal to the imagination which reflected his literary power, his literary appreciation, and his splendid hold on written speech."

Van Amringe was a great teacher, especially of undergraduate men. He was never converted to a belief in coeducation. He did not believe in, and he did not employ, the lecture method with undergraduates. He was convinced that one of the great desiderata is to teach students to read solid books understandingly, and so he assigned them daily definite lessons in a chosen book and required them to report, usually in the form of classroom recitations. Idleness was not tolerated; industry and achievement were praised generously and discriminatingly. If a student, having tried, failed to understand, he was not overwhelmed by explanations, but was led, by the too rare art of skillful questioning and suggestion, into the presence of the truth. But it was not his method nor his subject that made Van Amringe a great teacher. The secret lay in his personality, deep, rich, sympathetic, generous, upright, sturdy and profoundly devoted to the higher things of the spirit. What Professor Van Amringe taught was men. With him mathematics was not an end; it was a means, an agency for making his presence and personality effective in the enlightenment and edification of young men. And he had his reward: the great and increasing gratitude, admiration, and love of generation after generation of students and colleagues, manifested continuously and everywhere in private and in public. Despite all this he preserved throughout that modesty, simplicity and democracy of spirit that we like to fancy essential in the character of a great man. Indeed, his successor in the office of Dean, Dr. Keppel, has justly characterized "Van Am" as "absolutely unspoilable."

Of technical mathematical ability, Professor Van Amringe had much. Fortunately, he did not devote it to mathematical research and publication, for, had he done so, he could not have rendered the manifold other services which he did render and which but few producers of mathematics are qualified to render. He did indeed edit a now obsolete series of the mathematical works of Davies, and in 1874 he published a valuable pamphlet on "The Theory and Practice of Life Insurance." But, in the main, his great abilities expressed themselves in other forms of service: as a powerful helper in the establishment and direction of the Columbia School of Mines; as a contributor towards the establishment and early conduct of graduate mathematical instruction in Columbia University; as first president, and subsequently as member of the council, of the American Mathematical Society; as the author of a history of Columbia College; as churchman and publicist in the matter of good counsel, especially in the management of hospitals and schools; as chief organizer, leader and stimulator of alumni associations; but above all and most joyously of all as an exemplar for a half century of how mathematics may be made the means of releasing the faculties of young men and winning their allegiance to rectitude of thought and life. Professor and Dean Van Amringe was one of the greatest figures in the collegiate history of American education.

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BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Plane Analytic Geometry. By MAXIME BÔCHER. Henry Holt and Company, New York, 1915. xiii + 235 pages. $1.60.

In reading this book the first thing to impress one is the pleasing style. Everything is so simply and clearly, yet at the same time so accurately stated, that the average student should be able to read the text by himself and understand even the finer points. It is exactly the sort of book that those who have been fortunate enough to hear Professor Bôcher's lectures might have expected him to write. To give an example of this accuracy of statement, in deriving the equation of a locus, the author points out very clearly that all we have shown is that the moving point must satisfy the derived equation, and hence that the required locus may form only a part of what we have found. Again, he is at some pains to show that for the ellipse, when we rationalize the equation $FP + F'P = 2a$, the resulting equation is really equivalent to the original one—something the reviewer remembers seeing in no other text. Then, too, in deriving the equations of a straight line, care is taken to show, not merely that any point on the line satisfies the equation, but also the converse. Something, again, that is not done in every text.

In the introduction it is stated that "the one aim should be to put the student into possession of an instrument which he can himself use in proving new geometrical theorems or solving new problems." As a result of the observance of this principle, many will feel that there are too many theorems to be proved analytically and not enough drill problems. Whether this feeling is justified or not depends, of course, on the class and the instructor.

Here, as well as in the author's *Trigonometry,* a great deal of matter has been put in fine print to be omitted by the average class. The principal topic so relegated, besides oblique coördinates, is the normal form of the equation of a straight line. The distance from a point to a line is found by a direct method, and the question of the sign of this distance is treated later in fine print. Many will doubtless disagree with this procedure, but it has much to recommend it. The usual treatment of this distance seems a very roundabout and artificial one to students and usually presents difficulty; in fact the normal form often seems to them a most abnormal one. Omitting it tends to make the work seem more direct. On the other hand, the normal form is the only one to which the equation of *any line whatever* may be reduced, and it is almost indispensable in any work using abridged notation. This does not form an argument, however, for its being studied in a first course; and the reviewer is in hearty agreement with the arrangement in the text.

At the end of the book are two chapters on the differential calculus—not "hashed fine," as the author says in the introduction, "but put squarely as a new subject," covering a surprisingly large amount of the subject with unusua, clearness. While these chapters are very carefully written and form an excellent

introduction to the calculus, there is plenty of material in the rest of the book for those who prefer not to take up the calculus at this time.

Many who teach analytic geometry to students of whom it is required, will think the book too difficult, but others will find it well adapted to their needs. In any case, it is a book to be considered in choosing a text, and presents a distinct advance over the usual textbook.

The general appearance and typography of the book are excellent.

It should be added that Professor Bôcher has written a syllabus for a course in solid analytic geometry following the same lines as this book.

ELIJAH SWIFT.

UNIVERSITY OF VERMONT.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

448. Proposed by W. D. CAIRNS, Oberlin College.

In the *Washington* (D. C.) *Times*, Mr. W. A. Dayton called attention some weeks ago to a curious repetition of digits in the decimal value of 1/115. If this decimal, which we print in the form $0.\dot{8}6956521739130\ 4347826\dot{0}$ be divided by two, the result is $0.\dot{4}3478260\ 8695652173913\dot{0}$, the fourteen-digit and eight-digit groups having been thus interchanged. A similar result, as he points out, is obtained if the original decimal value is divided by four. Mr. Dayton asks that this curiosity be explained.

449. Proposed by FRANK IRWIN, University of California.

Sum the expression

$$1 + 2\binom{k+1}{k} + 3\binom{k+2}{k} + \cdots + (n-k+1)\binom{n}{k}.$$

Also show how to sum

$$1\cdot 2 + 2\cdot 3\binom{k+1}{k} + 3\cdot 4\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)\binom{n}{k},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4\binom{k+1}{k} + 3\cdot 4\cdot 5\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)(n-k+3)\binom{n}{k},$$

etc., where $\binom{l}{k}$ is used to denote the coefficient of x^k in $(1+x)^l$.

GEOMETRY.

479. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the locus of the point whose polars (polar planes) with respect to two given conics (quadrics), are perpendicular to each other.

480. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Of equal quadrilaterals on the same base, that which has the least perimeter must have the angles not adjacent to the base equal to each other.

CALCULUS.

400. Proposed by H. S. UHLER, Yale University.

The axis of a prism whose right-section is a regular polygon of apothem a and of n sides passes through the center of a sphere of radius R. Show that, in general, the volume may be expressed by the formula:

$$V = \frac{4}{3}\pi R^3 + \frac{2}{3}a^2 n\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n} + \frac{1}{3}an(3R^2 - a^2)\sin^{-1}\left[\frac{2a\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}}{R^2 - a^2}\right] - \frac{4}{3}nR^3\sin^{-1}\left[\frac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right].$$

Also discuss the two special cases where

$$a = R\cos\frac{\pi}{n}, \text{ and } n = \infty.$$

401. Proposed by LAENAS G. WELD, Pullman, Illinois.

Given a continuum of triangles whose sides are in arithmetical progression, the common difference being h: (*a*) The ratio of the mean value of the areas of all the triangles, the mean of whose three sides is not greater than μ, to the area of the triangle, the mean of whose three sides is equal to μ, is $(\mu + 2h)/3\mu$. Indicate the limiting values of this ratio and show that, when it is equal to $\frac{1}{2}$, the triangle whose mean side is μ is right angled. (*b*) The ratio of the mean value of the areas of the circles inscribed in all of these triangles, the mean of whose three sides is not greater than μ, to that of the circle inscribed in the triangle, the mean of whose three sides is equal to μ, has the limiting values $\frac{1}{2}$ and $\frac{1}{3}$. When the triangle whose mean side is μ is right angled, the ratio in question is 4/9. (*c*) Of the circles circumscribed about these triangles the minimum has the radius $2h$.

MECHANICS.

319. Proposed by LAENAS G. WELD, Pullman, Illinois.

A hexagonal pencil lies upon the inclined top of a drawing table and is on the point of either rolling or sliding. Find the angle between its direction and the horizontal edge of the table, the coefficient of friction being μ.

320. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A heavy uniform chain of length l is hung over a rough horizontal cylinder of radius r. Show that one end of the chain will be

$$\frac{2\mu r}{\mu^2 + 1}(e^{\mu\pi} + 1) + l(e^{\mu\pi} - 1)$$

units lower than the other, just when the chain begins to move, the coefficient of friction being μ.

NUMBER THEORY.

237. Proposed by NORMAN ANNING, Chilliwack, B. C.

Prove that for three numbers, x, y, z,

$$9\Sigma(x - y)^4 = \Sigma(2x - y - z) = 2\square.$$

238. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the rational value of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

436. Proposed by WALTER H. DRANE, Cumberland University, Tenn.

The product of two numbers, p and q, may be obtained by dividing p by 2 successively, discarding remainders, until the quotient 1 is obtained and then multiplying q by 2 successively and adding those products, $2^i q$, which correspond to those quotients of p that are odd numbers.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

The meaning of the proposition is a little clearer if we write "those products, $2^i q$, $(i = 0, 1, 2, \ldots)$ which correspond to the odd numbers of the set composed of p and the quotients."

We have:

$$p = x_0 + 2n_0, \text{ where } x_0 = 1 \text{ or } 0 \text{ according as } p \text{ is odd or even;}$$

$$n_0 = x_1 + 2n_1, \text{ where } x_1 = 1 \text{ or } 0 \text{ according as } n_0 \text{ is odd or even;}$$

. . . .

. . . .

$$n_{i-1} = x_i + 2n_i, \text{ where } x_i = 1 \text{ or } 0 \text{ according as } n_{i-1} \text{ is odd or even,}$$

and so on until we reach $n_k = 1$, so that $n_{k-1} = x_k + 2$.

Substituting,

$$p = x_0 + 2x_1 + 2^2x_2 + 2^3x_3 + \cdots + 2^ix_i + \cdots + 2^kx_k + 2^{k+1}$$

$$qp = x_0(q) + x_1(2q) + x_2(2^2q) + \cdots + x_i(2^iq) + \cdots + x_k(2^kq) + 2^{k+1}q.$$

This, in view of the definition of the x's, proves the proposition, for the process associates

$$(p, q), (n_0, 2q), (n_1, 2^2q), \cdots, (n_{i-1}, 2^iq), \cdots.$$

Also solved by N. P. PANDYA, S. A. JOFFE, A. H. HOLMES, FRANK IRVING, and NORMAN ANNING.

437. Proposed by C. N. SCHMALL, New York City.

Given that $s_1, s_2, s_3, \cdots, s_k$ are the sums of k arithmetical series, each taken to n terms. The first terms are, respectively, 1, 2, 3, $\cdots$, k, and the common differences are 1, 3, 5, $\cdots$ $(2k - 1)$. Show that

$$s_1 + s_2 + s_3 + \cdots + s_k = \frac{nk(nk + 1)}{2}.$$

SOLUTION BY N. P. PANDYA, Sojitra, India.

We have

$$s_k = k + \{k + (2n - 1)\} + \{k + 2(2n - 1)\} + \cdots + \{k + (n - 1)(2n - 1)\}$$

$$= \frac{2n^2k - n^2 + n}{2}.$$

Hence,

$$\sum_{i=1}^{i=k} s_i = n^2 \sum_{i=1}^{i=k} i - \frac{kn^2}{2} + \frac{kn}{2} = \frac{n^2k(k+1) - kn^2 + kn}{2} = \frac{nk(nk+1)}{2}.$$

438. Proposed by WALTER C. EELLS, U. S. Naval Academy, Annapolis, Maryland.

In Hardy's *Pure Mathematics* (page 14, Nos. 2, 3) occurs the problem: "Show that if m/n is a good approximation to $\sqrt{2}$, then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions, *e. g.*, 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, $\cdots$." Find (*a*) other approximations for $\sqrt{2}$ of the same type, *i. e.*,

$$\frac{m'}{n'} = \frac{am + bn}{cm + dn}, \quad (a,\ b,\ c,\ d,\ m,\ n, \text{ integers}).$$

(*b*) Similar approximations for the square roots of other integers.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Assume that $(m/n) < \sqrt{2}$. We wish to find a, b, c, d so that

$$(1) \qquad 0 < \frac{am + bn}{cm + dn} - \sqrt{2} < \sqrt{2} - \frac{m}{n}.$$

Assume a, b, c, d positive. The second inequality leads, after simple reductions and squaring both sides, to

$$(2) \qquad \begin{aligned} c^2m^4 + 2(ac + cd)m^3n + (a^2 + 2ad + d^2 + 2bc - 8c^2)m^2n^2 \\ + 2(ab + bd - 8dc)mn^3 + (b^2 - 8d^2)n^4 < 0. \end{aligned}$$

If we had assumed $m/n > \sqrt{2}$, inequality (1) would have been reversed, and hence (2) also. But if the right-hand side of (2) changes sign with $m - \sqrt{2}\,n$, it must have $m - \sqrt{2}\,n$ and hence also $m^2 - 2n^2$ as a factor. This leads to the following equations connecting a, b, c, d.

$$(3) \qquad ab + bd + 2ac - 6cd = 0.$$

$$(4) \qquad b^2 - 6d^2 + 2a^2 + 4ad + 4bc - 12c^2 = 0.$$

These equations have for a solution $a = d$, $b = 2c$; and this is the only solution in positive integers.

The first inequality of (1) leads, after substitution of a for d, and $2c$ for b, to $(a^2 - 2c^2)(m^2 - 2n^2) > 0$. Hence, $a^2 < 2c^2$. Finally, in (2) putting $a = d$, $b = 2c$, we obtain, after factoring, $[m^2 - 2n^2][c^2m^2 + 4acmn + (4a^2 - 2c^2)n^2] < 0$. Hence, $c^2m^2 + 4acmn + (4a^2 - 2c^2)n^2 > 0$, or $(cm + 2an)^2 > 2c^2n^2$, that is, $m/n > \sqrt{2} - (2a/c)$. If $(2a/c) \geqq 1$, m/n need only be as large as .5 to satisfy this inequality.

If we had assumed $m/n > \sqrt{2}$, we should obviously be led to the same results.

If, then, $m'/n' = (am + 2cn)/(cm + an)$, and $2a \geqq c$, $a^2 < 2c^2$, or in other words $8c^2 > 4a^2 \geqq c^2$, we shall have satisfied the conditions of the problem. Such

sets of values are $\begin{cases} c = 1 \\ a = 1 \end{cases}$, $\begin{cases} c = 2 \\ a = 1 \end{cases}$, $\begin{cases} c = 3 \\ a = 2 \text{ or } 4 \end{cases}$, etc. The first set is the one given in the problem. The second gives $m'/n' = (m + 4n)/(2m + n)$ which yields the series 1/1, 5/3, 17/13, 69/47, ...

(b) Almost the same work leads to similar results in the case of any surd, $\sqrt{k}$. We find $a = d$, $b = kc$, $2a \geqq c(k - 1)$, $kc^2 > a^2$. These possess, among others, the following solutions for the case where $k = 3$.

$$\begin{cases} c = 1 \\ a = 1 \end{cases}, \begin{cases} c = 2 \\ a = 3 \end{cases}, \begin{cases} c = 3 \\ a = 4 \text{ or } 5 \end{cases}, \text{etc.}$$

The first set gives $m'/n' = (m + 3n)/(m + n)$, which yields the series 1/1, 4/2 = 2/1, 5/3, 14/8 = 7/4, 19/11, ...

Also solved by NORMAN ANNING.

GEOMETRY.

466. Proposed by HORACE OLSON, Chicago, Illinois.

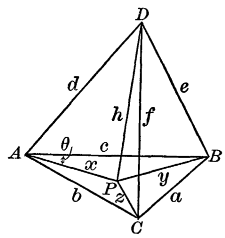
Given the edges of a triangular pyramid, find the radius of the inscribed sphere.

SOLUTION BY A. H. HOLMES, Brunswick, Maine.

Calling the radius of the sphere r, and using the notation in the figure, we have,

$$h^2 = d^2 - x^2 = c^2 - y^2 = f^2 - z^2,$$

$$\cos\theta = \frac{c^2 + x^2 - y^2}{2cx} = \frac{c^2 + d^2 - e^2}{2cx} = \frac{n}{x}. \qquad (x\cos\theta = n.)$$



$$\cos(A - \theta) = \cos A\cos\theta + \sin A\sin\theta = \frac{b^2 + x^2 - z^2}{2bx} = \frac{b^2 + d^2 - f^2}{2bx} = \frac{m}{x}.$$

Hence,

$$\sin\theta = \frac{m - n\cos A}{x\sin A}, \qquad x\sin\theta = \frac{m - n\cos A}{\sin A}.$$

Hence,

$$x^2 = n^2 + \frac{(m - n\cos A)^2}{\sin^2 A}, \qquad \text{and} \qquad h = \frac{\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2}}{\sin A}.$$

Hence,

$$3 \text{ times contents of pyramid} = \frac{bc}{2}\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2},$$

and therefore,

$$r = \frac{bc\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2}}{2(ABC + ACD + ABD + BCD)},$$

in which

$$m = \frac{b^2 + d^2 - f^2}{2b}, \qquad n = \frac{c^2 + d^2 - e^2}{2c}, \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\sin A = \sqrt{1 - \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2}}, \qquad ABC = \sqrt{s(s-a)(s-b)(s-c)},$$

and similarly for ABD, ACD and BCD.

Also solved by WALTER C. EELLS and J. W. CLAWSON.

CALCULUS.

380. Proposed by C. N. SCHMALL, New York City.

Show that

$$\int_0^\infty \left[\frac{1}{1^4 + x^2} + \frac{1}{2^4 + x^2} + \frac{1}{3^4 + x^2} + \cdots\right] dx = \frac{\pi^3}{12},$$

where the series in the brackets is infinite.

SOLUTION BY A. M. HARDING, University of Arkansas.

For all values of x in the interval $(0, \infty)$ the nth term of the given series has the property

$$\frac{1}{n^4 + x^2} \leqq \frac{1}{n^4}.$$

Now the series $\sum\limits_{n=1}^{\infty} 1/n^4$ converges. Hence, the given series is uniformly convergent in the interval $(0, \infty)$.

Each term of the series is continuous in the interval $(0, \infty)$.

Hence, it may be integrated term by term. Hence,

$$\int_0^\infty \left[\frac{1}{1^4 + x^2} + \frac{1}{2^4 + x^2} + \frac{1}{3^4 + x^2} + \cdots\right] dx = \frac{\pi}{2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right].$$

It can be easily shown that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Hence, the given integral $= \dfrac{\pi^3}{12}$.

Also solved by S. A. JOFFE, and J. A. CAPARO.

299. Proposed by B. F. FINKEL, Drury College.

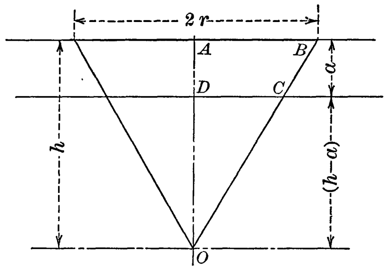
A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid; to find how much its density must be increased that it may rest with its base in the common surface of the fluids. (From Walton's *Hydrostatical Problems.*)

SOLUTION BY J. F. BRACHO, University of Notre Dame.

Let, w = density of cone in first position, w_1 = density of cone in second position, d_1 = density of upper fluid, d_2 = density of lower fluid. We have:

$$DC = \frac{AB + OD}{OA} = \frac{r(h-a)}{h}.$$

By Archimedes' principle, we may write,



$$\frac{1}{3}\pi r^2 hw = \left[\frac{1}{3}\pi r^2 h - \frac{1}{3}\pi\frac{r^2(h-a)^3}{h^2}\right]d_1 + \frac{1}{3}\pi\frac{r^2(h-a)^3}{h^2}d_2;$$

Then,

$$w = d_1 + \frac{(h-a)^3}{h^3}(d_2 - d_1).$$

For the second position of the cone we must have,

$$w_1 = d_2.$$

Hence, increase of

$$w = w_1 - w = d_2 - d_1 - \frac{(h-a)^3}{h^3}(d_2 - d_1) = (d_2 - d_1)\left[1 - \frac{(h-a)^3}{h^3}\right].$$

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

At the time of making up copy for this issue no replies had yet been received for Questions numbered 4, 8, 12, 13, 16, 20 and 28.

REPLIES.

27. A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytic geometry and differential and integral calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours?

Note.—Last month we published a reply to this question by a professor in a middle western college. The reply given below was received while the December issue was still in press. It therefore gives an entirely independent view by an instructor in an eastern university. We hope that our readers will compare the two replies and note the points of agreement.—EDITOR.

REPLY BY R. B. ROBBINS, Sheffield Scientific School, Yale University.

To deal intelligently with the question it must be made more definite. Let us assume that just one year of calculus is presupposed and that an additional twelve-hour curriculum is desired which can be repeated without much modification year after year. With only the interests of prospective students of pure mathematics in mind it seems that such a college might hope to attain two fairly definite ends (I and II below) by means of the courses outlined below:

I. Introduce the student to the general subject matter of mathematics by courses in (A) topics in analysis—3 hours; (B) modern geometry—3 hours.

II. Give the student some of the standard tools for more advanced work by courses in (C) advanced calculus and solution of differential equations—3 hours; (D) infinite series, definite integrals and topics in advanced algebra—3 hours.

This doubtless looks rather formidable for a third-year college student; therefore a few words of explanation. In his junior year the student should take courses B and C, reserving courses A and D for his senior year.

Course C might well contain quite a variety of material. If the student does not have access to a course in mechanics in some other departments it certainly should contain some of the applications of the calculus to mechanics. This is the place to make partial differentiation and certain topics in the integral calculus concrete by application to solid analytic geometry. The solution of the commoner types of differential equations should also be included here.

Course B can be made as difficult and dull or as easy and inspiring as the instructor desires. If the student is led gradually into the meaning and importance of the transformations involved and is led to see why homogeneous coördinates are so useful, he will appreciate the power of his methods and will see in them a genuine beauty probably new to him in mathematical experience.

Courses A and D can best be studied together. In course A should be included an introduction to the theory of functions of a real and of a complex variable. It might also contain some of the easier parts of the theory of ordinary differential equations. Course D will include material useful in course A and in fact in any further work in pure mathematics. The importance of this course as an introduction to further work in mathematics cannot be over-emphasized and yet the material can be grasped by a college senior whose ability justifies continuing the study of mathematics.

An immediate objection to such a set of courses is the difficulty of finding text-books. Course C would not be so troublesome in this respect. As for course B, it seems to the writer that it is time for a book to appear dealing with methods in modern geometry from the bottom up (from the student's point of view) rather than from the top down. The material of courses A and D is such that if the instructor knows it well enough to teach it without a book, the student will be much better off if he makes his own text-book from careful notes; while if the instructor does not know it thoroughly, a text would be of little value and he better not attempt to give the courses.

There is room for much more coöperation than at present exists between the colleges of the type in question and the graduate schools in the study of mathematics. Since the only equipment essential for the mathematics department is the instructor and the blackboard, there is no reason why first class introductory work in advanced mathematics should not be done in a great many of our colleges. On the other hand, the demand for graduate instruction in mathematics is so small and the opportunities at the few large graduate schools are such that it is rarely wise for a graduate student to study at a place where he commands the

attention of very few men who are giving their best efforts to research in mathematics. For this reason, the type of college here considered has a most important work in opening the eyes of the student to the possibilities ahead of him and in giving him definite, essential tools with which to work; but such a college blunders if, instead, it tries to give more specialized courses which demand for their full appreciation a preparation which the student cannot have had.

29. While studying the problem of two equal rough bodies, connected by an inelastic wire, resting on an inclined plane, Professor Clifford N. Mills of South Dakota State College met the following interesting expression. If $1/a$ and $1/(a+1)$ are the coefficients of friction, the tension of the wire when the bodies are about to descend becomes a multiple of $\frac{1}{2}/[1/a + 1/(a+1)]$. This, when simplified, becomes $(2a+1)/2a(a+1)$. If $2a+1$ and $2a(a+1)$ represent the base and altitude of a right triangle, the hypothenuse is $2a^2+2a+1$. Therefore, this gives a series of numbers which satisfy the relation $x^2+y^2=z^2$, if a is given any value whatsoever. Professor Mills desires to know if this will give all the integers which satisfy the condition that the sum of the squares of two integers equals the square of an integer.

Reply by S. Lefschetz, University of Kansas.

If the integers x, y, z, form a Pythagorean set, that is, if they satisfy the relation $x^2+y^2=z^2$, then it is well known* that there can be found three other integers m, n, p, such that $x=p(m^2-n^2)$, $y=2pmn$, $z=p(m^2+n^2)$. For $m=a+1$, $n=a$, $p=1$ the special solution indicated by Professor Mills is obtained.

NOTES AND NEWS.

Edited by D. A. Rothrock, Indiana University.

Professor Paul Painlevé, of the department of mathematics in the University of Paris, is minister of education in the present French cabinet.

Henry Holt and Company have just published a "Plane Analytic Geometry," by Professor Maxime Bôcher. The last two chapters are devoted to calculus.

Ginn and Company have published "Problems in the Calculus," by Dr. D. D. Leib of the Sheffield Scientific School, Yale University, also "The Theory of Invariants," by Professor O. E. Glenn, of the University of Pennsylvania.

The sixteenth in the series of Wiley's Mathematical Monograph Series appears as "Diophantine Analysis," by Professor Robert D. Carmichael. The same firm has recently published an "Analytic Geometry," by Professor H. B. Phillips, of the Massachusetts Institute of Technology.

Junior Professors Peter Field, L. C. Karpinski and T. R. Running have been promoted to associate professorships of mathematics at the University of Michigan, and Drs. T. Fort and T. H. Hildebrandt to assistant professorships

* Cf. Bachman, *Zahlentheorie*, Vol. I, p. 192. Also Carmichael, *Diophantine Analysis*, pp. 8–13.

of mathematics. DR. A. L. NELSON has been appointed to an instructorship in mathematics.

In *Science* for October 22 appears a paper by PROFESSOR C. N. MOORE in which he points out the danger of assuming that the coefficient of correlation is necessarily a satisfactory measure of all forms of relationship between two variable quantities, at the same time suggesting a method of attack for determining in what way a particular relationship depends on the value of this coefficient.

The United States Bureau of Education has recently published a bulletin entitled "Mathematics in the lower and middle commercial and industrial schools of various countries represented in the International Commission on the Teaching of Mathematics." It is furnished to teachers of mathematics on application to the Bureau.

"Cubic surfaces and their nodes" is the title of an article by Dr. S. LEFSCHETZ in the *Bulletin* of the University of Kansas, Volume IX, number 6. This bulletin is issued at irregular intervals. The present volume contains 290 pages and 76 plates. It is a science number, containing 21 articles, the one mentioned above being the only one on a mathematical subject. The exchange editor is Dr. U. G. MITCHELL, of the MONTHLY staff.

PROFESSOR H. L. RIETZ, University of Illinois, was appointed a member of a committee of four to investigate the operation of all pension laws enacted in the State of Illinois, "together with the present and future cost thereof, and to collect information as far as possible in regard to the operation of similar laws in other states, and countries, and to make recommendation upon this subject to the next General Assembly."

At the High School Conference held at the University of Illinois on November 18–20, the following mathematical papers were presented: "Algebra from the utilitarian standpoint," by Dr. A. R. Crathorne; "The experimental determination of standards in first year algebra," by Dr. H. O. Rugg; "Report of the committee investigating high school mathematical libraries," by Professor E. H. Taylor; "Graphs in elementary algebra," by Mr. H. C. Zeis; "The function notion in elementary algebra," by Professor J. F. Millis.

The part of the *Encyclopédie des Sciences Mathématiques* which treats the theory of domains of algebraic numbers appeared on June 18, 1915. The first three pages conclude the article on transcendental propositions relating to the theory of numbers and the last seven pages begin the article on complex multiplication. The remaining 86 pages are devoted to algebraic number realms, while the German edition devotes only about 23 pages to this subject. The present part is issued as Tome I, volume 3, fascicule 5, and consists of 96 pages.

The third regular meeting of the Association of Mathematics Teachers of New Jersey was held at Stevens Institute of Technology, November 20, under the Presidency of DEAN HENRY B. FINE. Papers were read as follows: "Mathematics and insurance," by P. C. H. PAPPS; "The proper functioning of a high school course in geometry," by R. T. LE VALLEY: "A review of Bourlet's plane geometry," by B. B. STRANG; and "A high school course in strength of materials," by G. D. ORNER.

The Mathematical Club of the University of Illinois is divided into two sections, known as the Graduate Section and the Undergraduate Section, respectively. The former meets bi-weekly for the consideration of papers involving new results or new methods, while the latter meets monthly for the consideration of questions of general mathematical interest and the solutions of problems. Each meeting lasts about one hour. The Undergraduate Section usually arranges for one social evening meeting annually. For the current year Professor G. A. Miller has been elected chairman of the Graduate Section, and Mr. G. W. Smith, chairman of the Undergraduate Section.

A "mathematics contest" which was held between the Hyde Park High School and University High School of Chicago is described by Mr. RALEIGH SCHORLING in *School Science and Mathematics* for December. In the same issue appear "The proofs of the law of tangents," by R. M. Mathews, "A simple and effective method of solving a polynomial," by C. H. FORSYTH, a "Graphical method for cubic equations," by ALFRED RITTER, and "Historical notes in textbooks on secondary mathematics," by PROFESSOR G. A. MILLER, the last-named article criticizing the accuracy of the historical references made in a recently published high school text.

The Mathematics Club of the University of Kansas is an organization of students with Dr. E. B. STOUFFER as faculty adviser. It was started in 1911 and now has twenty-six members. The Club meets bi-weekly and is carrying out the following program during the present year: Fermat's theorem and allied topics, Dr. S. LEFSCHETZ; "Non-Euclidean geometry," Miss JESSIE JACOBS; "Line construction," Miss ADA WEST; "Who's who in mathematics in America," Dr. U. G. MITCHELL; "Mathematical reference books," Dr. E. B. STOUFFER; "Curve tracing," Mr. P. W. HORNLEY; "Methods of computing errors," Dr. H. E. JORDAN; "Quadratic forms in number theory," Miss WILMA ARNETTE; "Elements of orbits of heavenly bodies," Miss CORA SHINN; "Mathematical fallacies," Mr. J. B. RAMSEY; "Some definite integrals," Mr. A. W. LARSEN; "The origin of the calculus," Mr. L. L. STEIMLEY; "Finite geometry," Mr. CYRIL NELSON; "Review of *Memorabilia Mathematica*," Miss FLORENCE SCHEIDENBERGER.

Beginning with this issue the MONTHLY becomes the Official Journal of THE MATHEMATICAL ASSOCIATION OF AMERICA, and will be subject to the control of the Council of the ASSOCIATION. Thus the MONTHLY enters a broad field of

usefulness and is assured of wise and efficient support. The Constitution provides that the official journal shall be directly controlled by a Committee on Publication, consisting of a Managing Editor and two other members, who are appointed by the Council and thereby become *ex officio* members of the Council. The Editorial Board consists of the three members of the Publication Committee, together with twelve Associate Editors selected by them. The editorial work is divided among various subcommittees, and all are now laying plans for immediate aggressive action. They will welcome suggestions from all sources, looking toward the strongest possible equipment of the MONTHLY for the important responsibilities now laid upon it.

The Editorial Board for 1916 is as follows:

NOTES ON THE COLUMBUS MEETING.

A most remarkable circumstance occurred at Columbus, in connection with the choosing of the name for the new organization. A committee of three had been chosen to sift the eighteen proposals and make a recommendation. The committee agreed to act independently and each to make his choice by himself. They did so and each made the same choice. Moreover, five other members, after discussing the matter informally by themselves, also came to the same conclusion. This seemed so remarkable that the final adoption seemed inevitable.

Another interesting feature of the Columbus meeting was the remarkable unanimity of purpose displayed by representatives of all interests concerned. There seemed to be no doubt as to the importance of the opportunity presented to this new organization in its chosen field. There was no lack of serious interest and of genuine enthusiasm.

Great confidence was placed in the judgment and wisdom of the nominating committee for the new Association. All will realize how difficult a task they had to perform in distributing the various positions of responsibility among the districts and interests far more numerous than the number of nominees. How well they performed this task remains to be shown by the fidelity and efficiency with which these officers discharge their duties, and by the clearness of vision which shall animate them as they contemplate their opportunities for genuine service to the cause of mathematics in America.

Particular attention is called to the second section in the By-laws on the nomination and election of officers. The success of such a thoroughly democratic plan of procedure will depend entirely upon the extent to which the members appreciate its importance and uphold its dignity by conscientious and thoughtful use of the franchise thus provided.

Many questions have been asked with reference to what will be the attitude of THE MATHEMATICAL ASSOCIATION OF AMERICA to the existing secondary associations in various parts of the country. The official answer to these questions is section 2 in the first article of the Constitution. It may be further stated unofficially that the spirit of the new ASSOCIATION, so far as it has had opportunity to manifest itself, will surely be one of friendly coöperation. It is fully recognized that these associations have most important and far-reaching problems in their own field and that they are doing efficient service toward the solution of these problems. But the new ASSOCIATION also has its peculiar problems in a field hitherto quite unoccupied, and, in entering upon its responsibilities, it invites the coöperation of all individuals, or groups of individuals, who may in any way be interested in the field of collegiate mathematics.

Blanks for admission to membership in THE MATHEMATICAL ASSOCIATION OF AMERICA are sent under separate cover to all persons to whom this issue of the MONTHLY goes. All who return these with the annual dues before April 1, 1916, will be entitled to admission without the initiation fee and will constitute the Charter Membership of the Association. How large this initial list of members shall be depends upon the readiness of response of individuals. Thus far, about seven hundred persons have manifested direct and personal interest by signing and returning cards, thus giving rise to the hope that the charter membership may reach that number. Shall it be so?

The annual dues in THE MATHEMATICAL ASSOCIATION OF AMERICA have been fixed at THREE DOLLARS, including a yearly subscription to THE AMERICAN MATHEMATICAL MONTHLY, this being the lowest possible figure at which the expenses of the ASSOCIATION can be met on a membership basis of less than one thousand. The MONTHLY alone has actually cost about this amount per subscriber for the past three years, the deficit being provided for by subsidies from fourteen universities and colleges. These subsidies will, of course, no longer be continued.

VOLUME XXIII FEBRUARY, 1916 NUMBER 2

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

PUBLISHED BY THE ASSOCIATION

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1912. FROM 1912 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE AMERICAN MATHEMATICAL MONTHLY

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A TENTATIVE PLATFORM OF THE ASSOCIATION.

No man can speak with authority concerning the future of this new ASSOCIATION which was created by those who met at Columbus last December. Its future lies with those who constitute its membership. Any statement must be rather a history of past events than a prediction for the future.

What were the causes which led so many to wish for, to exert themselves and to struggle for a new society in the mathematical field? What motives lay behind the movement which culminated in this organization? These are questions which are distinctly answerable. I shall try to show for those who did form the Association what were their purposes and what is now their aim. If these purposes or aims are wrong or insufficient, they will perish and newer and better policies will supplant them. The great fact which we cannot overlook is that we now have a large and representative body of men and women interested in mathematics joined together in this association to foster whatever they believe to be worthy and beneficial.

The chief motive may well be said to be that of service to the whole body of teachers of mathematics in American colleges. If I am right, the Association will not stop at anything which will serve this body of men.

Perhaps there is one exception. The majority of those responsible for the new organization are themselves members of the American Mathematical Society. This older organization is itself bound by its constitution to promote the interests of mathematics in this country. That there should be any conflict between the two organizations would defeat the ends of both, and would not give the maximum service which can be rendered to American mathematicians. The American Mathematical Society has chosen, through the action of its Council,[1] to restrict its activities to the field of pure research in mathematics, and to the promotion of those phases of mathematics which are commonly associated with

[1] *Bulletin of the American Mathematical Society*, volume 21, page 482 (July, 1915); this MONTHLY, volume 22, page 252 (October, 1915).

that word. Those responsible for the new organization are by no means at variance with this determination, and it is their aim to carry out in good faith the separation of fields of activity provided for by the action just mentioned. This one limitation to the activities of the Association should therefore be mentioned prominently.

Another restriction which is imposed, not by any agreement but by the dictates of good judgment, is that matters dealing with secondary and elementary schools should be left to the organizations already in existence devoted to that field. The new organization will not undertake to discuss or to print papers specially dealing with the details of secondary instruction, though it may well undertake to discuss and define questions concerning the preparation of students who enter colleges, particularly with respect to training in mathematics.

This question of college entrance is one of such vital importance that some leadership of national standing is desirable to crystallize and to formulate the views of mathematicians of all grades of schools. Such questions cannot be said to be the primary function of any one class of organization, and it is thought that secondary school teachers will be the first to welcome a strong national leadership in this matter. It may be well to add that the Council of the American Mathematical Society decided specifically about a year ago not to undertake work along this line, in its relation to the attack upon mathematics in the secondary schools now being made in various quarters.

In general, however, the activities of the Association will be centered strongly in the collegiate field, and it is expected that the great majority of the work fostered by the Association will be on questions directly affecting collegiate courses in mathematics. That the range of topics which may be concerned is rather large, and that the considerations which may be presented are varied and complex is reasonably forecast by the papers which have appeared in the MONTHLY since its reorganization three years ago.

There will doubtless be many articles of historical interest. These will deal with topics which may lie anywhere in the entire range of mathematics. They may be said to be allied with the collegiate courses on the history of mathematics. Thus, the interesting paper by Professor Karpinski presented before the Columbus meeting was entitled "The Story of Algebra." It might be thought that this paper was therefore of secondary character. But the merest inspection of it will suffice to demonstrate that it lies beyond the secondary field and that its association is strictly with the history of mathematics. Other papers, such as Professor Cajori's remarkable series of articles on the History of Logarithms, and the various interesting papers on Number Systems which have appeared in the MONTHLY, are further indication of the intention to deal with matters of this type.

That elementary college courses are still open to serious reconsideration is evidenced by the appearance of several important papers in the MONTHLY during the last two years which deal with subjects taught in the freshman year. Other weighty contributions of this character are in type awaiting their turn.

Perhaps more deserving of mention is the fact that advanced college subjects

should properly fall within the field of this association, and that discussions which affect such topics as projective geometry, second courses in calculus, the elementary theory of functions, and other courses commonly given to undergraduates, are properly subjects for discussion.

One more idea would seem to me to clarify the situation very materially. There have appeared in the MONTHLY from time to time articles which cannot be said to be of research character from the standpoint of the common acceptation of that word, but which nevertheless represent a great deal of labor of a purely investigational sort which would seem quite worthy of being called research in a broader interpretation of that word. This again is well illustrated by the historical papers mentioned above, all of which certainly constitute a very dignified form of research in this broader sense, though they may not satisfy the stricter interpretation placed ordinarily on the word research. The same thing can be said of a number of other papers which have appeared in the MONTHLY which deal essentially with college subjects. It is held that a dignified discussion which involves investigation from a scientific point of view is worthy of the name research in its broader application. While the new association will recognize fully the prior right of the American Mathematical Society in all questions which would ordinarily be termed research under the common interpretation, the attitude of this Association will be to encourage and to dignify all investigations of the character which have here been called research in the broader sense.

If I have tried to say what seems to me to be the policy of those responsible for the organization of the ASSOCIATION, I should perhaps add a word concerning the questions distinctly avoided thus far, which may be said to be not within our present intentions. One such which certainly deserves mention is the general notion of pedagogy in its more restricted interpretation. All of those questions which are termed pedagogical in the strict sense of that word have been held, and are held, by those responsible for the organization of the Association to belong to the field of education and to be wholly outside the field of the present association. Just as research will be held to be within its province only if the word is given a broad interpretation, it may be said also that the discussions which this association will foster may be termed pedagogical only if that word is used in a much broader sense than is common. I may define this broader sense to include those questions affecting instruction in which a *professional knowledge of the subject-matter* is a necessary element toward the formation of any dignified conclusion. That there is no doubt about the existence of such questions is amply proved by the files of the MONTHLY during the last two years.

This statement does not pretend to be exhaustive or infallible. The intention is to give as clear an idea as may be in a short space of characteristic topics which this Association will discuss. That the policy of the Association may be changed in the future and that the statements of this article are by no means binding upon the Association will be quite evident upon even a casual examination of the Constitution.

E. R. HEDRICK, PRESIDENT.

COLUMBIA, MISSOURI, February 25, 1916.

"QUANTITY OF MATTER" IN DYNAMICS.[1]

By L. M. HOSKINS, Stanford University.

I. Introductory.

1. *Different Views Regarding Mass.*—In explaining the significance of the quantity commonly called *mass* the term "quantity of matter" has been freely used by many writers on the laws of motion, including such high authorities as Newton, Maxwell, Kelvin, Tait and Clifford. These writers have assumed, either explicitly or tacitly, that the words have a meaning apart from dynamical laws. From this point of view the second Newtonian law (or "law of acceleration") is a statement of the way in which the *acceleration* of a body depends upon the two factors, (1) the *force* acting on the body and (2) the *quantity of matter* in the body. There are, however, those who dissent from this view, maintaining that the word mass as used in dynamics is fully defined by the law of acceleration itself; that the statement that "the mass of a body is a measure of its quantity of matter" contributes nothing to the definition. Because of these differing views it seems profitable to consider somewhat carefully what is really involved in the conception of quantity of matter.

In this study we may ask first what meaning, if any, may properly be attached to the words quantity of matter independently of the laws of dynamics, and secondly, whether the use of these words contributes anything to the definition of the quantity usually called mass in the formulation of dynamical principles.

2. *Postulates Regarding Acceleration and Force.*—To answer the second question requires an analysis of the logical import of the laws of motion; and such an analysis, if exhaustive, must involve not only the meaning of mass and force but also the question of bases of reference for estimating acceleration. The present object does not, however, require an exhaustive analysis of these questions. As regards acceleration, we shall assume as granted either the Newtonian view of absolute motion or the view that, at any rate, there exists a base for which the Newtonian laws are true.[2] As regards force, we shall treat separately two assumptions—(1) that force may be accepted as an exactly measurable magnitude independently of the laws of motion, and (2) that the definition of force as a measurable magnitude is contained in those laws; in accordance with these two assumptions parallel sets of propositions will be given, designed to illustrate the significance of "quantity of matter" in the law of acceleration.

II. Preliminary Quantitative Notions Regarding Matter.

3. *Definition of Body.*—We regard matter as consisting of individual parts which are indestructible; and we define a body as *any definite connected portion of matter.*

[1] Paper presented to the American Mathematical Society, August 3, 1915, at Stanford University, Calif.

[2] Such a base may be called a Newtonian base. (See W. H. Macaulay, *Mathematical Gazette,* October, 1900; *Bull. Am. Math. Soc.*, July, 1897, and April, 1898.)

Explanations of the laws of dynamics are often stated as applying to "particles" of matter or "material points." A more practical presentation must, however, deal with aggregates of matter of finite size. We shall here mean by a body any connected aggregate of matter; and when reference is made to the acceleration of a body, it is to be understood that all parts of the body have equal accelerations, *i. e.*, that the motion is a translation.

4. *Common Notion About Quantity of Matter.*—That a quantitative view of matter is one of the common notions associated with every-day experience is evident on a very little reflection. The statement that one of two gold coins contains twice as much of that particular material as the other is certainly not meaningless, quite apart from any consideration of inertia, or of weight, or of volume, or of value; although a precise explanation of the statement would be as difficult as would be a precise explanation of the statement that one body has twice the bulk of another. The notion of quantity, associated with particular kinds of matter, is in fact employed daily by the many persons who are engaged in buying and selling useful commodities. Whatever method may be used for the actual comparison of quantities, the object sought is to determine definite amounts of certain kinds of matter, the utility of which is not generally due to their inertia, or weight, or bulk.

5. *Definite Quantitative Notions Applicable to Matter.*—There are, moreover, certain very definite quantitative comparisons which are applicable to matter, without restriction to bodies of the same homogeneous substance, and without appeal to dynamical principles or to particular physical laws such as the law of gravitation. We recognize the applicability to matter of the notions that

(I) *The whole is greater than any part, and the whole is equal to the sum of its parts.*

And thus, quite independently of any kinetic considerations, we accept the truth of the following propositions:

(II) *If a body A be divided into two bodies B and C, the quantity of matter of A is greater than that of B or that of C; the sum of the quantities of matter of B and C is equal to that of A.*

(III) *If any two distinct bodies B and C be combined into a single body A, the quantity of matter of A is greater than that of B or that of C; the sum of the quantities of matter of B and C is equal to that of A.*

III. Significance of Quantity of Matter in the Law of Acceleration, Assuming Force to be Defined Non-kinetically.

6. *First Assumption Regarding Force.*—We now assume that force is a quantity which can be exactly measured, or at least defined as an exactly measurable magnitude, without any appeal to the laws of motion. Thus we may accept as satisfactory a method of comparing force-magnitudes based upon the deformations of springs or of any elastic bodies. It will be seen, however, that in the first set of illustrative propositions given below the only requirement regarding the measurement of forces is that it is possible to apply equal forces to any different

bodies; while in the second set it is assumed possible to apply forces of any desired relative magnitudes to any bodies at pleasure.

7. *The Effects of Equal Forces Applied to Different Bodies.*—To illustrate the way in which the notion of quantity of matter enters into our interpretation of the law of acceleration, we may state a number of propositions referring to suppositious cases in which equal forces act upon different bodies.

(*a*) If a force be applied to a body B, and an equal force be afterward applied to a body A formed by adding matter to B, the acceleration of A will be less than that of B.

(*b*) If, when equal forces are applied to two distinct bodies A and B, the acceleration of A is less than that of B, then by the removal of matter from A there may be produced a body A' such that equal forces applied to A' and B will cause equal accelerations; and by the addition of matter to B there may be produced a body B' such that equal forces applied to A and B' will cause equal accelerations.

(*c*) If two distinct bodies have equal accelerations when acted upon by equal forces, then an acceleration half as great will be caused by an equal force acting upon the body formed by combining the two.

(*d*) If any number of distinct bodies have equal accelerations when acted upon by equal forces, then an acceleration one nth as great will be caused by an equal force acting upon a body formed by combining n of the given bodies.

(*e*) If two distinct bodies, when acted upon by equal forces, have accelerations a' and a'', then the body formed by combining them will, if acted upon by an equal force, have an acceleration a such that $1/a = 1/a' + 1/a''$.

(*f*) If any number of distinct bodies, when acted upon by equal forces, have accelerations a', a'', $\cdots$, then the body formed by combining them will, if acted upon by an equal force, have an acceleration a such that $1/a = 1/a' + 1/a'' + \cdots$. (It will be noticed that this case really includes the five preceding.)

These propositions are consequences of the interpretation we put upon the following principle, which is a part of the law of acceleration:

(*A*) *Different bodies acted upon by equal forces have accelerations inversely proportional to their masses.*

A part of the import of proposition (*A*) undoubtedly consists in giving precision to the definition of mass; but if the meaning of mass were wholly contained in it, not one of the propositions (*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) would necessarily follow from it. The reason we accept these as consequences of (*A*) is that we have, independently of (*A*), a conception of mass as a measure of quantity of matter and as satisfying propositions (I), (II) and (III).

8. *Case of Different Bodies Having Equal Accelerations.*—The part played by the quantitative notion of matter in our interpretation of the second law of motion is evident also in cases in which different bodies are supposed to be acted upon by forces such as to cause equal accelerations. This is brought out by the following propositions, which will be seen to have a correspondence to the propositions (*a*) $\cdots$ (*f*) given above.

(a') If a body A be formed by adding matter to a body B, the force required to give A any certain acceleration is greater than that previously required to give B an equal acceleration.

(b') If A and B are two distinct bodies such that a greater force must be applied to A than to B to cause equal accelerations, then by the removal of matter from A there may be produced a body A' such that equal forces must be applied to A' and B to cause equal accelerations; and by the addition of matter to B there may be produced a body B' such that equal forces must be applied to A and B' to cause equal accelerations.

(c') If two distinct bodies have equal accelerations when acted upon by equal forces, then a body formed by combining the two must be acted upon by twice as great a force in order to have an equal acceleration.

(d') If any number of distinct bodies have equal accelerations when acted upon by equal forces, then the body formed by combining any n of them must be acted upon by a force n times as great in order to have an equal acceleration.

(e') If the forces required to give two distinct bodies equal accelerations are F' and F'', then the body formed by combining the two must be acted upon by a force $F' + F''$ in order to have an equal acceleration.

(f') If the forces required to give any number of distinct bodies equal accelerations are F', F'', $\cdots$, then the body formed by combining them must be acted upon by a force $F' + F'' + \cdots$ in order to have an equal acceleration. (This includes the five preceding cases.)

These propositions are accepted as consequences of the following principle, which expresses part of the import of the law of acceleration:

(B) *In order that different bodies may have equal accelerations, they must be acted upon by forces whose magnitudes are proportional directly to the masses of the bodies.*

Assuming that force as a measurable magnitude is defined independently of this principle, a part of its import consists in giving precision to the meaning of mass; but if the whole meaning of mass were included in (B) no one of propositions (a') $\cdots$ (f') would necessarily follow from it. We accept these as consequences of (B) because we have, independently of (B), the conception that mass is a measure of quantity of matter and satisfies propositions (I), (II) and (III).

9. *Effects of Different Forces Applied at Different Times to the Same Body.*—Besides principles (A) and (B), each of which expresses a part of the import of the law of acceleration, there is a third principle of equal interest which may be stated as follows:

(C) *Different forces applied at different times to the same body cause accelerations having the same ratios as the magnitudes of the forces.*

This is not independent of (A) and (B); in fact, careful reflection shows that any one of the three propositions (A), (B) and (C) may be inferred from the other two.

10. *General Statement of Law of Acceleration.*—The general principle of which (A), (B) and (C) are particular cases may be stated as follows:

(*D*) *Any bodies acted upon by any forces will have accelerations proportional directly to the force-magnitudes and inversely to the masses* (*quantities of matter*) *of the bodies.*

Principle (*D*) may be called the general law of acceleration; it is equivalent to Newton's second law of motion, so far as this refers to forces acting singly.

IV. Significance of Quantity of Matter When Force is Defined Kinetically.

11. *Second View of Force.*—It is generally recognized that, while the measurement of forces by their effects in deforming elastic bodies (as by the spring balance) is of very great practical use, the exact comparison of force-magnitudes for the purposes of dynamics must make use of the law of acceleration. If this view is adopted a different form must be given to the explanation of the rôle of mass or quantity of matter in the laws of motion. To this we proceed.

12. *Formulation of the Laws of Motion.*—A rigorous logical analysis shows that the second and third laws of Newton (the law of acceleration and the law of action and reaction) are not independent principles. It will be seen, however, that the import of both is covered by the following propositions:

(1) *The acceleration of a body is always due to the influence of other bodies.*

(2) *If the acceleration of a body is due to the influence of more than one other body, it is equal to the vector sum of components, each due to some one of the bodies.*

(3) *If A and B are any two individual portions of matter, the acceleration of A due to B and that of B due to A are always oppositely directed, and their magnitudes are in a constant ratio.*

Definition.—The term "acceleration-ratio of A with respect to B" will be used to denote the ratio:

$$(\text{acceleration of } A \text{ due to } B)/(\text{acceleration of } B \text{ due to } A).$$

(4) *If A, B and C are any three distinct bodies, the acceleration-ratio of A with respect to B is equal to the acceleration-ratio of A with respect to C divided by the acceleration-ratio of B with respect to C.*[1]

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[1] In connection with (3) it should be added that if A and B are "particles,"—i. e., if their dimensions are vanishingly small in comparison with their distance apart,—their mutually-caused accelerations are directed along the line AB.

The "law of composition," expressed by (2), is sometimes held to mean that the acceleration of a body A due to a body B is not influenced by any third body C; in other words, that if a' is the acceleration of A due to B when C is absent, and a'' the acceleration of A due to C when B is absent, then when both B and C are present the acceleration of A will be the vector sum of a' and a''. This cannot be accepted as a valid interpretation of the law of composition; all that is implied by this law is that the acceleration of every particle can at every instant be resolved into components regarded as individually "due to" other particles in the sense that they satisfy (2), (3) and (4). The supposition that the acceleration of A due to B may be changed by the presence of C is entirely consistent with these laws. (See K. Pearson, "The Grammar of Science," second edition, page 317.)

The import of the laws is, in fact, summed up in the statement that the total momentum of a system of bodies is not changed by the action of these bodies upon one another (total momentum being understood in its broadest sense as a localized vector quantity). The definition of momentum must of course depend upon that of mass given in 13.

13. *Definition of Mass.*—These principles imply that *it is possible to assign to all bodies individual constant numbers such that the acceleration-ratio of any two bodies is the inverse ratio of their assigned constants.* Constants satisfying this condition are called the *masses* of the bodies. The actual values of the mass-constants are fixed as soon as any one of them is assigned; this is equivalent to choosing a unit of mass.[1]

14. *Mass as Quantity of Matter.*—That there is any relation between the quantity thus defined as mass and matter regarded as a quantity is not immediately apparent from principles (1), (2), (3) and (4). That there is such a relation, and that mass as just defined is an appropriate measure of quantity of matter, is illustrated by the following propositions referring to suppositious cases. It will be seen that there is a correspondence between these propositions and those above designated as $(a) \cdots (f)$.

(a'') If a body A be formed by adding matter to a body B, then the acceleration-ratio of A with respect to any third body D will be less than that of B with respect to D.

(b'') If A and B are two distinct bodies such that the acceleration-ratio of A with respect to a third body is less than that of B with respect to that body, then by the removal of matter from A there may be produced a body A' such that the acceleration-ratios of A' and B with respect to any third body are equal; and by the addition of matter to B there may be produced a body B' such that the acceleration-ratios of A and B' with respect to any third body are equal.

(c'') If the acceleration-ratios of two bodies B and C with respect to a third body D have equal values r, then if B and C be combined into a single body A the acceleration-ratio of A with respect to D will be $r/2$.

(d'') If there be any number of bodies having equal acceleration-ratios r with respect to a body D, then if any n of them be combined into a single body its acceleration-ratio with respect to D will be r/n.

(e'') If two distinct bodies have acceleration-ratios r', r'' with respect to another body D, then the body formed by combining the two will have with respect to D an acceleration-ratio r such that $1/r = 1/r' + 1/r''$.

(f'') If any number of distinct bodies have, with respect to a body D, the acceleration-ratios r', r'', $\cdots$, then the body formed by combining them will have, with respect to D, an acceleration-ratio r such that $1/r = 1/r' + 1/r'' + \cdots$. (This includes the five preceding cases.)

These propositions are consequences of the principle that

(E) *The acceleration-ratios of any bodies A, B, C, $\cdots$, with respect to any one other body are in the inverse ratios of the quantities of matter of A, B, C, $\cdots$.*

A part of the import of (E) is a precise *definition* of quantity of matter; but unless we had, independently of (E), a quantitative conception of matter satisfy-

[1] The above method of formulating the laws of motion and of defining mass is substantially that which has been used by a number of writers, among whom may be mentioned E. Mach, K. Pearson, G. Kirchhoff, P. Appell, A. E. H. Love, W. H. Macaulay. Some of these authors have explicitly recognized that the conception of mass which we actually use includes the notion of a quantitative measure of matter satisfying propositions (i), (ii) and (iii).

ing propositions (I), (II) and (III), we should not recognize that (a''), $\cdots$ (f'') are consequences of (E); and we certainly should not recognize that they are consequences of (1), (2), (3) and (4).

Proposition (E) shows that the quantity above defined as *mass* is an appropriate measure of *quantity of matter*.

15. *Definition of Force.*—From the present point of view the import of the Newtonian laws is completely covered by principles (1), (2), (3), (4) and (E); and it will be noticed that the word force is not used in the statement of these propositions. A definition of force may, however, be associated with them. Thus (1) and (2) suggest the following:

A force is an action exerted by one body upon another, the effect of which is to give the latter body an acceleration-component.

With this definition it is seen that (2) expresses the law of vector composition of forces.

An exact quantitative definition of force, consistent with the above principles, may be stated as follows:

A force is an action exerted by one body upon another, measured by the product of the acceleration produced and the mass of the body acted upon.

With this definition, and that of mass given above, principle (3) is seen to be equivalent to the law of action and reaction in its ordinary form:

Whenever one body exerts a force upon another, the second exerts an equal and opposite force upon the first.

V. Mass in the Law of Gravitation.

16. *Laws of Dynamics Independent of Law of Gravitation.*—Because the weights of bodies in the same locality are proportional to their masses, weight[1] plays an important part not only in the practical work of comparing the masses of bodies but in explanations of the principles of dynamics. It must not be forgotten, however, that the laws of dynamics are quite independent of the law of gravitation.

17. *Significance of Mass in the Law of Gravitation.*—The law of gravitation states that every portion of matter exerts an attractive force upon every other portion, and that the magnitude of the attractive force exerted by one particle[2] upon another depends in a definite way upon the masses of the particles and their distance apart. It is of interest to remark that all the questions which have been raised regarding the significance of mass in the law of acceleration might be raised with equal pertinence with regard to the meaning of mass in the law of gravitation. Thus the statement that "two particles which are at equal distances

[1] The word weight is often used popularly in the sense of quantity of matter, and some writers recommend the general adoption of this usage. There would be no objection to this if it did not lead to the confusing of two distinct things; but it is necessary to use some word to denote the gravitational pull of the earth upon a body, and scientific usage is nearly unanimous in assigning this meaning to the word weight. This usage is followed in the present paper.

[2] It is convenient here to refer to particles rather than to bodies of finite dimensions in order to avoid complexities of statement due to the fact that forces combine by vector addition.

from a third particle are attracted by it with forces proportional to their masses" might be held to be merely a definition of mass. It might be asserted that "the statement that A's mass is three times as great as B's means merely that if A and B are equally distant from C the attractive force of C upon A is three times that of C upon B."

It is not difficult to show, however, that the conception of mass as a measure of quantity of matter is an essential part of the interpretation we put upon the law of gravitation. Thus if, when A and B are equally distant from C, the attraction of C upon A is three times that of C upon B, we say that it is because A has three times as much matter as B; and we infer that A might be divided into three parts each having a mass equal to that of B.

The law of gravitation thus derives an essential part of its significance from a notion of mass which is independent of that law—the notion that mass is a measure of the matter of which bodies are composed and satisfies the fundamental quantitative relations expressed by (I), (II) and (III). At the same time it must be recognized that the law itself gives precision to the definition and furnishes a method of comparing quantities of matter with exactness.

18. *Agreement of Two Methods of Estimating Quantity of Matter.*—It has been seen that the law of acceleration and the law of gravitation furnish two independent methods of applying exact quantitative measurements to matter. That the results of the two methods are in apparently exact agreement is a matter of experimental knowledge rather than of *a priori* necessity. If the fact were otherwise, weighing would not be an available method of comparing the masses of bodies for the purposes of dynamics.

GEOMETRICAL AND OTHER ILLUSTRATIONS OF INDETERMINATE FORMS.

By W. V. LOVITT, Purdue University.

In the AMERICAN MATHEMATICAL MONTHLY, Vol. III, 1893, pp. 207–8, there is given by B. F. Finkel a geometrical proof that $0 \times \infty$ is indeterminate. In the present paper additional illustrations, geometrical and otherwise, are given of this and other indeterminate forms.

I. The area generated by a segment of a straight line AB (Fig. 1) revolving about an axis CD in its plane is given by the formula:

$$\text{Area } AB = 2\pi \cdot AB \cdot MO = 2\pi \cdot CD \cdot MR,$$

where CD is the projection of the segment AB upon the line CD, M is the middle point of AB, O is the center of the segment CD, and MR is perpendicular to AB. Let AB revolve about M and become perpendicular to CD. Then MR becomes parallel to CD and $CD \cdot MR$ becomes $0 \times \infty$.

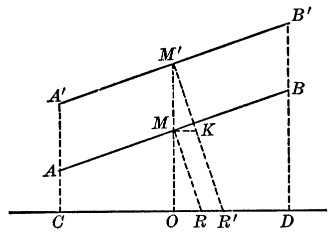


FIG. 1.

But $$CD \cdot MR = AB \cdot MO = K_1 \text{ (constant)}.$$

Let us take a line segment $A'B' = AB$ and parallel to AB. Let M' be the middle point of $A'B'$, $M'R'$ perpendicular to $A'B'$ and parallel to MR. Then

$$\text{Area } A'B' = 2\pi \cdot CD \cdot M'R'.$$

Rotate $A'B'$ about M' until it is perpendicular to CD. Then $CD \cdot M'R'$ becomes $0 \times \infty$. But

$$CD \cdot M'R' = A'B' \cdot M'O = K_2 \text{ (constant)}.$$

We note that $K_1 \neq K_2$; that is, our form $0 \times \infty$ takes on different values in these two instances and can thus be made to vary at will.

Draw MK parallel to CD. Consider the difference

$$\text{Area } A'B' - \text{Area } AB = 2\pi CD \cdot (M'R' - MR) = 2\pi CD \cdot M'K.$$

Let $A'B'$ and AB become perpendicular to CD. Then $M'R'$ and MR become parallel to CD and

$$M'R' - MR \text{ becomes } \infty - \infty.$$

But

$$M'R' - MR = M'K$$

and $M'K$ becomes infinite as $A'B'$ becomes perpendicular to CD. Hence, in this case the indeterminate form $\infty - \infty$ becomes infinite.

Incidentally, we have in the product $CD \cdot M'K$ an indeterminate form of the type $0 \times \infty$.

An additional illustration of the form $\infty - \infty$ is furnished by the hyperbola. This curve is defined as the locus of a point P, the difference of whose distances from two fixed points, F' and F, is a constant, $2a$. That is

$$F'P - FP = 2a.$$

As $F'P$ increases indefinitely, FP also increases indefinitely and

$$F'P - FP \text{ takes the form } \infty - \infty.$$

II. Consider now (Fig. 2) a triangle with a fixed base a, with sides x and y as indicated, the vertex P being on a line l which is parallel to the base. The

area of this triangle is constant and will be denoted by c. Then

$$c = \frac{a}{2} x \sin\theta, \qquad \text{and} \qquad \frac{2c}{a} = x \sin\theta = x/\csc\theta.$$

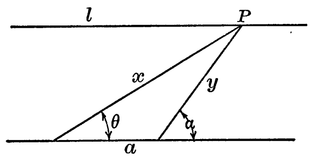


FIG. 2.

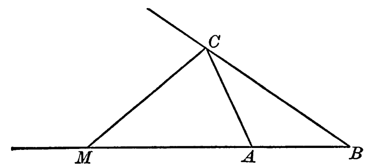


FIG. 3.

As P runs off to infinity on the line l, $x \sin\theta$ becomes $\infty \times 0$, while $x/\csc\theta$ becomes ∞/∞. From the law of sines,

$$\frac{x}{y} = \frac{\sin\alpha}{\sin\theta}.$$

As P moves off to infinity on the line l, the left and right hand members of the last equation take on the forms ∞/∞ and $0/0$ respectively. Thus we have two illustrations of the change from one indeterminate form to another.

III. The following theorem from plane geometry furnishes examples of the indeterminate forms ∞/∞ and $\infty - \infty$.

The bisector CM (Fig. 3) of an exterior angle of a triangle ABC divides the opposite side externally into segments which are proportional to the adjacent sides. That is,

$$\frac{MA}{MB} = \frac{CA}{CB}.$$

Let BC and AB remain fixed in length, and let AC increase, approaching BC in length. Then CM becomes parallel to AB, while MA/MB becomes ∞/∞. But

$$\lim_{CA \to BC} \frac{MA}{MB} = \lim_{CA \to BC} \frac{CA}{BC} = 1.$$

The difference $MB - MA$, which is always equal to the constant AB, becomes $\infty - \infty$.

IV. An additional illustration of the form $0 \times \infty$ is furnished by inversion with respect to a given circle, $x^2 + y^2 = a^2$. Let P and P' be inverse points and O the center of the given circle, then

$$\overline{OP} \cdot \overline{OP'} = a^2.$$

As OP approaches zero, OP' becomes indefinitely large and $\overline{OP} \cdot \overline{OP'}$ becomes $0 \times \infty$. In particular this form occurs in finding the inverse of a circle through the origin. The inverse is a straight line.

V. We add here two problems taken from works on hydraulics.

The theoretical efflux Q (Fig. 4) through a triangular orifice in a thin vertical

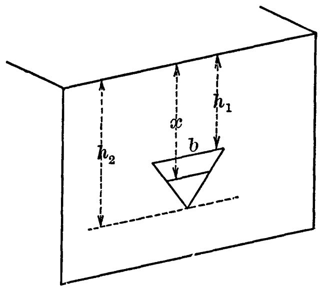


FIG. 4.

plate or wall, with base, b, horizontal, is given by the following formula:

$$Q = \frac{2b\sqrt{2g}}{15}\,\frac{[2h_2^{5/2} - 5h_2h_1^{3/2} + 3h_1^{5/2}]}{(h_2 - h_1)}.$$

Let $h_1 = h_2$; then certainly $Q = 0$, while the expression for Q in the formula takes on the indeterminate form 0/0.

Again, in hydrostatics the head h' on the center of pressure of a submerged rectangular plane, with edges parallel to the surface, one end having a head h_2, the other a head h_1, is

$$h' = \frac{2}{3}\cdot\frac{h_2^3 - h_1^3}{h_2^2 - h_1^2}.$$

When $h_2 = h_1$, $h' = h$, while the expression for h' in the formula becomes 0/0.

VI. We conclude with a few simple arithmetical illustrations.

(1) Type $\frac{0}{0}$: $\quad \frac{1}{9}; \frac{.1}{.9}; \frac{.01}{.09}; \cdots.$ $\quad$ Constantly equal to $\frac{1}{9}$.

(2) Type $\frac{\infty}{\infty}$: $\quad \frac{3}{10}; \frac{33}{100}; \frac{333}{1000}; \cdots.$ $\quad$ Approaches $\frac{1}{3}$.

Any repeating decimal furnishes an additional example.

(3) Type $\infty - \infty$: $\quad 1.3 - 1;\ 2.33 - 2;\ 3.333 - 3;\ \cdots.$ $\quad$ Approaches $\frac{1}{3}$.

(4) Type $0 \times \infty$: $\quad 1.2;\ \frac{1}{2}\cdot 4;\ \frac{1}{4}\cdot 8;\ \cdots.$ $\quad$ Constantly equal to 2.

(5) Type ∞^0: $\quad (2^2)^{1/2};\ (2^3)^{1/3};\ (2^4)^{1/4};\ \cdots.$ $\quad$ Constantly equal to 2.

(6) Type 0^0: $\quad (2^{-2})^{1/2};\ (2^{-3})^{1/3};\ (2^{-4})^{1/4};\ \cdots.$ $\quad$ Constantly equal to $\frac{1}{2}$.

A TRIBUTE TO SAMUEL WALKER SHATTUCK.

Professor Shattuck was administrative head of the department of mathematics in the University of Illinois for nearly forty years (1868–1906), having come to the University as the first and the only instructor in mathematics when the entire faculty of the institution consisted of but eleven members. He lived to see the time when the faculty of his department was twice as large as the entire faculty of the University had been when he entered it. The remarkable growth of the University during the years that he was actively connected with it was largely due to his devotion and integrity, for he was not only in charge of the mathematical department during the greater part of this period but also very influential in directing the finances of the institution, acting as business agent and manager during the years 1873–1905, and as comptroller from 1905 to 1912 when he retired on a Carnegie pension on account of failing health.

Notwithstanding the fact that his university duties were unusually heavy and that he performed unselfishly the many other services to society which his prominence in the community made possible, he found time to write a mimeographed work on calculus, consisting of two volumes and giving evidences of much thought. He was regarded as a good teacher both in the classroom and outside of it, for his training as a soldier gave him a dignified bearing which led the former President Draper to remark that "if he did nothing else it would be worth while to the state of Illinois to pay him his salary just to have him on the campus as a visible example to young men."

The great and faithful services which Professor Shattuck rendered the University of Illinois during a long period of years helped to endear him to all interested in its welfare, but his pleasing personality combined with a readiness to be helpful in every good work and to stand for justice and the strictest impartiality added very much to this endearment.

During the last few years of his life he witnessed numerous evidences of appreciation on the part of the alumni and the faculty, and since his death on February 13, 1915, two appreciative biographical sketches, relating to his services, written by Professor S. A. Forbes and Dean T. A. Clark respectively, have appeared in the *Alumni Quarterly* of the University of Illinois.

Professor Shattuck was an army officer during the Civil War, and belonged to a family of soldiers, his father, grandfather, and great-grandfather having all been officers in the colonial or national armies. He was born at Groton, Massachusetts, on February 18, 1841, and came to the University of Illinois from Norwich University, Vermont, where he had been professor of mathematics and military tactics. Although he entered the faculty of the University of Illinois less than half a century ago it is interesting to note that he was appointed assistant professor of mathematics, instructor in military tactics, and first commander of the University corps of cadets. These appointments indicate that the professor in the University of Illinois in those days did not enjoy sufficient leisure to become the scholar which the more recent opportunities have made

possible. The varied duties of these early professors tended, however, to bring the students and the faculty into more frequent contact and thus they enabled the unusually strong members of the faculty to leave a more lasting impression on the minds and the hearts of their students than is possible under the more modern conditions.

In his method of teaching, Professor Shattuck exhibited his usual willingness to be helpful to all who were likely to profit by such help, but he expected the student to do his part first. That is, he was willing to help all those who had made reasonable efforts to follow the explanations of the textbook and who had met with difficulties, but he did not, as a rule, lecture to his classes on the general bearing of the subject under consideration. His examination questions were usually rather difficult but he was not very severe in grading the answers. His students knew that they were dealing with a man who expected them to do serious work and who was likely to make few concessions for irregularities.

While his many other duties prevented Professor Shattuck from taking an active part in the development of mathematics, he frequently exhibited a keen interest in these developments. In particular, he was one of the first to join the American Mathematical Society and he continued this membership to the end of his life. As evidence of his deep interest in teaching we may cite the fact that when the President of the University suggested to him that he might devote himself entirely to the financial interests of the University the President received the reply: "When I give up teaching I shall give up the University."

In brief, Professor Shattuck devoted himself without ostentation but with complete devotion to what appeared to him to be the most important work of the time. When, as a young man, his country seemed to need his services in the defense of a great principle he enlisted twice in the United States Army and bore bravely the hardships of the battlefield and the serious wound received while at the front. When he came to the University of Illinois and found that a few men had to shoulder a large number of important outside duties he assumed his full share and performed all his work punctually and with great care. As a member of the church, as a citizen, and as the head of a family he led an exemplary life, and thus extended his teaching beyond his classroom. In particular, his careful management of the finances of the University inspired widespread confidence and this confidence has been one of the greatest assets in winning for the institution the loyal support which has in recent years been given to it by the state.

G. A. Miller.

University of Illinois.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

Send all communications to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

GEOMETRY.

481. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the locus of the intersection of a pair of perpendicular normals to a parabola $y^2 = 4px$ is the parabola $y^2 = p(x - 3p)$.

482. Proposed by ROBERT G. THOMAS, The Citadel, Charleston, S. C.

In laying out a kite-shaped mile race-track, composed of a circular arc and two intersecting tangents at the ends of the arc, determine the angle at the center of the arc (a) when the length of the arc equals the sum of the two tangents, and (b) when the arc is equal to the length of each tangent.

CALCULUS.

402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2) if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x\,\partial y} = 0,$$

where (x', y') are the current coördinates of points on the tangents.

NOTE.—In an early issue, we will publish all the unsolved problems in Number Theory proposed from January, 1913, to December, 1915. EDITORS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

439. Proposed by A. M. KENYON, Purdue University.

If k, n are natural numbers, $n > 2k$, show that

$$\frac{2^k}{\lfloor k}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\frac{1}{\lfloor 2i+1\,\lfloor n-k-2i} = \frac{2^n}{\lfloor n+1}\sum_{i=0}^{k}\binom{n-i}{n-k},$$

where $I(n/2)$ denotes the integral part of $n/2$ and $\binom{n}{k}$ is the coefficient of x^k in $(1 + x)^n$.

SOLUTION BY FRANK IRWIN, University of California.

On the right side

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\binom{n+1}{n-k+1}.$$

This formula expresses in symbols the well-known property of the Pascal triangle, that any term is equal to the sum of all terms above it in the preceding column. (See, for instance, LUCAS, *Théorie des Nombres*, page 6.)

Or it may be proved as follows:

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\sum_{i=0}^{k}\left[\binom{n-i+1}{n-k+1}-\binom{n-i}{n-k+1}\right]=\sum_{i=0}^{k}\binom{n-i+1}{n-k+1}-\sum_{i=1}^{k+1}\binom{n-i+1}{n-k+1}.$$

Here all the terms but one cancel in pairs, leaving $\binom{n+1}{n-k+1}$.

On the other hand, the left side of our given equation may be written

$$\frac{2^k}{\lfloor n-k+1 \ \lfloor k}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\binom{n-k+1}{2i+1}.$$

Here the summation sign takes in the 2d, 4th, $\cdots$, *all* the even-placed coefficients,

$$\binom{n-k+1}{1},\quad \binom{n-k+1}{3},\ \cdots$$

in the development of $(1+x)^{n-k+1}$; the sum of which is well known to be 2^{n-k}.

Our formula, then, reduces to the obviously true form:

$$\frac{1}{\lfloor n-k+1}\,\frac{2^k}{\lfloor k}\,2^{n-k}=\frac{2^n}{\lfloor n+1}\binom{n+1}{n-k+1},$$

Also solved by the Proposer.

440. Proposed by W. D. CAIRNS, Oberlin College.

n being a positive integer, find the sum of the series

$$2n^2+4(n-1)^2+2(n-2)^2+4(n-3)^2+2(n-4)^2+\cdots,$$

where the succeeding coefficients are alternately 4 and 2; or, more generally, the series

$$an^2+b(n-1)^2+a(n-2)^2+b(n-3)^2+a(n-4)^2+\cdots.$$

L'Intermédiaire, July, 1913.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The problem as originally published in the September, 1915, issue contained two misprints. The question is indefinite. These series are not convergent, and do not break off after a finite number of terms. However, it is easy to find the sum of $2k$ terms.

Expanding, we can write the second series as

$$\begin{aligned}S&=a\{n^2+n^2-4n+4+n^2-8n+16+n^2-12n+36+\cdots+n^2-2(2k-2)n+(2k-2)^2\}\\&\quad+b\{n^2-2n+1+n^2-6n+9+n^2-10n+25+\cdots+n^2-2(2k-1)n+(2k-1)^2\}\\&=a\{kn^2-4n(1+2+3+\cdots+k-1)+(2^2+4^2+6^2+\cdots+[2k-2]^2)\}\\&\quad+b\{kn^2-2n(1+3+5+\cdots+2k-1)+(1^2+3^2+5^2+\cdots+[2k-1]^2)\}.\end{aligned}$$

Summing the series in parentheses we obtain

$$S = a\left\{ kn^2 - 2nk(k-1) + \frac{4k^3 - 6k^2 + 2k}{3} \right\} + b\left\{ kn^2 - 2nk^2 + \frac{4k^3 - k}{3} \right\},$$

which may be written as

$$(a+b)kn^2 - 2nk(ak + bk - a) + \left\{ \frac{4}{3}k^3(a+b) - 2ak^2 + \frac{k}{3}(2a - b) \right\}.$$

Also solved by H. C. FEEMSTER, HORACE OLSON and the PROPOSER.

441. Proposed by W. D. CAIRNS, Oberlin College.

Prove that the equation $(e-1)x = e^x - 1$ has two and only two real roots.

I. SOLUTION BY H. S. UHLER, Yale University.

Let $y = e^x - 1 - (e-1)x$ and observe that $y = 0$ for $x = 0$ and $x = 1$. It remains to show that there can be no more real roots.

$$\frac{dy}{dx} = e^x - e + 1, \quad (1) \qquad \frac{d^2y}{dx^2} = e^x. \quad (2)$$

Equation (1) shows that the slope of the tangent is positive for all values of x greater than $\log_e(e-1)$ and negative for all values of x less than this value ($x_0 = 0.541325$). Since e^x is essentially positive, equation (2) indicates formally that the single stationary point, indicated by equation (1), corresponds to a minimum value of y. The coördinates of the minimum are $x_0 = \log_e(e-1)$ and $y_0 = (e-1)(1-x_0) - 1 = -0.211867$. It is clear, therefore, from the properties of the graph that the curve cannot cut the axis of x in more than two points and hence the given equation has two and only two real roots.

II. SOLUTION BY GRACE M. BAREIS, Ohio State University.

Writing e in series form in each member and transposing, the equation becomes

$$x\left[\frac{x-1}{\underline{|2}} + \frac{x^2-1}{\underline{|3}} + \frac{x^3-1}{\underline{|4}} + \cdots\right] = 0, \quad \text{or} \quad x(x-1)\left[\frac{1}{\underline{|2}} + \frac{x+1}{\underline{|3}} + \frac{x^2+x+1}{\underline{|4}} + \cdots\right] = 0.$$

Hence, $x = 0$, $x - 1 = 0$, or $\frac{1}{\underline{|2}} + \frac{x+1}{\underline{|3}} + \frac{x^2+x+1}{\underline{|4}} + \cdots = 0$. Since each term of the left member of the last equation is positive, this equation can have no real positive root.

To prove that the original equation can have no real negative root, put it in the form

$$x = \frac{e^x - 1}{e - 1},$$

or $x = e^{x-1} + e^{x-2} + e^{x-3} + \cdots$, a convergent series. Any real negative number makes the left member negative but the right member positive, and hence there are no real negative roots. Hence, 0 and 1 are the only real roots.

Solutions were also received from W. L. AGARD, F. L. GRIFFIN, H. C. FEEMSTER, WALTER C. EELLS, HORACE OLSON, C. E. HORNE, IRBY C. NICHOLS, and FRANK IRWIN.

GEOMETRY.

467. Proposed by E. T. BELL, Seattle, Washington.

It is well-known that if i, j, k, l are concyclic points, W_i the Wallace line (frequently, and erroneously, called the Simson line), of i with respect to the triangle jkl, then W_i, W_j, W_k, W_l are concurrent, say in the point $\{i, j, k, l\}$. If 1, 2, 3, $\cdots$ denote concyclic points, prove that:

(i) $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$ are concyclic; say on the circle $[1, 2, 3, 4, 5]$;

(ii) Starting with 1, 2, 3, 4, 5, 6, omitting each point in turn, by (i), six circles, are found; these are concurrent, say in the point $\{1, 2, 3, 4, 5, 6\}$;

(iii) Starting with 1, 2, 3, 4, 5, 6, 7, seven points of the kind in (ii) are found; these lie on a circle.

(iv) Continuing thus indefinitely, there is, in each case, finally a unique point or circle according as the number of initial points is even or odd. Also, at any stage, the point of concurrence on the circle bears a simple relation to the initial points: what is it?

I. Solution by J. W. Clawson, Collegeville, Pa.

If H_i is the orthocenter of triangle jkl, O the center of the circle, radius R, on which the points i, j, k, l lie,—

iH_j = twice distance of O from $kl = jH_i$. Also iH_j is parallel to jH_i; hence H_iH_j is equal to, and parallel to, ij. It follows that $H_iH_jH_kH_l$ is inversely congruent with $ijkl$, the corresponding sides of the quadrilaterals being equal and parallel, but arranged in opposite orders.

Again, iH_i, jH_j, kH_k, lH_l have a common mid-point; and, since iH_i bisects W_i and so on, this common point is the point of concurrency of W_i, W_j, W_k, W_l. The point of concurrency $\{i, j, k, l\}$ bisects the join of O to the center of the circle circumscribing $H_iH_jH_kH_l$.

(i) If H_{123} is the orthocenter of triangle 123, H_{123}, H_{234}, H_{341}, H_{412} are four vertices of a pentagon inversely congruent with 12345. Moreover the sides of this pentagon are parallel to the corresponding sides of the pentagon 12345. Call the center of the circumcircle of this pentagon, P_5. Similarly H_{123}, H_{235}, H_{351}, H_{512} and three other sets of orthocenters determine four other congruent pentagons with parallel sides. Each of the orthocenters occurs as a vertex of two of the pentagons. It follows that P_5, P_4, P_3, P_2, P_1, the centers of the circles circumscribing these five pentagons, form a pentagon which is congruent with each of the pentagons and hence inversely congruent with the original pentagon 12345. The points P_5, P_4, P_3, P_2, P_1 are therefore concyclic. Call the center of their circle, O_6.

It follows at once that the points (1234), (2345), (3451), (4512), (5123) are concyclic on a circle of half the radius of the given circle, its center bisecting the join of O to O_6.

(ii) The circle, center O_6, passes through P_5, the center of a circle determined by the orthocenters of the triangles obtained by selecting triads of points from 1, 2, 3, 4. In the same way if we consider the five points 1, 2, 3, 4, 6 we shall obtain a circle, center O_5, which passes through P_5 and through four new points. In this way we can get six hexagons, one of which has P_1, P_2, P_3, P_4, P_5 for five of its vertices, each inversely congruent with the hexagon 123456 and five vertices of each hexagon (orthocenters of triangles obtained by selecting three points from the six) belonging also to a different hexagon of the system. It follows that $O_6O_5O_4O_3O_2O_1$ forms a hexagon which is congruent with each of the hexagons and hence is inversely congruent with the original hexagon 123456. Hence O_6, O_5, O_4, O_3, O_2, O_1 are concyclic; and the circles with O_6, O_5, O_4, O_3, O_2, O_1 as centers and with R as radius concur at a point, say Q_7.

It follows at once that circles (1, 2, 3, 4, 5), $\cdots$ concur at a point halfway between Q_7 and O.

(iii) This process can be continued indefinitely as stated in the problem.

II. Solution by Norman Anning, Chilliwack, B. C.

Letting $e^{i\phi} = \cos\phi + i \sin\phi$, then $Re^{i\theta_j}$ $(j = 1, 2, 3, 4 \cdots)$ are points on a circle of radius R, whose center is at the origin. The W line of 4 quâ 123 passes through the foot of the perpendicular from 4 on line 12 and makes with this perpendicular an angle equal to that between 13 and 14. All points on W_4 are given by

$$Re^{i\theta_4} + ke^{i\frac{\theta_1+\theta_2}{2}} + xe^{i\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}},$$

where x is a running coördinate on the line and k, the distance from 4 to 12, is to be determined. Projecting on the radius vector midway between 1 and 2,

$$k + R\cos\left(\theta_4 - \frac{\theta_1+\theta_2}{2}\right) = R\cos\frac{\theta_1-\theta_2}{2}, \qquad k = 2R\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_4-\theta_1}{2}.$$

$$W_4 \text{ is } Re^{i\theta_4} + 2R\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_4-\theta_1}{2}e^{i\frac{\theta_1+\theta_2}{2}} + xe^{i\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}}.$$

$$W_3 \text{ is } Re^{i\theta_3} + 2R\sin\frac{\theta_3-\theta_2}{2}\sin\frac{\theta_3-\theta_1}{2}e^{i\frac{\theta_1+\theta_2}{2}} + ye^{i\frac{\theta_1+\theta_2-\theta_3+\theta_4}{2}}.$$

To find the common point of these lines equate the two expressions, separate reals and imaginaries and solve for x and y. Result:

$$x = y = 2R \cos \frac{\theta_4 + \theta_3 - \theta_2 - \theta_1}{2}.$$

When the exponential values are inserted for the sines and cosines, the expression for the common point of W_4 and W_3 becomes

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4}).$$

The symmetry of this expression shows that the point is the common point of the four W lines.

$$\{1, 2, 3, 4\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_5}),$$

$$\{1, 2, 3, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_4}),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\{2, 3, 4, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_1}).$$

These 5 points lie on the circle whose center is the point, $\frac{R}{2}\sum_{p=1}^{p=5} e^{i\theta_p}$, and whose radius is $R/2$. Where ϕ is variable, the circle

$$[1, 2, 3, 4, 5] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5}) + \frac{R}{2}e^{i\phi},$$

$$[1, 2, 3, 4, 6] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$[2, 3, 4, 5, 6] \text{ is } \frac{R}{2}(e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi}.$$

These 6 circles have the common point,

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}).$$

In the next step the 7 points, of which $\{1, 2, 3, 4, 5, 6\}$ is one, lie on the circle whose center is $\frac{R}{2}\sum_{p=1}^{p=7} e^{i\theta_p}$ and whose radius is $R/2$. Then follows a group of 8 such circles having the common point, $\frac{R}{2}\sum_{p=1}^{p=8} e^{i\theta_p}$, etc., etc.

In general, with an even number of initial points this method yields a point. The vector to this point from the origin is half the sum of the vectors to the initial points. With an odd number of initial points we have a circle whose radius is half that of the original circle and the vector to whose center is half the sum of the vectors to the initial points. The " W " points on this circle are in perspective relation to the initial points.

CALCULUS.

363. Proposed by B. F. FINKEL, Drury College.

The axis of a right prism whose cross-section is a regular polygon of n sides coincides with the diameter of a sphere of radius R. Find the surface of the sphere included within the prism.

I. SOLUTION BY H. S. UHLER, Yale University.

Let a denote the apothem of the right-section of the prism. The required area will be expressed as a function of a, n, and R. This area may be analyzed as $4n$ triangles on the surface of

the sphere. Each triangle is bounded by arcs of two great circles and one small circle. The plane of one great circle contains the axis of the prism and the apothem to the chosen lateral face. The other great circle is determined by the same axis and a lateral edge of the selected face. The small circle is the intersection of the same lateral face and the spherical surface.

An element of surface of the sphere may be written $R^2 \sin \phi \, d\phi d\theta$, where ϕ is the polar angle reckoned from the axis of the prism and θ is the azimuth with respect to the plane containing the apothem. From the geometry of the problem it is readily seen that the total area required is expressed by

$$A \equiv 4nR^2 \int_0^{\pi/n} d\theta \int_0^{\Phi} \sin \phi \, d\phi, \text{ where } \sin \Phi = \frac{a}{R} \sec \theta;$$

whence

$$A = 4nR^2 \int^{\pi/n} \left(1 - \frac{1}{R} \sqrt{R^2 - a^2 \sec^2 \theta}\right) d\theta$$

or

$$A = 4\pi R^2 - 4nR \int_0^{\pi/n} (\sqrt{R^2 - a^2 \sec^2 \theta}) d\theta.$$

The indefinite integral may be evaluated as follows. Let $y = \sin \theta$, then

$$\int (\sqrt{R^2 - a^2 \sec^2 \theta}) d\theta = \int \frac{(\sqrt{(R^2 - a^2) - R^2 y^2}) dy}{1 - y^2}$$

$$= \frac{1}{2} \int \frac{(\sqrt{(R^2 - a^2) - R^2 y^2}) dy}{1 + y} + \frac{1}{2} \int \frac{(\sqrt{(R^2 - a^2) - R^2 y^2}) dy}{1 - y}.$$

The integral

$$U \equiv \int \frac{(\sqrt{\alpha^2 - y^2}) dy}{1 + y} = \int \frac{(\sqrt{\alpha^2 - 1 + 2z - z^2}) dz}{z} \quad \text{if} \quad z = 1 + y.$$

Formula 187 of B. O. Peirce's *A Short Table of Integrals* gives

$$U = \sqrt{\alpha^2 - 1 + 2z - z^2} + \int \frac{dz}{\sqrt{\alpha^2 - 1 + 2z - z^2}} - (1 - \alpha^2) \int \frac{dz}{z \sqrt{\alpha^2 - 1 + 2z - z^2}}.$$

The first and second of these integrals can be reduced by formulae 161 and 183 respectively, so that

$$U = \sqrt{\alpha^2 - 1 + 2z - z^2} + \sin^{-1}\left(\frac{z - 1}{\alpha}\right) - (\sqrt{1 - \alpha^2}) \sin^{-1}\left(\frac{z + \alpha^2 - 1}{\alpha z}\right).$$

In like manner, the substitution $u = 1 - y$ gives

$$\int \frac{(\sqrt{\alpha^2 - y^2}) dy}{1 - y} = - \sqrt{\alpha^2 - 1 + 2u - u^2} - \sin^{-1}\left(\frac{u - 1}{\alpha}\right) + (\sqrt{1 - \alpha^2}) \sin^{-1}\left(\frac{u + \alpha^2 - 1}{\alpha u}\right).$$

Returning to θ and noting that $\alpha = (1/R)\sqrt{R^2 - a^2}$ we find

$$\int (\sqrt{R^2 - a^2 \sec^2 \theta}) d\theta = R \sin^{-1}\left(\frac{R \sin \theta}{\sqrt{R^2 - a^2}}\right)$$

$$+ \frac{a}{2} \sin^{-1}\left[\frac{R^2 (1 - \sin \theta) - a^2}{R \sqrt{R^2 - a^2}(1 - \sin \theta)}\right] - \frac{a}{2} \sin^{-1}\left[\frac{R^2 (1 + \sin \theta) - a^2}{R \sqrt{R^2 - a^2}(1 + \sin \theta)}\right].$$

Making use of the relation $\sin^{-1} \beta - \sin^{-1} \gamma = \sin^{-1} (\beta \sqrt{1 - \gamma^2} - \gamma \sqrt{1 - \beta^2})$ the following form (which I have verified by differentiation) is obtained, namely

$$\int (\sqrt{R^2 - a^2 \sec^2 \theta}) d\theta = R \sin^{-1}\left(\frac{R \sin \theta}{\sqrt{R^2 - a^2}}\right) - \frac{a}{2} \sin^{-1}\left[\frac{2a (\tan \theta)(\sqrt{R^2 - a^2 \sec^2 \theta})}{R^2 - a^2}\right].$$

Finally $\infty > n \geqq 3$

$$A = 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R \sin \frac{\pi}{n}}{\sqrt{R^2 - a^2}}\right) + 2anR \sin^{-1}\left[\frac{2a\left(\tan \frac{\pi}{n}\right)\left(\sqrt{R^2 - a^2 \sec^2 \frac{\pi}{n}}\right)}{R^2 - a^2}\right].$$

REMARKS. In the preceding analysis it has been tacitly assumed that we are dealing with the general case where the small circles intersect in real points thus forming a scalloped polar zone. Keeping n and R constant and increasing a, from a sufficiently small value, a time will come when the small circles will be externally tangent to one another. This will happen when the lateral edges of the prism are tangent to the sphere, that is, when the cross-section of the prism becomes inscribed in a *great* circle. Then $a = R \cos(\pi/n)$. This equation also follows at once from the second term of the final formula for A since the greatest value attainable by the sine of an angle is unity. In other words, the equation

$$\frac{R \sin(\pi/n)}{\sqrt{R^2 - a^2}} = 1$$

leads to $a = R \cos(\pi/n)$. Again, this relation affords a means of checking the formula for A. Substituting $R \cos(\pi/n)$ for a in function A and taking π for the value of $\sin^{-1} 0$, we find

$$A = 4\pi R^2 - 2\pi nR^2 + 2\pi nR^2 \cos(\pi/n) = 4\pi R^2(1 - n \sin^2(\pi/2n)).$$

Precisely the same result is obtained from the theorem that the area of a zone of one base is equal to the product of its altitude $(R - R \cos(\pi/n))$ by the circumference of a great circle. When a starts from zero and increases to

$$\frac{R \cos(\pi/n)}{\sqrt{1 + \sin^2(\pi/n)}}$$

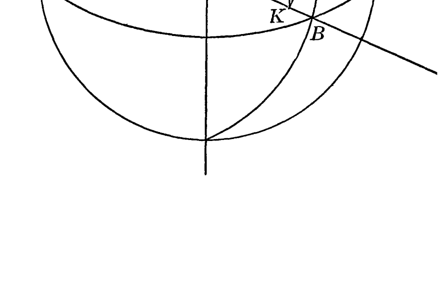
the expression in brackets in the last term of the formula for A increases from 0 to 1. Then as a continues to increase up to $R \cos(\pi/n)$ the expression between the brackets decreases from 1 to 0. Therefore the angle increases continuously from 0 to π. When a exceeds $R \cos(\pi/n)$ the small circles do not intersect, the formula for A does not apply, and the problem degenerates to the calculation of areas of zones of one base. It may also be remarked that the formula for A can be rationalized by making use of the relation $2 \sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

A very convincing check on the correctness of the formula for A is afforded by making n infinite while keeping a and R constant. We should then find $A = 4\pi R^2 - 4\pi R(R^2 - a^2)^{\frac{1}{2}}$ for the sum of the areas of the two zones cut out by a coaxial cylinder of radius a. By expanding in infinite series the functions involving n and then substituting $n = \infty$ we obtain at once the limits $-4\pi R^3(R^2 - a^2)^{-\frac{1}{2}}$ and $+4\pi a^2R(R^2 - a^2)^{-\frac{1}{2}}$ for the second and third terms of the expression for A, respectively. But $-4\pi R^3(R^2 - a^2)^{-\frac{1}{2}} + 4\pi a^2R(R^2 - a^2)^{-\frac{1}{2}} = -4\pi R(R^2 - a^2)^{\frac{1}{2}}$, as required.

II. SOLUTION BY THE PROPOSER.

Let O, the center of the given sphere, be the origin of Cartesian coördinates; the axis OC, of the prism, the z-axis; the line OX through L, the mid-point of a side of the regular polygon formed by the intersection of the prism and a plane through O perpendicular to the axis of the prism, the x-axis; and OY, $\perp OA$ and OC, the y-axis. Call OL, the apothem of this polygon, b.

The equation of the sphere is then $x^2 + y^2 + z^2 = R^2$ and the equation of the line OB is $y = x \tan(\pi/n) = mx$, say. Then the surface of the sphere included by the prism is $4n$ times the area of the triangle PCN. The required surface is, therefore,



$$S = 4n\int_0^b\int_0^{mx}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy = 4n\int_0^b\int_0^{mx}\frac{Rdxdy}{\sqrt{R^2-x^2-y^2}}$$

$$= 4nR\int_0^b \sin^{-1}\frac{y}{\sqrt{R^2-x^2}}\Big]_0^{mx} dx = 4nR\int_0^b \sin^{-1}\left(\frac{mx}{\sqrt{R^2-x^2}}\right)dx,$$

$$= 4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}} - mR\int\frac{xdx}{(R^2-x^2)\sqrt{R^2-(1+m^2)x^2}}\right]_0^b,$$

by integrating by parts.

To integrate the last expression, let $x = R\sin\theta/\sqrt{1+m^2}$. The integral then becomes

$$-mR\int\frac{\sin\theta d\theta}{1+m^2-\sin^2\theta} = mR\int\frac{-\sin\theta d\theta}{m^2+\cos^2\theta} = R\tan^{-1}\left(\frac{\cos\theta}{m}\right) = R\tan^{-1}\left[\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right].$$

Hence,

$$S = 4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}} + R\tan^{-1}\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right]_0^b$$

$$= 4nR\left\{b\sin^{-1}\left(\frac{mb}{\sqrt{R^2-b^2}}\right) - R\left[\tan^{-1}\left(\frac{1}{m}\right) - \tan^{-1}\frac{\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right]\right\},$$

$$= 4nR\left\{b\sin^{-1}\frac{mb}{\sqrt{R^2-b^2}} - R\sin^{-1}\left[\frac{m}{1+m^2}\left(\frac{R-\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right)\right]\right\},$$

$$= 4nR\left\{b\sin^{-1}\tan\left(\frac{\pi}{n}\cdot\frac{b}{\sqrt{R^2-b^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-b^2\sec^2\frac{\pi}{n}}}{\sqrt{R^2-b^2}}\right]\right\}.$$

When $n = 4$,

$$S = 8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}} - 2R\sin^{-1}\left(\frac{1}{2}\cdot\frac{R-\sqrt{R^2-2b^2}}{\sqrt{R^2-b^2}}\right)\right],$$

$$= 8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}} - R\sin^{-1}\left(\frac{b^2}{R^2-b^2}\right)\right],$$

a result agreeing with that in Osborne's *An Elementary Treatise on the Differential and Integral Calculus*, p. 270, example 5. When $n \doteq \infty$, the prism $\doteq$ a cylinder and

$$S \doteq 4R\left[\frac{\pi b^2}{\sqrt{R^2-b^2}} - \pi R\left(\frac{R-\sqrt{R^2-b^2}}{\sqrt{R^2-b^2}}\right)\right] = 4\pi R\left[\frac{b^2-R^2}{\sqrt{R^2-b^2}}+R\right] = 4\pi R(R-\sqrt{R^2-b^2}),$$

that is, twice the circumference of a great circle of the sphere times the altitude of the zone, or the surface of two zones of altitude $(R - \sqrt{R^2-b^2})$.

Also solved by J. W. CLAWSON.

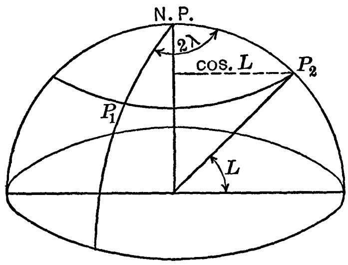
381. Proposed by ELBERT H. CLARK, Purdue University.

Of all points having the same latitude and a constant difference α in their longitudes, to find the latitude of the two so situated that the distance between them, measured along their common parallel of latitude, shall exceed the distance between them measured on their great circle by the greatest possible amount.

SOLUTION BY W. C. EELLS, U. S. Naval Academy.

Let L be latitude of the points, 2λ (in radians) their difference in longitude, $2d$ and $2D$ the small circle latitude and great circle distances between them, and $y = d - D$. Let the earth be a sphere of radius unity. Since the radius of the latitude circle is $\cos L$,

$$d = \lambda\cos L. \qquad (1)$$



Dropping a perpendicular from the north pole to the great circle through the two points, we have two right triangles. Whence, by Napier's rules of circular parts,

$$\sin D = \sin \lambda \cos L. \tag{2}$$

From (1) and (2), $y = \lambda \cos L - \text{arc} \sin (\sin \lambda \cos L)$, and it is desired to find the maximum value of y.

Let

$$\frac{dy}{dL} = -\lambda \sin L + \frac{\sin L \sin h}{\sqrt{1 - \sin^2 \lambda \cos^2 L}} = 0.$$

Then, $\sin L = 0$ is a solution, giving $L = 0$, obviously the minimum value of y.

The other solution is

$$\cos L = \pm \frac{\sqrt{\lambda^2 - \sin^2 \lambda}}{\lambda \sin \lambda},$$

the maximum. Whence, L can be found for any given λ. As λ varies from 0° to 90°, 2λ varies from 0° to 180° and L varies from 90° to 39° 32′ 23″. For $\lambda = 30°$, $L = 53° 34' 42''$ and for $\lambda = 60°$, $L = 49° 31' 16''$.

Also solved by J. A. Caparo, Paul Capron, Horace Olson, C. N. Schmall.

382. Proposed by B. J. BROWN, Student in Drury College.

Discuss for what values of m and n, the integral, $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, is finite and show how this integral can be expressed by means of integrals of the form $\int_0^\infty e^{-x}x^{p-1}dx$.

Solution by H. L. Agard, Williams College.

1°. The integrand has no singular points for $m \geqq 1$ and $n \geqq 1$. The only singular points are $x = 0$ for $m < 1$ and $x = 1$ for $n < 1$.

Pierpont[1] states the

Theorem. Let $f(x)$ be regular in the interval (a, b), except at a. For some $0 < \mu < 1$, let $R \lim\limits_{x=a} (x-a)^\mu |f(x)|$ be finite. Then $f(x)$ is absolutely integrable in (a, b).

[The symbol $R \lim\limits_{x=a}$ indicates that $x \doteq a$ from the right. A similar meaning attaches to $L \lim\limits_{x=a}$. The theorem applies equally to a singularity at b.]

Consider

$$R \lim_{x=0} x^\mu |x^{m-1}(1-x)^{n-1}|, \quad m < 1.$$

Evidently, this limit is finite for $m > 0$ and $1 - m < \mu < 1$.

Consider

$$\text{L} \lim_{x=1} (1-x)^\mu |x^{m-1}(1-x)^{n-1}|, \quad n < 1.$$

Evidently, this limit is finite for $n > 0$ and $1 - n < \mu < 1$.

Hence, by the above theorem, the integral is absolutely convergent if both $m > 0$ and $n > 0$.

[1] Pierpont, *Theory of Functions of Real Variables*, Vol. I, p. 407.

2°. The integral under consideration is known as the Beta function, or first Eulerian integral, and is denoted by $B(m, n)$. This function is to be expressed in terms of the Gamma function, or second Eulerian integral, $\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx$. The required relation is

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

In

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{1}$$

set $x = y/(1+y)$, obtaining,

$$B(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy. \tag{2}$$

In

$$\Gamma(m) = \int_0^\infty e^{-z}z^{m-1}dz,$$

if we set $z = ax$, we get the formula,

$$\frac{1}{a^m} = \frac{1}{\Gamma(m)}\int_0^\infty e^{-ax}x^{m-1}dx. \tag{3}$$

In (3) replace a by $1 + y$ and m by $m + n$, obtaining,

$$\frac{1}{(1+y)^{m+n}} = \frac{1}{\Gamma(m+n)}\int_0^\infty e^{-(1+y)x}x^{m+n-1}dx. \tag{4}$$

(2) and (4) give

$$B(m, n) = \frac{1}{\Gamma(m+n)}\int_0^\infty dy\int_0^\infty e^{-(1+y)x}x^{m+n-1}y^{m-1}dx. \tag{5}$$

In (5) the order of integration may be inverted.[2] For, (1) the integrand, $f(x, y)$, is continuous in the first quadrant, except on the axes; (2) the integrals, $\int_0^\infty f(x, y)dx$ and $\int_0^\infty f(x, y)dy$ are uniformly convergent[3] in any intervals (a, b), $a > 0$, and (α, β), $\alpha > 0$, respectively; and (3) $\int_0^\infty dx\int_0^\infty f(x, y)dy$ exists, since

$$\begin{aligned}\int_0^\infty f(x, y)dy &= e^{-x}x^{m+n-1}\int_0^\infty e^{-xy}y^{m-1}dy \\ &= e^{-x}x^{m+n-1}\cdot x^{-m}\Gamma(m), \text{ by (3)} \\ &= e^{-x}x^{n-1}\Gamma(m)\end{aligned} \tag{6}$$

and, therefore, for $a > 0$, using (6),

$$\int_0^\infty dx\int_0^\infty f(x, y)dy = \Gamma(m)\lim_{a=0}\int_a^\infty e^{-x}x^{n-1}dx = \Gamma(m)\Gamma(n). \tag{7}$$

The conditions for inversion being fulfilled, (5) and (7) give the desired result,

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

383. Proposed by WILLIAM CULLUM, Albion, Mich.

Find the area of the curved surface of a right cone whose base is the asteroid, $x^{2/3} + y^{2/3} = a^{2/3}$, and whose altitude is h. From Townsend and Goodenough's *First Course in Calculus*, p. 288, Ex. 11.

Note. Among other methods, find the required area by means of the formula

$$\iint\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dydx.$$

EDITORS.

[2] See Nielson's *Handbuch d. Theorie d. Gamma Functionen*, pages 148–9. Pierpont, *loc. cit.*, § 680, 3.

[3] Pierpont, *loc. cit.*, § 663.

Two Solutions by F. L. Griffin, Reed College.

(I) *By a single integration.* The element of area is a triangle, whose base ds is an elementary arc of the asteroid, and whose altitude H is the perpendicular from the vertex of the cone upon the tangent to the arc ds. This perpendicular is the hypotenuse of a right triangle whose legs are the height of the cone h, and the perpendicular p from the center of the base upon the tangent. Hence

$$dS = \tfrac{1}{2}Hds, \qquad \text{where} \qquad H = \sqrt{h^2 + p^2}.$$

From the parametric equations of the asteroid $x = a\cos^3 t$, $y = a\sin^3 t$, we find $ds = \sqrt{dx^2 + dy^2} = \frac{3}{2}a\sin 2tdt$. Also, since $dy/dx = -\tan t$, the equation of the tangent line is $Y - a\sin^3 t = -\tan t(X - a\cos^3 t)$. On reducing this to the normal form we find

$$p = \frac{a\sin^3 t + \tan t \cdot a\cos^3 t}{\sec t} = \tfrac{1}{2}a\sin 2t;$$

whence

$$H = \sqrt{h^2 + \frac{a^2}{4}\sin^2 2t} = \tfrac{1}{2}\sqrt{(4h^2 + a^2) - a^2\cos^2 2t}.$$

Finally,

$$S = \frac{3a}{2}\int_0^{\pi/2}\sqrt{(4h^2 + a^2) - a^2\cos^2 2t}\cdot\sin 2tdt,$$

which reduces, on putting $\cos 2t = z$, to

$$S = \tfrac{3}{4}a^2\int_{-1}^{1}\sqrt{\frac{4h^2 + a^2}{a^2} - z^2}dz = \tfrac{3}{2}ah + \tfrac{3}{4}(4h^2 + a^2)\sin^{-1}\frac{a}{\sqrt{4h^2 + a^2}}.$$

[This result, by the way, has a simple interpretation: Three times the right triangle which forms half of the broadest vertical section of the cone, increased by three times a circular sector whose radius is the smallest slant height of the cone and whose angle is the angle between two such opposite smallest slant heights.]

(II) *By the formula* $\int\int\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}dydx$. The equation of the surface, taking the origin at the center of the base, is

$$z = h - \frac{h}{a}(x^{2/3} + y^{2/3})^{3/2}$$

whence

$$S = 4\int_0^a\int_0^{(a^{2/3}-x^{2/3})^{3/2}}\sqrt{x^{2/3}y^{2/3} + \frac{h^2}{a^2}(x^{2/3} + y^{2/3})^2}\,x^{-1/3}y^{-1/3}dydx.$$

Put $x = au^{3/2}$, $y = av^{3/2}$, $h/a = 1/k$, and we have

$$S = \frac{9a^2}{k}\int_0^1\int_0^{1-u}\sqrt{k^2uv + (u + v)^2}dvdu$$

$$= \frac{9a^2}{k}\int_0^1 du\left[\frac{2v + (2 + k^2)u}{8}\sqrt{(2v + \overline{2 + k^2}u)^2 - (k^4 + 4k^2)u^2}\right.$$
$$\left. - \frac{(k^4 + 4k^2)u^2}{8}\log\left(\frac{2v + \overline{2 + k^2}u + \sqrt{(2v + \overline{2 + k^2}u)^2 - (k^4 + 4k^2)u^2}}{2}\right)\right]_0^{1-u}$$

$$= I_1 - I_2 - I_3 + I_4,$$

where

$$I_1 = \frac{9a^2}{4k}\int_0^1(2 + k^2u)\sqrt{1 + k^2u - k^2u^2}du, \qquad I_2 = \frac{9a^2(k^2 + 2)}{4k}\int_0^1 u^2du = \frac{3a^2(k^2 + 2)}{4k},$$

$$I_3 = \frac{9a^2k(k^2 + 4)}{8}\int_0^1 u^2\log(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2})du,$$

$$I_4 = \frac{9a^2k(k^2 + 4)}{8}\int_0^1 u^2[\log u + \log(k^2 + 4)]du = \frac{a^2k(k^2 + 4)}{8}[\log(k^2 + 4)^3 - 1].$$

To evaluate I_1, complete the square and put

$$u = \frac{1}{2}\left(1 + \frac{\sqrt{k^2+4}}{k}\sin\theta\right).$$

Then

$$\sqrt{1+k^2u-k^2u^2} = \frac{\sqrt{k^2+4}}{2}\cos\theta,$$

and $2 + k^2u = \frac{1}{2}(k^2+4)(1+\sin\alpha\sin\theta)$ where $\alpha = \sin^{-1}(k/\sqrt{k^2+4}) = \cos^{-1}(2/\sqrt{k^2+4})$.
Hence,

$$I_1 = \frac{9a^2(k^2+4)^2}{32k^2}\int_{-\alpha}^{\alpha}(1+\sin\alpha\sin\theta)\cos^2\theta d\theta$$

$$= \frac{9a^2(k^2+4)^2}{32k^2}[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{1}{3}\sin\alpha\cos^3\theta]_{-\alpha}^{\alpha} = \frac{9a^2(k^2+4)^2}{32k^2}\left[\alpha + \frac{2k}{k^2+4}\right].$$

To evalute I_3, first integrate by parts, putting

$$U = \log(2+k^2u+2\sqrt{1+k^2u-k^2u^2}), \qquad dV = \frac{9a^2k(k^2+4)}{8}u^2du.$$

Then

$$dU = \frac{k^2du}{2R}\left\{1 - \frac{(k^2+4)u}{2+k^2u+2R}\right\}, \qquad V = \frac{3a^2k(k^2+4)}{8}u^3,$$

where $R = \sqrt{1+k^2u-k^2u^2}$. Then

$$I_3 = \frac{3a^2k(k^2+4)}{8}\left\{[u^3\log(2+k^2u+2R)]_0^1 - \frac{k^2}{2}\int_0^1\frac{u^3du}{R} + \frac{k^2(k^2+4)}{2}\int_0^1\frac{u^4du}{R(2+k^2u+2R)}\right\}$$

$$= \frac{3a^2k(k^2+4)}{8}\log(k^2+4) - I_5 + I_6,$$

where

$$I_5 = \frac{3a^2k^3(k^2+4)}{16}\int_0^1\frac{u^3du}{\sqrt{1+k^2u-k^2u^2}} = \frac{3a^2(k^2+4)^{\frac{5}{2}}}{128k}\int_{-\alpha}^{\alpha}(\sin\alpha+\sin\theta)^3d\theta,$$

$$= \frac{3a^2(k^2+4)^{\frac{5}{2}}}{128k}[2\alpha\sin^3\alpha + 3\alpha\sin\alpha - 3\sin^2\alpha\cos\alpha],$$

$$= \frac{3a^2(k^2+4)}{128}[(5k^2+12)\alpha - 6k],$$

and

$$I_6 = \frac{3a^2k^3(k^2+4)^2}{16}\int_0^1\frac{u^4du}{\sqrt{1+k^2u-k^2u^2}(2+k^2u+2\sqrt{1+k^2u-k^2u^2})}$$

$$= \frac{3a^2(k^2+4)^3}{128k^2}\int_{-\alpha}^{\alpha}\frac{(\sin\alpha+\sin\theta)^4d\theta}{\sin\alpha\sin\theta+\cos\alpha\cos\theta+1},$$

since

$$2+k^2u = \tfrac{1}{2}(k^2+4)(1+\sin\alpha\sin\theta) \quad\text{and}\quad 2R = \sqrt{k^2+4}\cos\theta = \tfrac{1}{2}(k^2+4)\cos\alpha\cos\theta.$$

Hence,

$$I_6 = \frac{3a^2(k^2+4)^3}{128k^2}\int_{-\alpha}^{\alpha}\frac{\left(2\sin\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}\right)^4d\theta}{2\cos^2\frac{\theta-\alpha}{2}}$$

$$= \frac{3a^2(k^2+4)^3}{16k^2}\int_{-\alpha}^{\alpha}\sin^4\left(\frac{\theta+\alpha}{2}\right)\cos^2\frac{\theta-\alpha}{2}d\theta.$$

By making imaginary exponential substitutions, or putting $\theta = 2\phi + \alpha$ and expanding, we find

$$I_6 = \frac{3a^2(k^2+4)^3}{512k^2}[\sin(2\theta+2\alpha) - 8\sin(\theta+\alpha) + 6\theta + \tfrac{1}{3}\sin(3\theta+\alpha)$$
$$+ \sin(\theta+3\alpha) - 2\sin 2\theta - 4\theta\cos 2\alpha + 6\sin(\theta-\alpha)]^{\alpha}_{-\alpha}$$
$$= \frac{3a^2(k^2+4)^3}{128k^2}[\tfrac{14}{3}\sin\alpha\cos^3\alpha - \tfrac{17}{3}\sin\alpha\cos\alpha + \alpha(1+4\sin^2\alpha)]$$
$$= \frac{a^2(k^2+4)}{128k^2}[-2k(17k^2+12) + (15k^2+12)(k^2+4)\alpha].$$

Collecting results we find, since $S = I_1 - I_2 - I_3 + I_4$,

$$S = \alpha\left\{\frac{9a^2(k^2+4)^2}{32k^2} + \frac{3a^2(k^2+4)(5k^2+12)}{128} - \frac{a^2(k^2+4)^2(15k^2+12)}{128k^2}\right\}$$
$$+ \frac{a^2}{64k}\{36(k^2+4) - 48(k^2+2) - 8k^2(k^2+4) - 9k^2(k^2+4) + (17k^2+12)(k^2+4)\}$$
$$= \frac{3}{2}\frac{a^2}{k} + \tfrac{3}{4}a^2\frac{k^2+4}{k^2}\alpha = \tfrac{3}{2}ah + \tfrac{3}{4}(4h^2+a^2)\sin^{-1}\frac{a}{\sqrt{4h^2+a^2}}.$$

Also solved by G. W. Hartwell and the Proposer.

398A. Proposed by EMMA GIBSON, Drury College.

[This problem was unnumbered in the December issue.]

Solve the differential equation

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)x\frac{dy}{dx} = 0.$$

Solution by Frederick Wood, University of Wisconsin.

The equation

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)x\frac{dy}{dx} = 0$$

reduces to

$$(x^3y^3 + 1)(ydx + xdy) + (x^2y^2 + xy)(ydx - xdy) = 0$$

and finally to

$$x^2y^2(ydx + xdy) + ydx + xdy - 2x^2ydy = 0$$

from which we get

$$ydx + xdy + \frac{ydx + xdy}{x^2y^2} - \frac{2dy}{y} = 0.$$

This becomes, after integration,

$$xy - \frac{1}{xy} - 2\log y = c, \qquad \text{or} \qquad x^2y^2 - 2xy\log c'y = 1.$$

Also solved by H. Polish, W. W. Beman, H. S. Uhler, Elijah Swift, Horace Olson, S. E. Rasor, G. Keuligan, H. L. Agard, Elmer Schuyler, and B. J. Brown.

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas.

REPLIES.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their

retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers was printed in a previous issue.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLY BY CHARLES N. MOORE, University of Cincinnati.

The best thing to do is to devote a little time and effort to pointing out the essential unsoundness of the arguments that have been advanced against the retention of mathematics as a required subject in the high schools. The movement against mathematics is, for the most part, confined to a group of educational theorists who feel that they must advocate something new in order to convince their readers that they are investigators. This group, however, has made up in volume of sound for what it has lacked in numbers and, in consequence, has deceived many people into thinking that it represents a widespread trend of thought.

The statement that there is at the present time much uncertainty as to the educational value of algebra and geometry will not bear examination. That the thinking public is as firmly convinced as ever of the high educational value of these subjects was conclusively shown by the answers to a questionnaire sent out recently by Professor Hancock. This questionnaire was sent to a group of the most prominent physicians, clergymen, lawyers and business men throughout the country; it was also sent to a similar group of residents of Cincinnati. In the first group 90 out of a total of 99 advocated a high school course in which mathematics was a required subject; in the second group 96 out of a total of 105 did the same thing.[1] In view of these replies it seems to the writer that the burden of proof rests upon those who wish to displace mathematics as a required subject for high school pupils.

The statement that recent investigations have thrown doubt on the disciplinary value of mathematical study is absolutely without justification. The chief disciplinary value of mathematics is in the training of the reasoning powers it affords. The writer has gone over the literature on transfer of training quite recently, and does not know of any experiments that involved training in logical reasoning. It is absurd to contend that experiments based on the marking of certain letters on a printed page or guessing the size of pieces of paper, will enable one to draw valid conclusions with regard to the training afforded by the study of mathematics.

The statement that the value of mathematics as an instrument has been greatly exaggerated, is also without foundation. As a matter of fact the majority of people do not realize the wide extent of the applications of mathematics. The writer is continually meeting people who are deploring their lack of mathematical training because their deficiency in this respect renders it difficult or impossible to pursue some particular study or calling in which they are interested. In a recent article in *Science*,[2] Professor S. G. Barton pointed out the fact that in

[1] For a more detailed account of Professor Hancock's investigation, *cf. School and Society*, Vol. I (1915), p. 893.

[2] Volume 40 (1914), page 697.

104 articles of the Encyclopedia Britannica, eleventh edition, use was made of the symbols of the infinitesimal calculus. Only about one fourth of these articles could be classed as dealing with pure mathematics.

The statement that the great majority of the teachers of mathematical subjects in high schools are not informed as to "the ultimate educational values" of these studies, seems much exaggerated. Undoubtedly some of these teachers are not so informed, and the reason for it lies in the deficiency of their own mathematical training. The remedy for this lies in demanding a greater amount of preliminary study of mathematics as a qualification for the teaching of that subject in the high schools.

The suggestion that mathematics might be made more cultural by introducing purely descriptive courses in this subject, is in line with the effort in certain quarters to convert our high schools into kindergartens for students of high school age. The idea that mathematics could be "interpreted" to a student that never had a mathematical idea in his head, would be amusing if it were not advocated as a serious proposal.

DISCUSSIONS.

RELATING TO THE "SIMSON LINE" OR "WALLACE LINE."

By Roger A. Johnson, Western Reserve University.

(Cf. statement in this Monthly for June, 1915, p. 201, problem No. 467.)

It is a well-known theorem that the feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumscribed circle of the triangle. This theorem has always been known as Simson's, and the line as Simson's line, until recent times, when an attempt has been made to replace Simson's name by that of Wallace. See, for instance, this Monthly, June, 1915, p. 201, problem 467. The purpose of this note is to submit the proposition that the name "Simson Line" should be retained. It would be desirable, perhaps, to have the views of readers of the Monthly on the matter.

Let us first determine the facts. Some thirty years ago, the inquiry was made as to where in Simson's works the theorem in question was to be found. Search was made, especially by that able investigator J. S. Mackay of Edinburgh, who read all known writings of Simson, without finding an allusion to the theorem. Further investigation persuaded him that the fact was discovered by one William Wallace about 1798; it seems to have appeared in print for the first time in that year, in Leybourn's Mathematical Repository.[1] Mackay also[2] traces the source of the erroneous ascription to Simson. Early in the century, Servois[3] cites the theorem, introducing it in the following words, "le théorème suivant, qui est, je crois, de Simson." A little later, Poncelet in his "Propriétés Projectives," 468, reproduced this statement without the qualification, and his authority was evidently suf-

[1] Cf. Muir, *Proceedings of the Edinburgh Math. Society*, Vol. 3, p. 104.
[2] Mackay, *Proceedings of the Edinburgh Math. Society*, Vol. 9, pp. 83–91.
[3] Gergonne's *Annales de Math.*, Vol. IV, p. 250.

ficient to perpetuate the error. Certainly, until the last decade of the nineteenth century, no one doubted that the theorem was Simson's. Now, however, there seems no reason to believe that he ever heard of it.

If, then, the theorem is Wallace's, should it not be given his name, in tardy recognition of the debt we owe him? After Mackay's researches had been published, he and other Scotch writers made the innovation, unwisely it seems to the present writer. At present, custom is divided. No author dares use one name without mentioning the other.

In any such situation, it is obvious that an attempt to effect a change from a universally established term to a new one will involve such an amount of confusion that we must convince ourselves that the advantages are sufficient to outweigh the inconveniences. The most powerful, and perhaps only reason for changing is that of justice, to assure to every discoverer the credit due him. This seems inadequate. In general, the fact that a theorem is named for a man is neither a sufficient indication that the theorem was discovered by him, nor, on the other hand, a definite characterization of the theorem. The classic example of an unjustly named theorem is Cardan's solution of the cubic, discovered, as is well known, by Tartaglia; several cases of the same kind will occur to the reader. On the other hand, the expression "Steiner's Theorem," for instance, may mean any of numerous familiar theorems. If, in particular, we continue to designate the line under discussion as Simson's line, or rather as the Simson line, we shall not be implying that it was discovered by Robert Simson, but that his name is attached to it by long-established custom.

There is, however, another reason for opposing the change, more cogent than this. The term "Wallace's line" already has a well defined meaning in another field of science, namely Zoögeography. According to Webster's New International Dictionary, it was named for Dr. Alfred Russel Wallace, and is "an imaginary line separating the Oriental and Australian regions. . . . The faunas on either side of the line are remarkably distinct." Now it is not likely that this line will be confused with the one mentioned in our theorem, but it would seem that since we are in a position to choose, the wisest choice is that which avoids all possibility of confusion.

CORRESPONDENCE.

To the Board of Editors:

Thinking in retrospect of the organization meeting at which the Mathematical Association of America was launched, I desire to congratulate you upon the very successful issues of the meeting, and upon the very auspicious beginning of the new Association.

One thought has occurred to me that I have not yet heard expressed. It relates to the toast on "The economic value of the young" to which Professor Hedrick so aptly responded at the luncheon tendered by the mathematics faculty of the University of Ohio. Now, economic considerations are of very vital import

to the average teacher of mathematics. His salary, as well as his rank and professional standing, are very closely related to the way in which he is regarded by other mathematicians, in his own institution and throughout the country. His advancement usually depends upon a change from one institution to another, or on the prospect of such a change. Until the present time the only way of claiming the regard and good opinion of mathematicians outside of one's own institution has been through research. This has resulted in great emphasis on activities in research and small emphasis on effective teaching, when appointments and promotions are considered. The new association should become an avenue through which an effective teacher may gain the good opinion of the profession at large because of the quality of his work and through contributions to the solution of the problems of the teacher. In this way salary, rank, etc., may come to be more closely related to service rendered. May not the new born Association thus become a very material economic asset to many worthy teachers of mathematics?

The economic motive may become an influence of considerable proportions in promoting activities in the new Association, as I believe it has been in promoting research which finds its expression through the old Society.

RALPH E. ROOT.

U. S. NAVAL ACADEMY,
January 4, 1916.

ECHOES FROM THE COLUMBUS MEETING.

All will be interested to know that the membership list of the Association is growing apace, over one hundred per week being the record for the time since the announcements were distributed. It is taken for granted that all who signed the call for the organization meeting will come in as charter members. In fact, the membership committee of the Council has already elected all of these persons to membership so that there will be no delay when their names are received. Large numbers of them are already in, as well as many of those whose names were received too late to appear in the call.

Attention is again called to the provision of the Constitution that in the case of all who join the Association before April 1, 1916, the initiation fee is to be waived. Doubtless, many who fully intend to take advantage of this provision may find themselves characterized in the following note just received from a prominent member of the faculty of a great New England University:

> You may have wondered why I did not join the Mathematical Association earlier, if you thought about it at all. It has been negligence and not lack of appreciation for the movement you have had so much to do with. I feel a large personal debt to the MONTHLY, and I believe it will do even more good in the future than in the past.

It is confidently believed that such declarations of allegiance, just at this time, to the Mathematical Association of America, will prove to be as important a service to the cause of mathematics in this country as may be rendered by those who are in any way related to the collegiate field. The establishment of a widely

representative membership of large proportions within a period of sixty days (from the distribution of the announcements) will set a record and create a momentum which should sweep all obstacles from the path.

Institutional memberships are coming in as rapidly as could be expected in view of the time required to make such arrangements. One of the universities on the former subsidy list has shown its confidence in the cause by subscribing for an institutional membership for a period of five years in advance. It is believed that many institutions will consider it an honor to be thus connected with this Association.

A correspondent asks whether a delegate sent by an institutional member to a meeting would be counted as an individual member. This is clearly not contemplated by the Constitution. Institutional members are those who support the Association by the payment of the individual dues, and only the names of such persons will appear in the address list of individual members to be printed later. There will also be printed a list of institutional members, but this list will not be accompanied by the names of delegates, since these will vary from time to time. The names of delegates actually in attendance at any meeting will be printed with the report of that meeting. An individual member may, of course, be delegated by an institution to represent it at any meeting. The two copies of the MONTHLY to be sent to institutional members will in all cases be mailed directly to the library of the institution, and not to individual members of the faculty.

Inquiries are coming in with respect to the formation of state sections of the Association. One of particular interest at this time is from the state of Georgia. It is hoped and believed that many new organizations of this character will be formed as soon as there is time for the plan to become sufficiently well known; and also that many existing organizations will be directed into this channel. Emphasis is laid upon the state as a convenient geographical subdivision for a section, because of the fact that there are already mathematical organizations in many cases in connection with the general state teachers' associations, and these can be readily transformed into sections of the new Association. Moreover, the meetings of such sections can be held at the same time as the general state meetings, thus making it possible for larger numbers to get together, and avoiding unnecessary duplication of places and times of meeting. The Committee on the organization of sections is working out the details and will have a report ready to publish in the March issue.

Suggestions have already been made for starting a library for the Association. For the present it will be in charge of the Secretary, and any who wish to contribute books or journals may send them directly to Professor W. D. CAIRNS, Oberlin, Ohio. The Managing Editor wishes to announce that through the kindness of Professor E. H. MOORE, he has been able to make up a complete set of the MONTHLY. These volumes will be bound and placed in the Library at an early date. This is especially fortunate, since it is almost impossible now to secure complete sets of the MONTHLY. Some publishers have also contributed certain books. A complete list of these will be published soon.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University.

A late volume of the Proceedings of the Indiana Academy of Sciences contains an extended treatment of "Some properties of binomial coefficients," by PROFESSOR A. M. KENYON, of Purdue University.

"The theory of invariants," by PROFESSOR O. E. GLENN, of the University of Pennsylvania, has been published by Ginn and Company.

MISS GERTRUDE I. MCCAIN has been appointed professor of mathematics in the Oxford College for Women at Oxford, Ohio.

John Wiley & Sons announce the publication of a book of 390 pages on the "Theory and Applications of Finite Groups," by PROFESSORS G. A. MILLER, H. F. BLICHFELDT, and L. E. DICKSON.

MR. ALTON MILLER, absent on leave from the University of Michigan, is studying under Professor Segre at the University of Torino. Mr. Miller holds a fellowship in mathematics from Harvard University.

The Wyoming School Journal for January, 1916, contains an article by C. EBIN STROMQUIST, professor of mathematics in the University of Wyoming, on "The correlation of mathematics in grades seven to ten."

"The college teacher's function" is the title of a thoughtful article in *School and Society* for January 15, by Professor ARTHUR J. TODD, University of Minnesota.

"College entrance requirements" is a topic discussed with considerable illumination in *School and Society* for January 8, 1916, by Professor LOUIS W. RAPER, of Pennsylvania State College.

"Robert of Chester's Latin translation of the algebra of Al-Khowarizmi," by PROFESSOR L. C. KARPINSKI, has just appeared from the Macmillan Press. Besides an English translation the work contains an introduction giving a brief history of algebra.

"Mathematics and efficiency" is the title of an article in the January number of *School Science and Mathematics*, by FLETCHER DURELL, Headmaster at the Lawrenceville School, New Jersey. In the same issue, Professor G. A. MILLER writes on "Reform in teaching mathematics."

The major part of the October issue of the *Mathematical Gazette* is devoted to "A study of the life and writings of Colin Maclaurin," by Charles Tweedie,

F.R.S.E., including four full-page plates and a critical estimate of his writings. A condensation of this article will be given in a later issue of the MONTHLY.

The *Bollettino di Bibliografia e Storia delle Scienze Matematiche* continues to arrive with commendable regularity in view of the extraordinary conditions created by the war. The closing number of 1915, No. 4, was published Nov. 20, 1915. Besides numerous reviews, the volume for 1915, Vol. XVII, includes an article by L. GODEAUX on a Belgian mathematician of the sixteenth century; articles by J. H. GRAF on the correspondence between Ludwig Schläfli and Italian mathematicians of his period, including to date letters from D. Chelini and E. Beltrami, and an article on Lingi Forni, a mathematician of Pavia, by G. VIVANTI.

Two articles in *Science* for September 11 and October 22, 1915, by Professor C. N. Moore, of the University of Cincinnati, will be of interest to readers of the MONTHLY. The one is on "Correlation and disciplinary values" and the other on "The coefficient of correlation as a measure of relationship." The former gives the results of some statistical studies with reference to the disciplinary value of certain high school subjects, including algebra and geometry, which are being attacked in many quarters. See also Professor Moore's reply to Question 24 in this issue.

A reprint from the Napier Tercentenary memorial volume, published by the Royal Society of Edinburgh, contains the address by Professor FLORIAN CAJORI on "Algebra in Napier's day and alleged prior inventions of logarithms." The three main conclusions are (1) "That John Napier enjoys the all-important right of priority of publication"; (2) "that Joost Bürgi is entitled to the honor of independent invention"; (3) "that Joost Bürgi constructed his table some time between 1603 and 1611, and that John Napier worked on logarithms probably as early as 1594—that, therefore, Napier began working on logarithms probably much earlier than Bürgi."

In the twelfth volume, series III, 1915, of the *Periodico di Matematica* which completed the thirtieth year of publication with the issue of September, 1915, several articles are found which may be of interest to readers of the MONTHLY. Two articles, to be continued, by G. LAZZERI treat of static moments, moments of inertia, and moments of higher order; two articles by D. KRYJANOVSKY, translated from the Russian into Italian apparently by the author himself, treat of maxima and minima of plane figures; an article by Q. PAOLINA discusses the relations between arithmetic and geometric means, based upon the very ancient proposition that the geometric mean between two quantities is less than the arithmetic mean.

The American Mathematical Society held its twenty-second annual meeting in New York, December 27–28, 1915. There were seventy-two members present at the four sessions and thirty-seven papers were presented. The total member-

ship of the Society is now 736. The number of members attending at least one meeting of the Society or its sections during the year 1915 was 253. The total number of registrations at all meetings during the year was 418. The president of the Society is Professor E. B. Brown, of Yale University, and the two vice-presidents are Professor E. R. Hedrick, of the University of Missouri, and Professor Virgil Snyder, of Cornell University.

In the September–November issue of *L'Enseignement Mathématique* PROFESSOR G. A. MILLER writes an article on "The preparation of mathematics teachers in the United States of America," in which he presents a report of the many changes which the American universities have inaugurated during the last twenty-five years, with a view of providing better facilities for students who are teachers (through summer sessions) or who expect to become teachers (through schools of education). He points out, further, the inadequacy of the number of properly equipped teachers to fill all the available positions, the relatively low but increasing salary scale, the wholesome influence of THE AMERICAN MATHEMATICAL SOCIETY on the teachers of higher mathematics, and the awakening of interest among the teachers of secondary mathematics through their numerous teachers' associations and through two periodicals devoted to their interests.

There is a bill before Congress to make the use of the Centigrade thermometer scale obligatory in all government publications, in the hope of bringing about its adoption for all purposes in place of the Fahrenheit scale. This is a move in a good direction, but it raises again the larger question of the metric system as a whole, and we wonder whether the United States will be the last of the civilized nations to adopt that system. An article in *The Scientific Monthly* for December, 1915, by Dr. Joseph V. Collins, of Stevens Point, Wis., discusses the question under the title: "A metrical tragedy," showing that at least two thirds of a year for every child in the land is wasted in the study of our cumbersome system of weights and measures, and that this waste entails an economic loss of possibly three hundred millions of dollars annually. It is an opportune time for all friends of progress in this direction to act, especially as requests have been made for all available data bearing upon the thermometer phase of the question.

The Chicago Section of the American Mathematical Society held its thirty-sixth regular meeting at Columbus, Ohio, on Thursday, Friday and Saturday, December 30–31, 1915, and January 1, 1916. There were sixty-seven members in attendance and twenty-five papers were read, aside from the retiring address of the chairman, Professor E. J. WILCZYNSKI, of the University of Chicago, who spoke at the joint session with Section A of the American Association on "The historical development and future prospects of differential geometry of plane curves." At the same session Professor H. S. WHITE, of Vassar College, gave his retiring address as Vice-President of Section A on the topic: "Poncelet Polygons." Practically all members of the American Mathematical Society who were present at Columbus

joined in the organization meeting of the THE MATHEMATICAL ASSOCIATION OF AMERICA. The dinner on Thursday evening, in fact, resolved itself largely into a congratulatory occasion in honor of the birth of the new Association. The officers of the Chicago Section for the next two years are: *Chairman,* Professor W. B. FORD, University of Michigan; *Secretary,* Professor ARNOLD DRESDEN, University of Wisconsin; *Third member of program committee,* Professor H. L. RIETZ, University of Illinois.

Please note the following addresses of chairmen of committees to whom communications should be sent on the respective subjects:

Problems Proposed and Solved, to Professor B. F. FINKEL, 1228 Clay St., Springfield, Mo.

Questions and Discussions, to Professor U. G. MITCHELL, University of Kansas, Lawrence, Kan.

Notes and News, to Professor D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Books to be Reviewed, to Professor W. H. BUSSEY, University of Minnesota, Minneapolis, Minn.

General Editorial Correspondence, to Professor H. E. SLAUGHT, Managing Editor, 5548 Kenwood Ave., Chicago, Ill.

Business Correspondence concerning both the MONTHLY and the ASSOCIATION, to Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

IMPORTANT NOTICE TO PRESENT MONTHLY SUBSCRIBERS.

Henceforth, the subscription price of the MONTHLY will be *three dollars net to all non-members of the* ASSOCIATION. The following adjustments for prospective members are proposed:

(1) Those who have already paid their subscriptions for the entire year 1916 are asked to send *one dollar* additional, which will entitle them to membership in the ASSOCIATION.

(2) Those who have not paid for 1916 are asked to send *three dollars,* which will entitle them to membership and include the MONTHLY. *No further subscriptions for 1916 will be received at the old rate of two dollars.*

In the case of subscriptions under (1) or (2) which expire *before* the end of 1916, please add *twenty cents extra* for each copy needed to complete the year. *Hereafter all subscriptions will date from January of each year.*

(3) An institution in which the Calculus is taught may become an *institutional member* of the ASSOCIATION by the payment of *five dollars* annually, which will entitle the library to receive *two copies of the* MONTHLY and the institution to send a voting delegate to all meetings of the ASSOCIATION. Institutions in which the Calculus is taught, whose libraries have already renewed their subscriptions for 1916, are asked to send *three dollars additional* and thus become institutional members of the ASSOCIATION.

Other institutions, and those not wishing to become institutional members, whose library subscriptions have already been renewed for 1916, are asked to send *one dollar additional* to complete the advanced price of the MONTHLY. No further subscriptions will be received at the old rate of two dollars, *and no discount from the advanced rate of three dollars will be allowed on subscriptions made through agencies.*

(4) The obligations of the MONTHLY for 1916 will, of course, be fulfilled on the former basis in the case of any individual or institution whose subscription has already been paid, and who may decline to make the adjustment on the new basis.

(5) Please note that all subscriptions to the MONTHLY and dues in the ASSOCIATION are to be paid to the SECRETARY-TREASURER, Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

If you have not already returned the membership blank, please do so at once. Delay may make it impossible to secure the back issues of the MONTHLY.

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.
2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.
2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.
2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.
3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.
4. The Council shall have authority to fill vacancies *ad interim*.

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.
2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.
2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.
2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

The initiation fee shall be waived in case of those who join the Association before April 1, 1916, and this clause shall be dropped after its provisions have been fulfilled.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list or charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the othef being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

Twelve members of the Council shall be elected at the first meeting of the Association, and the secretary shall draw lots to determine which four of those elected shall serve for one, for two, and for three years respectively. (This clause shall be dropped after its provisions have been fulfilled.)

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

This shall not be construed to affect existing contracts with any subscribers or news agencies for the year 1916, who may decline to readjust on the new basis. (This clause shall be dropped after its provisions have been fulfilled.)

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

IMPORTANT ANNOUNCEMENT

TO

ALL INTERESTED IN MATHEMATICAL PROGRESS

THE AMERICAN MATHEMATICAL MONTHLY, since its reorganization in January, 1913, has endeavored to fulfill its mission as "A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS."

A selection from the Tables of Contents thus far includes articles on—
The History of Mathematics, such as the following:

"History of the Exponential and Logarithmic Concepts," by PROFESSOR FLORIAN CAJORI of Colorado College;

"The Foundation Period in the History of Group Theory," by JOSEPHINE BURNS, Graduate Student at the University of Illinois;

"Errors in the Literature on Groups of Finite Order," by PROFESSOR G. A. MILLER, University of Illinois;

"Number Systems of the North American Indians," by PROFESSOR W. C. EELLS, United States Naval Academy;

"The Algebra of Abu Kamil," by PROFESSOR L. C. KARPINSKI, University of Michigan;

"Centers of Similitude of Circles and Certain Theorems Attributed to Monge. Were they known to the Greeks?" by PROFESSOR R. C. ARCHIBALD, Brown University;

"The History of Zeno's Arguments: Phases in the Development of the Theory of Limits," by PROFESSOR FLORIAN CAJORI, Colorado College.

Pedagogical Considerations, such as the following:

The "Foreword" concerning Collegiate Mathematics, by PROFESSOR E. R. HEDRICK, University of Missouri;

"Some Things we wish to know," by PROFESSOR E. R. HEDRICK;

"Mathematical Literature for High Schools," by PROFESSOR G. A. MILLER;

"Mathematical Troubles of the Freshman," by PROFESSOR G. A. MILLER;

"Minimum Courses in Engineering Mathematics,"by PROFESSOR SAUL EPSTEEN, University of Colorado;

"Incentives to Mathematical Activity," by PROFESSOR H. E. SLAUGHT, University of Chicago;

"Synthetic Projective Geometry as an Undergraduate Study," by PROFESSOR W. H. BUSSEY, University of Minnesota;

"Retrospect and Prospect," by PROFESSOR H. E. SLAUGHT;

"Note on a Memory Device for Hyperbolic Functions," by F. S. ELDER, Central High School, Kansas City, Mo.;

"A Plea for less Formal Work in Mathematics," by F. M. MORGAN, Dartmouth College;

"A Simple Algebraic Paradox," by PROFESSOR J. L. COOLIDGE, Harvard University;

"Note on Simple Algebraic Equations," by PROFESSOR H. L. SLOBIN, University of Minnesota;

"On Courses in Synthetic Projective Geometry," by PROFESSORS LAO G. SIMONS, Normal College of the City of New York, C. E. STROMQUIST, University of Wyoming, T. G. RODGERS, Normal School of New Mexico, R. D. CARMICHAEL, and D. N. LEHMER;

"On the Cultural Value of Mathematics," by PROFESSORS W. T. STRATTON, Kansas State Agricultural College, and D. N. LEHMER;

"On Courses in the History of Mathematics," by PROFESSORS W. T. STRATTON and G. A MILLER;

"Remarks on Klein's Famous Problems of Elementary Geometry," by PROFESSOR R. C. ARCHIBALD, Brown University;

"On the Trisection of an Angle and the Construction of Regular Polygons of 7 and 9 Sides," by PROFESSOR L. E. DICKSON, University of Chicago;

"An Equation Balance for Class-Room Use," by PROFESSOR E. W. PONZER, Stanford University;

"A Cardioidograph," by C. M. HEBBERT, University of Illinois;

"Coördinated Courses in High School Mathematics," by EDITH LONG, Lincoln, Neb., and ROY CUMINS, Columbia University;

"Conference Periods for Students," by PROFESSOR C. R. MCINNES, Princeton University, and PROFESSOR C. S. ATCHISON, Washington and Jefferson College;

"Determinant Formula for Coplanarity of Four Points," by PROFESSOR A. M. KENYON, Purdue University;

"What can the Colleges do toward Improving the Teaching of Mathematics in the Secondary Schools?" by PROFESSOR C. N. MOORE, University of Cincinnati.

General Mathematical Information, such as the following:

"The Third Cleveland Meeting of the American Association for the Advancement of Science," by PROFESSOR G. A. MILLER;

"Western Meetings of Mathematicians," by PROFESSOR H. E. SLAUGHT;

"Summer Meeting of the American Mathematical Society," by PROFESSOR H. E. SLAUGHT;

"Notes and News" of events pertaining to mathematics, under the direction of a committee of which PROFESSOR FLORIAN CAJORI is chairman;

"The Napier Tercentenary Celebration," by PROFESSOR FLORIAN CAJORI, Colorado College;

"The Paris Report on Calculus in the Secondary Schools," EDITORIAL;

"California Teachers of Mathematics," EDITORIAL;

"Book Reviews" and announcements of new books in Mathematics, under the direction of a committee of which PROFESSOR W. H. BUSSEY, University of Minnesota, is chairman.

Fifty-four books have thus far been reviewed, each by a selected expert in his field.

Topics Involving a Minimum of Technical Treatment, such as the following:

"Maximum Parcels under the New Parcel Post Law," by PROFESSOR W. H. BUSSEY;

"Precise Measurements with a Steel Tape," by PROFESSOR G. R. DEAN, Missouri School of Mines;

"A Direct Definition of Logarithmic Derivative," by PROFESSOR E. R. HEDRICK;

"A Simple Formula for the Angle Between Two Planes," by PROFESSOR E V. HUNTINGTON, Harvard University;

"On the Solutions of Linear Equations having Small Determinants," by PROFESSOR F. R. MOULTON, University of Chicago;

"The Accuracy of Interpolation in a Five-Place Table of Logarithms of Sines," by PROFESSORS A. M. KENYON and G. JAMES, Purdue University;

"A Theorem about Isogonal Conjugates," by DAVID F. BARROW, Harvard University;

"The Significance of the Weierstrass Theorem," by PROFESSOR E. R. HEDRICK;

"On the Impossibility of Certain Diophantine Equations and Systems of Equations," by PROFESSOR R. D. CARMICHAEL, Indiana University;

"A Computation Formula in Probability," by E. C. MOLINA, New York City;

"Two Geometrical Applications of the Method of Least Squares," by PROFESSOR J. L. COOLIDGE, Harvard University;

"A Puzzle Generalized," by PROFESSOR R. P. BAKER, University of Iowa;

"On Certain Diophantine Equations having Multiple Parameter Solutions," PROFESSOR R D. CARMICHAEL;

"A Geometrical Discussion of the Regular Inscribed Hexagon," by J. Q. MCNATT, Florence Colo., and S. A. JOFFE, New York City;

"A Theorem in Number Theory connected with the Binomial Formula," by Professor D. N. LEHMER;

"An Application of Partial Derivatives to the Ellipse," by PROFESSOR M. O. TRIPP, Muncie, Ind.;

"A Curious Convergent Series," by PROFESSOR A. J. KEMPNER, University of Illinois;

"Optical Interpretations in Higher Geodesy," by PROFESSOR W. H. ROEVER, Washington University;

"A Problem in Number Theory," by PROFESSOR G. A. OSBORNE, Massachusetts Institute of Technology;

"Perfect Magic Squares for 1914," by V. M. SPUNAR, Chicago, Ill., and PROFESSOR B. L. REMICK, Manhattan, Kan.;

"The Construction of Conics under given Conditions," by Dr. B. M. WOODS, University of California;

"A Simple Method of Constructing the Normals to a Parabola," by PROFESSOR S. G. BARTON, University of Pennsylvania;

"Some Properties of the Normals to a Parabola," by PROFESSOR S. G. BARTON;

"Apparent Size of a Cube," by PROFESSOR A. M. HARDING, University of Arkansas;

"Residues of Certain Sums of Powers of Integers," by PROFESSOR T. M. PUTNAM, University of California;

"Groups of Figures in Elementary Geometry," by PROFESSOR G. A. MILLER, University of Illinois;

"On the Use of Partial Derivatives in Plotting Equations from their Curves," by PROFESSOR A. M. KENYON, Purdue University;

"A Method of Solving Numerical Equations," by S. A. COREY, Hiteman, Iowa;

"Sur un Paradoxe Algébrique Apparent," par G. LORIA, Université de Gêne;

"The Theorem of Rotation in Elementary Mechanics," by PROFESSOR E. V. HUNTINGTON, Harvard University;

"Groups of Subtraction and Division with Respect to a Modulus," by PROFESSOR G. A. MILLER, University of Illinois;

"Questions and Discussions," under the direction of PROFESSOR U. G. MITCHELL, University of Kansas;

"Problems Proposed and Solved," under the direction of PROFESSORS B. F. FINKEL, Drury College, and Professor R. P. BAKER, University of Iowa.

Topics Involving Somewhat More Technical Treatment, designed to stimulate mathematical activity on the part of ambitious students and teachers. Such articles have occupied only about one-sixth of the entire space; for example, such as the following:

"The Remainder Term in a Certain Development of $F(a + x)$," by PROFESSOR R. D. CARMICHAEL;

"A Geometric Interpretation of the Function F in Hyperbolic Orbits," by PROFESSOR W. O. BEAL, Illinois College;

"Certain Theorems in the Theory of Quadratic Residues," by PROFESSOR D. N. LEHMER, University of California;

"Some Inverse Problems in the Calculus of Variations," by DR. E. J. MILES, Yale University;

"Amicable Number Triples," by PROFESSOR L. E. DICKSON, University of Chicago;

"The Probability Integral," by PROFESSOR E. L. DODD, University of Texas;

"A Note on the Solution of Linear Differential Equations," by DR. C. R. MACINNES, Princeton University;

'A Graphical Solution of the Differential Equation of the First Order," by PROFESSOR T. R. RUNNING, University of Michigan;

"The Curve of Light on a Corrugated Dome," by PROFESSOR W. H. ROEVER, Washington University;

"The Cube Root of a Binomial Surd," by PRINCIPAL ARTHUR C. JOHNSON, Hopedale, Mass.

"The Tactical Problem of Steiner," by PROFESSOR W. H. BUSSEY;

"On Some Geometric Properties of Circular Transformations," by PROFESSOR ARNOLD EMCH, University of Illinois;

"A Note on Plane Kinematics," by PROFESSORS ALEXANDER ZIWET and PETER FIELD; University of Michigan;

"A Theorem in the Modern Plane Geometry of the Abridged Notation," by PROFESSOR R. E. BRUCE, Boston University;

"On a purely Projective Basis for the Theory of Involution," by PROFESSOR D. N. LEHMER;

"A Formula for the Sum of a Certain Type of Infinite Power Series," by ELBERT H. CLARKE, Purdue University;

"On a Special Case of the Tetrahedral Complex," by PROFESSOR D. N. LEHMER, University of California;

"General Formula for the Valuation of Securities," by PROFESSOR J. W. GLOVER, University of Michigan.

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THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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PUBLISHED BY THE ASSOCIATION

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1912. FROM 1912 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the Managing Editor, H. E. Slaught 5548 Kenwood Avenue, Chicago, Ill.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 55 East Lorain Street, Oberlin, Ohio.

The last of these not only has remarkably small coefficients but also has a root more nearly equal to π than we had any right to expect, namely $x = 3.1415925$.

The last four equations may be combined to satisfy various further conditions. For example, $F_1 - 2D_1$ gives us the third degree equation

$$41x^3 - 87x^2 - 89x - 133 = 0.$$

Or we may obtain an equation making a still closer approximation. Thus, $3D_1 - F_1$ gives the equation

$$4x^4 - 72x^3 + 125x^2 + 135x + 185 = 0,$$

which has a root equal to 3.1415926557, the value of π being 3.1415926536. Or various other conditions might be imposed.

For the degree of accuracy used here the computation is rather laborious. If, however, only a single third degree equation is desired and not more than four or five significant figures in the root, the work is not long, especially if the powers of x at the beginning are found by logarithms.

NAPIER'S LOGARITHMIC CONCEPT: A REPLY.

By FLORIAN CAJORI.

In the *Mathematical Gazette* of May, 1915, page 78, Professor H. S. Carslaw quotes the following passage from my article: "A History of the Exponential and Logarithmic Concepts," in the AMERICAN MATHEMATICAL MONTHLY of January, 1913, page 7:

Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{cccccc} v, & v\left(1-\frac{1}{v}\right), & v\left(1-\frac{1}{v}\right)^2, & \cdots, & v\left(1-\frac{1}{v}\right)^n, & \cdots \\ 0, & 1, & 2, & \cdots, & n, & \cdots. \end{array}$$

The numbers in the upper series represent successive values of the *sines*; the numbers in the lower series stand for the corresponding logarithms. Thus $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log [10^7(1 - 10^{-7})^n] = n$, where $n = 0, 1, 2, \cdots$.

Professor Carslaw says: "This statement is incorrect. In Napier's Tables the logarithm of $(10^7 - 1)$ is not 1. It lies between 1 and 1.0000001, and he takes it as the mean between these two numbers, namely 1.00000005."

In reply to this I desire to make the following remarks: (1) In my article I did not explain at all Napier's *computation*; I aimed to explain his logarithmic *concept*. Napier's *theory* rests on the establishment of a one-to-one correspondence between the terms of a geometric series and the terms of an arithmetic series.[1] But, *it is not possible to write down two such series which represent exactly the numbers arising in Napier's computations*. Professor Carslaw himself admits that, in Napier's computations, "the numbers are not exactly in geometrical

[1] See Napier's *Constructio* (Macdonald's edition), page 19.

progression" (p. 82). Professor Carslaw remarks also that "when he (Napier) comes to define the term *logarithm,* he takes a new point of view altogether, and, though his logarithms nearly agree with those defined above, they do not do so absolutely."

I wrote down the two series quoted above, believing that they exhibited, all things considered, Napier's *theory* in its simplest and truest light. Professor Carslaw says that my exposition is "incorrect."

(2) I claim that the two series exhibit correctly the meaning of the word *logarithm*—"the number of the ratios."[1] The logarithm n indicates the number of the ratios in the antilogarithm $v(1 - 1/v)^n$.

(3) I claim further that even according to Napier's computations, as explained in his *Constructio,* page 21, my statement is not "incorrect." When speaking of upper and lower limits of the logarithm of a given sine, Napier says:

> "The limits themselves differing insensibly, they or anything between them may be taken as the true logarithm. Thus, in the above example, the logarithm of the sine 9999999 was found to be either 1.0000000 or 1.00000010, or best of all 1.00000005. For since the limits themselves, 1.0000000 and 1.0000001, differ from each other by an insensible fraction like 1/10000000, therefore they and whatever is between them will differ still less from the true logarithm lying between these limits."

Napier here admits 1 as a value of log $(10^7 - 1)$ which differs "insensibly" from the true value. The true value demanded by his theory of moving points is neither 1 nor 1.00000005; it is a little greater than the latter.

(4) I cheerfully admit that taking log $(10^7 - 1) = 1.00000005$ would have been "best of all" for the purpose of exhibiting somewhat more closely the results of Napier's *computation.* The first 101 numbers from 10^7 to 9999900.0004950, found in Napier's "First Table" in the *Constructio* (pp. 13, 22), are in geometric progression; Napier assigns multiples of 1.00000005 as their respective logarithms. But for reasons mentioned above, taking 1.00000005 as the first term of an arithmetical progression does not accurately reproduce all of Napier's results of computation. Moreover, 1.00000005 would have been less satisfactory than 1 for the purpose of exhibiting the notion of "the number of the ratios." I still think that in a very brief exposition, such as I gave in the article quoted, Napier's logarithmic *concept* is made plainer by the two series as I gave them, than by the introduction of eight-place decimals in each term of the arithmetic series, which complicate the bird's-eye view, without showing with absolute accuracy Napier's tabular results.[2] That my exposition is "incorrect" I cannot admit.

[1] Professor Carslaw and I interpret the word "logarithm" as meaning "the number of the ratios." Some writers prefer another derivation, namely, "ratio-number" or "number associated with ratio."

[2] I seize this opportunity to state that the two Latin definitions of logarithms quoted in my article in the AMERICAN MATHEMATICAL MONTHLY, Vol. 20, 1913, page 7, are the phrasings given by Briggs and not by Napier.

A NOTE ON THE CALCULATION OF EULER'S CONSTANT.

By GOLDIE HORTON, University of Texas.

Euler's constant, which plays an important rôle in the theory of gamma functions, is usually defined by the relation

$$\gamma = \lim_{n=\infty} \sum_1^n \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right)\right].$$

Its direct calculation from this definition is impracticable, and it is actually computed by means of the asymptotic expansion

$$\gamma = \lim_{n=\infty} \sum_1^n \frac{1}{m} - \log n - \frac{1}{2n} + \frac{B_1}{2n^2} - \frac{B_2}{2n^4} + \cdots, \tag{1}$$

where the B's are the Bernoullian numbers, the error being less than the first term omitted.[1] The proof of (1) is a delicate matter, however, and it seems therefore worth while to call attention to the fact that γ can be calculated as readily as π or $\log n$ to three or four places of decimals, if one makes use of an idea employed in Cauchy's integral test. This method, which is applicable to all convergent series of positive monotone decreasing terms, depends upon the fact that from Cauchy's test it follows immediately that both

$$\sum_1^n u_m + \int_n^\infty u_m dm \qquad \text{and} \qquad \sum_1^n u_m + \int_{n+1}^\infty u_m dm$$

approximate $\sum_1^\infty u_m$ with an error less than $\int_n^{n+1} u_m dm$, and hence less than u_n.[1]

Applying this to

$$\gamma = \sum_1^\infty \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right)\right],$$

we see that

$$\sum_1^n \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right)\right] + \int_n^\infty \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right)\right] dm$$

or

$$\sum_1^n \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right)\right] + \left[-1 - \log\left(\frac{n}{n+1}\right)^{n+1}\right]$$

approximates to Euler's constant with an error less than

$$\log\left(\frac{n+1}{n+2}\right)^{n+2}\left(\frac{n+1}{n}\right)^{n+1}$$

For $n = 5$, this gives $\gamma = .58249$ with an error less than .01487; for $n = 10$, $\gamma = .57948$ with an error less than .00429; for $n = 20$, $\gamma = .57779$ with an error less than .00115.

[1] See Wm. Shank's paper, "On the Calculation of the Numerical Value of Euler's Constant." *Proceedings of the Royal Society of London*, Vol. XV, p. 429.

[1] In these integrals the definition of the function u_m is extended so that it relates to the continuous variable m in such wise that u_m is a monotone decreasing function of the continuous variable m.

THE RETIREMENT OF SIR THOMAS MUIR.

By C. T. LORAM, Durban, South Africa.

The retirement of Sir Thomas Muir, Kt., C.M.G., M.A., LL.D., F.R.S., etc., from the office of superintendent general of education for the province of the Cape of Good Hope, South Africa, which took place on June 30, 1915, is an event of interest, not only to educationists, but to the mathematical world. Sir Thomas was born in Lanarkshire, Scotland, on August 25, 1844. He was educated at Wishaw Public School and Glasgow University. From 1868 to 1871 he was sub-warden of College Hall and mathematical tutor at St. Andrews University, Scotland, from which position he proceeded to the principalship of the mathematical and science department of the famous Glasgow high school. In 1892 he was appointed superintendent-general of education for Cape Colony. In this capacity Sir Thomas had control over a system of some 4,500 schools with an enrolment of more than 250,000 pupils. His tenure of office was coincident with a period of social change, political unrest, and war unparalleled in the history of South Africa. That he was able to keep his health and energy unimpaired through this period is due partly to a native physical and mental strength, but largely, in his own opinion, to the fact that he was able to forget the cares of office in his mathematical studies. In addition to the great work of building up a sound and lasting educational system in South Africa, Sir Thomas rendered important services to art and letters as vice-chancellor of the University of the Cape of Good Hope, in which office he had the honor of conferring the degree of LL.D. upon King George V (then Duke of Cornwall and York); as chairman of the board of trustees of the South African Public Library and of the South African Art Gallery; as trustee of the South African museum; and as president of the South African Fine Arts Association and of the Capetown Chamber Music Union.

As a student of mathematics Sir Thomas has for nearly half a century done continuous and valuable original research in mathematics, and is possibly the greatest living authority on the subject of determinants. For his work in this field he received the degree of LL.D. from Glasgow, in 1882 was elected a Fellow of the Royal Society of London, and twice won the Keith Medal of the Royal Society of Edinburgh. He is also a Fellow of the Royal Scottish Geographical Society, and a past President of the South African Association for the Advancement of Science.

The great esteem in which Sir Thomas Muir was held by the people of South Africa is reflected in the text of the presentation address which was made to him on the eve of his retirement, and signed by 104 inspectors, instructors, and members of his office staff.

"On your relinquishing the post of superintendent-general of education, the members of the staff of your office and of the body of inspectors and instructors desire to record their high esteem of your work as their official head, and of your personal worth.

"The value and extent of your original work as a mathematical investigator and historian are known throughout the scientific world and have gained you the highest distinction.

"Your wide range of scientific knowledge and your discriminating appreciation of literature, music, and art have borne rich fruit in the control which you have exercised over the South African Public Library, the South African Museum, the Art Gallery, the Chamber Music Union, and similar institutions in the city of Cape Town.

"We do not venture to estimate the value of your labors as the head of the system of public education in this province to the government under which you have served, to the body of school managers and teachers who have been guided by you, and to the whole community of this province and of South Africa generally. A true estimate of the value of that work will be possible only at a later date, when sufficient time has passed to show on what sound and lasting principles you have raised the educational fabric of the Cape Province.

"It has been your duty to shape our school system during a period marked by social change and economic growth, by political unrest, and by the calamity of war. How faithfully and successfully, uninfluenced by any purely political or sectional interests, you have discharged that duty—by the exercise of wide knowledge, unswerving impartiality, clear judgment, thoughtful prevision, unflinching determination and unabating personal energy—will certainly be more fully recognized hereafter. We whose privilege it has been to be associated with you in the great work as subordinates, have had the opportunity of watching the inception of your wise plans, of observing their growth, and of noting the far-sighted prudence, resolute persistence, and administrative tactfulness with which they have been accomplished.

"To each of us our association, in however humble a capacity, with you in your work has afforded a constant revelation of intellectual power and force of will, and has ever provided an incentive to thoughtful consideration, sound judgment, and earnest endeavor.

"To all of us your retirement from your life-work is a matter of deep regret, more especially when we consider that your physical and mental powers display all the energy of youth. And we regret it yet more when we realize that at the present juncture in the history of South Africa there is needed for the wise coördination of its various school systems such a strength of purpose, width of knowledge, and ripeness of experience as we find in you alone.

"It is a cause of gratification to us to know that the release from official life which has been accorded to you finds you still so well able to enjoy and use the leisure which you have so richly earned. That leisure we trust you may for many years be able to devote to the lasting benefit of many of the interests of your adopted land.

"In taking leave of you as our official head we desire to assure you of our very deep respect, of our lasting esteem, and of our most cordial wishes for your personal happiness."

his own if he knows that he has made any. If no answers are given, the student works a problem and does not know whether it is right or wrong until he comes to class. There are problems, like problems in finding the integral roots of equations or problems in factoring in elementary algebra, which are of such a nature that the answers, if given, supply the student with too much information as to the method of working the problem. But there are not many such in the calculus. Furthermore, if the answers are not given, and if the same textbook is used for several years, the number of second-hand books in use gets to be considerable and these have many of the answers marked in by students. Then the students in the class are not all on the same footing with respect to answers. Some of them have the answers and some do not. This use of second-hand books, with many answers marked in, is likely to interfere with the use of this book of problems in tests as is suggested by the author in the quotation given above.

The book is much more pretentious than the "Problems in Differential Calculus" published by W. E. Byerly in 1895. A good bibliography of problem books in the calculus, mostly in German, is to be found in the *Bulletin of the American Mathematical Society*, June, 1914.

W. H. Bussey.

University of Minnesota.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and R. P. Baker.

[Send all Communications to B. F. Finkel, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

452. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Find in the form of a continued fraction the positive root of the equation $x^3 - 2x - 5 = 0$.

453. Proposed by A. J. KEMPNER, University of Illinois.

Is the series whose terms are the reciprocals of all positive integers not containing the figure 0, convergent or divergent?

454. Proposed by C. N. SCHMALL, New York City.

Prove that a number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

GEOMETRY.

483. Proposed by LAENAS G. WELD, Pullman, Illinois.

A circle is inscribed in a triangle. In each of the three spandrils exterior to the circle another circle is inscribed; in the remaining spandrils three other circles; and so on ad infinitum. Show that the sum of the areas of these circles is given by the formula:

$$\Sigma = \frac{\pi}{4}\frac{\Delta^2}{s^2}\left[\frac{1}{\sin(A/2)} + \frac{1}{\sin(B/2)} + \frac{1}{\sin(C/2)} - 2 + \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right].$$

484. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that when spheres of uniform size are packed in the closest possible manner there is, in the interior of the mass, about 26 per cent. of voids.

485. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the surface generated by the orthogonal projection of a given line upon a variable plane turning about a fixed axis.

CALCULUS.

403. Proposed by C. N. SCHMALL, New York City.

A paraboloid of revolution generated by the curve $x^2 = 4ay$, contains a quantity of water such that if a sphere of radius r be dropped to the bottom, it will just be covered by the water. Show that if the volume of water used in this experiment is to be a *minimum*, then we must have $a = r/6$.

404. Proposed by B. J. BROWN, Victor, Colorado.

Solve the differential equation, $(x^2 - y^2)(1 + dy/dx) = 2xy(1 - dy/dx)$.

MECHANICS.

321. Proposed by E. J. MOULTON, Northwestern University.

The attraction, A, in any given direction, due to a homogeneous sphere, on a particle at the center of the sphere, using the Newtonian law, is obviously zero. Find the error in the following method of computing A. Take cylindrical coördinates with origin at the center of the sphere; let the z-axis extend in the direction of the attraction to be computed, and let r, θ be the polar coördinates used. Let δ be the density and R the radius of the sphere, and k the constant of gravitation. Then

$$A = \int_{z=-R}^{z=R} \int_{r=0}^{r=\sqrt{R^2-z^2}} \int_{\theta=0}^{\theta=2\pi} \frac{k\delta rz d\theta dr dz}{[r^2 + z]^{3/2}} \tag{1}$$

$$= 2\pi k\delta \int_{z=-R}^{z=R} \left[\frac{-z}{(r^2 + z^2)^{1/2}} \right]_{r=0}^{r=\sqrt{R^2-z^2}} dz \tag{2}$$

$$= 2\pi k\delta \int_{-R}^{R} \left[\frac{-z}{R} + 1 \right] dz \tag{3}$$

$$= 4\pi k\delta R. \tag{4}$$

322. Proposed by FRANK R. MORRIS, Glendale, Calif.

A pole of uniform size and weight throughout its length stands in a vertical position. The height of the pole is h and weight w. It hinges at the base and falls, passing through a horizontal position. At the moment it reaches the horizontal position, how far from the base is the maximum vertical force tending to break the pole? How great is this force? What is the horizontal force at the same position in the pole?

323. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Two equal bodies are placed on a rough inclined plane, being connected by a light inelastic string; if the coefficients of friction are respectively $1/3$ and $1/4$, show that they will both be on the point of motion when the inclination of the plane is $\sin^{-1}(7/25)$.

NUMBER THEORY.

239. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

Give a general method for determining the solution in integers of the equation

$$x^r - 10xy - (n + 1) + y = 0,$$

where r and n are positive integers.

240. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^3 - px + q = 0$ are rational, prove that $4p - 3x^2$ is a perfect square.

241. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $a^2 + b^2 = c^2$, where a, b, and c are integers, then prove that abc will be a multiple of 60.

In the next issue we shall reprint all unsolved problems in Number Theory published since January, 1913. They are numbers 191, 192, 196, 198, 201, 202, 205, 208, 209, 211, 214, 217, 219, 221, 222, 223. Please have these in mind. EDITORS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

433. Proposed by B. J. BROWN, Student at Drury College.

Prove that, if all the quantities, a, b, etc., are real, then all the roots of the equati ns

$$\begin{vmatrix} a-x & h \\ h & b-x \end{vmatrix} = 0, \qquad \begin{vmatrix} a-x & h & g \\ h & l-x & f \\ g & f & c-x \end{vmatrix} = 0$$

are real; and generalize the proposition.

SOLUTION BY WM. E. HEAL, Washington, D. C.

The first equation may be written

$$\left[x - \left(\frac{a+b}{2}\right)\right]^2 = \frac{(a-b)^2}{4} + h^2.$$

Since the second member is the sum of two squares and so can never become negative, if a, b, and h are real, it follows that both roots are real.

The second equation is proved, in Salmon's *Modern Higher Algebra*, 4th edition, page 28, to have its roots all real.

The general equation, of which the above are special cases, is shown on page 48 of the same work to have all its roots real.

Thus also referred to by A. M. HARDING.

442. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Show that the sum of n terms of the series $1/2 - 1/3 + 1/4 - 1/6 + 1/8 - 1/12 + \cdots$ is $1/3[1 - (1/2)^{n/2}]$ when n is even, and $1/3[1 + 2\sqrt{2}(1/2)^{(n/2)+1}]$ when n is odd.

SOLUTION BY IRBY C. NICHOLS, Chicago, Ill.

(1) *When n is even.* Grouping the terms successively by twos, we have a series of $n/2$ terms from which the factor $(\tfrac{1}{2} - \tfrac{1}{3})$ may be removed, thus,

$$(\tfrac{1}{2} - \tfrac{1}{3})\left[\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots \frac{1}{2^{n/2-1}}\right].$$

The series in brackets is geometrical, and we have the sum

$$(\tfrac{1}{2} - \tfrac{1}{3})\left[\frac{1 - \tfrac{1}{2}\cdot\frac{1}{2^{n/2-1}}}{\tfrac{1}{2}}\right] = \tfrac{1}{3}[1 - (\tfrac{1}{2})^{n/2}].$$

(2) *When n is odd.* Then the sum of $n+1$ terms can be written by (1), using $n+1$ for n. If now we add the $(n+1)$th term to this sum, we shall have the sum of n terms, since the $(n+1)$th term is negative. This gives

$$\begin{aligned} S_n &= \tfrac{1}{3}[1 - (\tfrac{1}{2})^{(n+1)/2}] + (\tfrac{1}{2})^{[(n+1)/2]-1} \\ &= \tfrac{1}{3}[1 - 2^{1/2}(\tfrac{1}{2})^{n/2+1}] + 2^{3/2}(\tfrac{1}{2})^{n/2+1} \\ &= \tfrac{1}{3}[1 + \sqrt{2}(\tfrac{1}{2})^{n/2+1}]. \end{aligned}$$

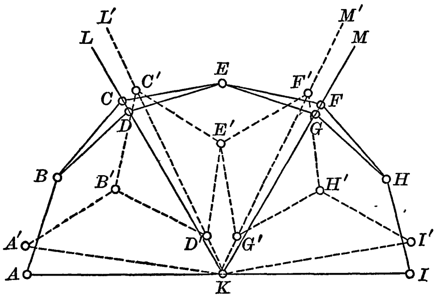
The last result was incorrectly stated by the Proposer.

Also solved by HERBERT N. CARLETON, J. A. CAPARO, F. L. GRIFFIN, FRANK R. MORRIS, G. W. HARTWELL, C. E. HORNE, H. C. FEEMSTER, HORACE OLSON, J. V. BALCH, H. L. AGARD, and the PROPOSER.

GEOMETRY.

427. Proposed by F. CAJORI, Colorado College.

In S. Gross's linkage for trisection of angles, shown in the figure (KL' and KM' being the trisectors of $A'KI'$), C is fixed on KL, also F on KM; at starting, C and D coincide, also F and G; D slides on KL, G slides simultaneously on KM; if E moves along a perpendicular to AI erected at K, find the loci of B and D.



SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Denote length of KC by b, $EC = CB = BA$ by a. Now $KA = b$, and, projecting the broken line $KCBA$ on KA,

$$b \cos \pi/3 + a \cos \pi/4 + a \cos 5\pi/12 = b.$$

Hence

$$b = 2a \cos \pi/12.$$

Denote length of KD' at any time by r, angle EKD' by θ, angle $E'C'K = E'D'C'$ by ψ. Then $r + 2a \cos \psi = b$. But, by the law of sines,

$$\frac{\sin(\theta + \psi)}{\sin \theta} = \frac{b}{a}.$$

Solving for $\cos \psi$,

$$\cos \psi = \frac{b}{a} \sin^2 \theta \pm \frac{\cos \theta}{a} \sqrt{a^2 - b^2 \sin^2 \theta}.$$

Hence,

$$r - b + 2b \sin^2 \theta = \mp 2 \cos \theta \sqrt{a^2 - b^2 \sin^2 \theta},$$

i. e.,

$$(r - b \cos 2\theta)^2 = 4 \cos^2 \theta (a^2 - b^2 \sin^2 \theta).$$

This is the polar (r, θ) equation of the locus of D' with respect to KE as initial line and K as pole.

Again, by the law of cosines, $KB'^2 = b^2 + a^2 - 2ab \cos \psi$. Denote length of KB' by ρ and $\angle EKB'$ by ϕ. The equation just written becomes

$$(\rho^2 - a^2 - b^2 \cos \phi)^2 = 2b^2(1 + \cos \phi) \left\{ a^2 - b^2 \frac{1 - \cos \phi}{2} \right\}.$$

This is the polar (ρ, ϕ) equation of the locus of B' where KE is initial line and K is pole.

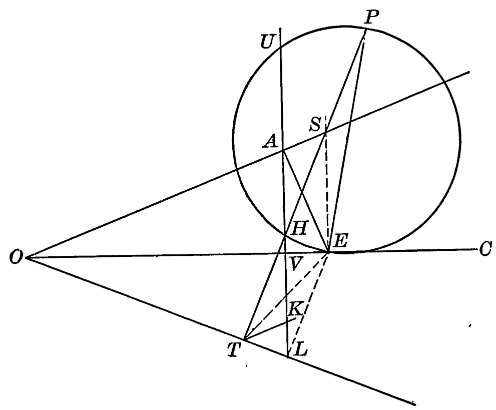
463. Proposed by NATHAN ALTSHILLER, University of Washington, Seattle.

Through a given point, to draw a line that cuts off on the sides of a given angle two segments the sum of which has a given value.

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let $\angle O$ be the given angle, P the given point, $OA = k$ half the required length.

Construction. Let OC bisect the given angle. Draw $AE \perp$ to OA and $AV \perp$ to OC, thus making $OL = OA$. On PE as a diameter describe a circle intersecting AL at U and H. Then PH and PU are the required lines.



Proof. Draw ES, ET and draw EL which will be $\perp$ to OL. The quadrilaterals $EHAS$ and $EHTL$ are inscriptible in circles since SAE and EHS; EHT and ELT are right angles.

Hence,

$$\angle SEA = \angle AHS = \angle THL = \angle TEL \quad \text{and} \quad \triangle ELT = \triangle EAS.$$

Then $ES = ET$, $HS = HT$ and $AS = TK = TL$, where TK is parallel to OA.

But, $OA + OL = 2k$ by construction. Hence,

$$OA + AS + OL - AS = 2k \quad \text{or} \quad OA + AS + OL - TL = 2k,$$

and finally,

$$OS + OT = 2k.$$

The line PU produced cuts from the given angle segments whose *difference* is $2k$. The proof is the same as the above.

An excellent solution was given by Professor F. L. GRIFFIN, of Reed College, the proof of which involves the calculus.

Using the same construction as in Professor Caparo's solution, he notes the fact that, after the point E is located, the rest is merely the well known construction of a tangent to a parabola from a given point. This parabola, which has OC as its axis, E as its focus, and AL as the tangent at its vertex, is the envelope of the whole family of lines cutting off on the sides of LO two segments whose sum is $2k$. Thus, the required line is merely that tangent to this parabola which passes through the given point P.

This point of view brings out the discussion that there is no solution, one solution, or two solutions, according as the given point P lies within, on, or without the parabola. If P lies outside the given angle, as in the figure above, then one of the lines, as PU, will cut one side of the angle produced through the vertex, but the sum will still be $2k$ if we call this segment negative. Professor Griffin also gives a solution involving analytic geometry but not the calculus. Others may be interested to work out both of these solutions. EDITORS.

Also solved by A. H. HOLMES, PAUL CAPRON, J. W. CLAWSON, and N. P. PANDYA.

468. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

SOLUTION BY NATHAN ALTSHILLER, University of Colorado.

Let r be the radical axis of the two given circles γ_1, γ_2, and P the point of intersection of r with the given line, l. The tangents drawn from any point of r, and in particular from P, to all the circles of the coaxial system determined by γ_1 and γ_2, are all of equal length, when measured from P to the respective points of contact. On the other hand, the centers of all the circles coaxial with the two given circles, lie on the line of centers c of these two circles.

The above suggests the following solution of the problem:

From the point P draw a tangent to one of the given circles. Let S denote the point of contact. On l lay off two segments PT and PT' such that $PT = PT' = PS$. The perpendiculars to l erected at T and T' will meet the line of centers c in the centers C and C' of the two circles satisfying the conditions of the problem, the respective radii being the segments CT and $C'T'$.

This construction is applicable whether the given circles are tangent to each other, or cut each other in real, or imaginary points.

The problem has, in general, two real solutions.

An interesting special case arises when the line l is parallel to the radical axis r, and hence perpendicular to the line of centers c. Since the center of the required circle is to be on c, the necessary and sufficient condition for it to be tangent to l is, in the present case, that the circle shall pass through the point of intersection of l with c. We are thus led to the problem:

Given two circles and a point, to draw a circle passing through the given point and coaxial with the two given circles.

This problem may be solved as follows: From an arbitrary point P of the radical axis r draw a tangent PR to one of the given circles, touching the circle at the point R. On the line PQ joining P to the given point Q find the point Q' such that $PQ \cdot PQ' = \overline{PR}^2$, Q and Q' being on the same side of the radical axis. The point Q' belongs to the required circle, which is now readily constructed.

This construction is valid for any point Q in the plane, and whatever the relative position of the two circles with respect to each other may be.

The problem has, in general, one real solution.

Also solved by C. N. SCHMALL, GEO. W. HARTWELL, FRANK IRWIN, HERBERT N. CARLETON, J. W. CLAWSON, and N. P. PANDYA.

469. Proposed by J. ALEXANDER CLARKE, West Philadelphia High School.

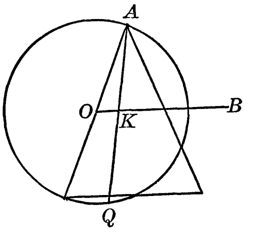
If in an isosceles triangle, a circle is described on one side as diameter, and a line is drawn through the mid-point of the side parallel to the base, the circle and the parallel will intercept on the trisector of the angle at the vertex a segment equal to the radius of the circle. Show how this can be used to trisect any angle.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let A be the vertex of the isosceles triangle. Bisect either arm at O. Draw the circle, center O, radius OA. Draw a line, OB, from O parallel to the base. Then if a line is supposed drawn to trisect the angle A, cutting the line OB at K and cutting the circle again at Q, we are to prove that $KQ = R$, the radius of the circle.

Now

$$AQ = 2R \cos\frac{A}{3}.$$



Also

$$AK = R\frac{\sin(90^\circ - A/2)}{\sin(90^\circ - A/2 + A/3)} = R\frac{\cos\dfrac{A}{2}}{\cos\dfrac{A}{6}}.$$

Hence

$$KQ = R\left[2\cos\frac{A}{3} - \cos\frac{A}{2}\sec\frac{A}{6}\right] = R\left[2\cos\frac{A}{3} - 4\cos^2\frac{A}{6} + 3\right]$$

$$= R\left[2\cos\frac{A}{3} - 2\left(1 + \cos\frac{A}{3}\right) + 3\right] = R.$$

Whence, to trisect an angle, construct an isosceles triangle having an arm of the angle for its side and the angle for its vertex. Draw a circle on the side of the isosceles triangle as diameter. Draw from the center of the circle a line parallel to the base of the triangle. Mark off on a straight edge a distance equal to the radius of the circle. Swing and slide the straight edge about until one end of the marked portion of the straight edge coincides with a point of the circle and the other with a point on the line while the straight edge passes through the vertex of the angle.

Also solved by S. A. JOFFE.

469A. Proposed by W. F. FLEMING, Chicago, Ill.

A pole whose length is l stands vertically against a vertical wall. A spider is at each end of the pole. The pole is drawn out from the wall in such a way that its upper end moves down the wall at a uniform rate. At the same time that the pole begins to move, the spiders begin to travel toward each other at rates equal to the rates at which the respective ends move. Determine the equations of the paths of the two spiders, in space.

SOLUTION BY H. S. UHLER, Yale University.

Let (x, y) and (x', y') denote, at any instant, the positions of the spiders which have started from the upper $(0, l)$ and lower $(0, 0)$ ends of the pole respectively. Since the spiders begin to recede from the ends of the pole at the same time that the pole starts to move, and as they crawl along the pole at the same rates as the associated ends of the pole slide along the coördinate axes, the distances which the spiders will have progressed along the pole will, at each instant, be equal to the distances which the corresponding ends of the pole have moved from their initial locations on the axes, quite regardless of whether the upper end of the pole moves uniformly or otherwise. In symbols

$$l - b = + [x^2 + (y - b)^2]^{1/2}, \tag{1}$$

$$a = + [(x' - a)^2 + y'^2]^{1/2}, \tag{1'}$$

where a and b denote the intercepts of the pole-line on the axes of x and y respectively.

Since the spiders are on the pole-line

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (2), \qquad \text{and} \qquad \frac{x'}{a} + \frac{y'}{b} = 1 \quad (2').$$

Since the length of the pole is l, $a^2 + b^2 = l^2$ (3).

The analysis is now reduced to the elimination of a and b from the two sets of equations (1), (2), (3) and (1′), (2′), (3).

Equation (1), when solved for b, gives

$$b = \frac{x^2 + y^2 - l^2}{2(y - l)}. \tag{4}$$

Substituting b from (4) in (2) we find

$$a = \frac{x(x^2 + y^2 - l^2)}{x^2 - (y - l)^2}. \tag{5}$$

Substitution of a and b from (5) and (4) in (3) leads to

$$(x^2 + y^2 - l^2)^2 = 4ly(y - l)^2$$

as the equation of the path of the spider which started at the upper end of the pole.

In like manner, we find

$$a = \frac{x'^2 + y'^2}{2x'}, \qquad b = \frac{y'(x'^2 + y'^2)}{y'^2 - x'^2},$$

and

$$(x'^2 + y'^2)^2 = 2lx'(y'^2 - x'^2),$$

which is the equation of the path of the second spider.

The spiders will pass at the point

$$x = \frac{l}{2}(\sqrt{2} - 1), \qquad y = \frac{l}{2},$$

which divides the pole into upper and lower segments equal to $(l/2)(2 - \sqrt{2})$ and $(l/2)\sqrt{2}$ respectively, the pole then forming an isosceles triangle with the axes.

Excellent solutions were received from FRANK IRWIN, J. A. CAPARO, H. C. FEEMSTER, and N. P. PANDYA.

CALCULUS.

384. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

In what time will a sum of money double itself at 6 per cent. interest per annum if compounded at indefinitely short intervals?

SOLUTION BY H. L. AGARD, Williams College.

If the interest is compounded k times a year, the amount after n years is given by the formula

$$A = P\left(1 + \frac{r}{k}\right)^{nk}.$$

When the interest is compounded at indefinitely short intervals,

$$A = \lim_{k=\infty} P\left(1 + \frac{r}{k}\right)^{nk} = \lim_{k=\infty}\left[P\left(1 + \frac{r}{k}\right)^{k/r}\right]^{nr} = Pe^{nr}. \tag{1}$$

In (1), setting $A = 2P$, $r = .06$ and solving for n, we have

$$n = \frac{\log_e 2}{.06} = \frac{.69315}{.06} = 11.5525 \text{ years.}$$

Also solved by H. C. FEEMSTER, W. W. BURTON, G. W. HARTWELL, C. E. FLANAGAN, J. W. CLAWSON, FRANK R. MORRIS, H. S. UHLER, ELIZABETH OVIN, F. FORDERO (Seville, Spain), and and HERBERT N. CARLETON.

386. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

C is a fixed point on the perpendicular bisector of a line segment AB. Locate a point D also on this bisector, such that $AD + BD + DC$ shall be a minimum.

SOLUTION BY H. C. FEEMSTER, York College, Nebraska.

Let the foot of the perpendicular be E, $CE = b$, $AE = EB = a$, and $DE = x$. Then $AD + BD + DC = 2\sqrt{a^2 + x^2} + b - x$, which is to be a minimum. Taking the derivative of this expression, setting it equal to 0, and solving for x, we have

$$x = \frac{a\sqrt{3}}{3} = DE, \text{ as required.}$$

MECHANICS.

301. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A wire is hanging from two points in the same horizontal plane. If the difference between the length of the wire and the actual distance between the supports is very small, show that

$$s = x\left(1 + \frac{x^2}{6c^2}\right),$$

where s is one half of the length of the wire, c is the tension at the lowest point divided by w the load per unit of horizontal distance, and x is the distance of the lowest point of the curve to the point of support.

SOLUTION BY A. M. HARDING, University of Arkansas.

A solution of this problem will be found in Jean's *Theoretical Mechanics*, pages 80–85. It is there shown that the length of the wire is given by

$$s = \frac{c}{2}\left(e^{x/c} - e^{-x/c}\right).$$

If we expand $e^{x/c}$ and $e^{-x/c}$ as power series in x and neglect all powers of x higher than the third we obtain the desired result.

Similarly solved by WALTER C. EELLS and R. M. MATHEWS.

302. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A ball is projected from a given point at a given inclination β towards a vertical wall; determine the velocity so that after striking the wall the ball may return to the point of projection.

SOLUTION BY PAUL CAPRON, Annapolis, Maryland.

Let v = the required velocity, a = the distance of the point of projection from the wall. After t seconds, the ball will have traversed x horizontally, y vertically, where

$$x = v\cos\beta\cdot t, \quad y = v\sin\beta\cdot t - \tfrac{1}{2}gt^2.$$

Its path will make the angle α with the horizontal, where $\tan\alpha = \tan\beta - (gt/v\cos\beta)$, and its component velocities will be $v\cos\beta$ horizontally, $v\sin\beta - gt$ vertically. Subscripting the values at the instant of impact, we have

$$x_1 = a, \quad t_1 = (a/v\cos\beta), \quad y_1 = a\tan\beta - (a^2g/2v^2)\sec^2\beta, \quad \tan\alpha_1 = \tan\beta - (ag/v^2)\sec^2\beta,$$

$$v_1^2 = v^2 - 2ag\tan\beta + (a^2g^2/v^2)\sec^2\beta = v^2 - ag\,(\tan\alpha_1 + \tan\beta).$$

The component velocities immediately after the impact will be

$$ev_1\cos\alpha_1 \text{ horizontally}, \quad ev_1(K\sin\alpha_1 - C\cos\alpha_1) \text{ vertically},$$

where, if e is the coefficient of restitution and μ is the coefficient of friction during the impact, $K = (5/7e)$ and $C = 0$ if $\mu > [2\tan\alpha_1/7(1+e)]$ or $K = 1/e$, $C = [\mu(1+e)/e]$ if $\mu < [2\tan\alpha_1/7(1+e)]$. [See Routh, *Elementary Rigid Dynamics*, Volume I, § 197. The ball is supposed homogeneous.]

Consequently, t seconds after the impact, the ball will have moved from the point of impact x horizontally and y vertically, where (supposing downward motion not to have begun) at the time of impact

$$x = ev_1\cos\alpha_1\cdot t, \quad y = ev_1(K\sin\alpha_1 - C\cos\alpha_1)t - \tfrac{1}{2}gt^2,$$

and it is required that when $x = a$, and therefore $t = (a/ev_1\cos\alpha_1)$,

$$y \text{ shall be } = -y_1 = -a\tan\beta + \frac{a^2g}{2v^2}\sec^2\beta,$$

i. e.,

$$a(K\tan\alpha_1 - C) - \frac{a^2g}{2e^2v_1^2}\sec^2\alpha + a\tan\beta - \frac{a^2g}{2v^2}\sec^2\beta = 0.$$

If we substitute in this equation the values of v_1^2 and $\tan\alpha_1$ given above, we shall have, after simplifying:

$$2e^2[(1+k)\tan\beta - C]v^6 - ag[(1+e^2+2Ke^2)\sec^2\beta + 4(1+K)e^2\tan^2\beta - 4e^2C\tan\beta]v^4$$
$$+ 2a^2g^2\sec^2\beta[(1+2e^2+3Ke^2)\tan\beta - e^2C]v^2 - a^3g^3(1+e^2+2Ke^2)\sec^4\beta = 0.$$

If we let

$$\frac{1+e^2+2Ke^2}{(1+K)\tan\beta - C} = 4e^2r \quad\text{and}\quad \frac{v^2}{ag} = x,$$

we shall have

$$x^3 - 2(r\sec^2\beta + \tan\beta)x^2 + \sec^2\beta(4r\tan\beta + 1)x - 2r\sec^4\beta \equiv$$
$$(x - 2r\sec^2\beta)(x^2 - 2\tan\beta x + \sec^2\beta) = 0.$$

The only real root is

$$x = \frac{v^2}{ag} = 2r \sec^2 \beta.$$

In case the ball at the time of impact has passed the highest point of its path ($v^2 \sin 2\beta < 2ag$), the problem is clearly impossible; this may be made to appear by putting $(-C)$ in place of (C) in the foregoing discussion.

There are two cases when $v^2 \sin 2\beta > 2ag$:

I. If

$$\frac{7}{2}\mu(1+e) > \tan \alpha_1, \quad v^2 = ag \csc 2\beta \left[\frac{7 + 10e + 7e^2}{e(7e+5)}\right].$$

II. If

$$\frac{7}{2}\mu(1+e) < \tan \alpha_1, \quad v^2 = ag \sec^2 \beta \frac{1+e}{2e(\tan\beta - \mu)}.$$

To these may be added:

III. If $\mu = 0$,

$$v^2 = ag \frac{1+e}{e} \operatorname{cosec} 2\beta.$$

By means of the relation

$$\tan \alpha_1 = \tan \beta - \frac{ag}{v^2} \sec^2 \beta = \tan \beta \left(1 - \frac{2ag}{v^2 \sin 2\beta}\right),$$

combined with the values of v^2 in the two cases, we find that Case I (rolling impact) or Case II (rolling and sliding at impact) occurs, according as $\mu \cot \beta$ is greater than or less than $[2(1-e)/7 + 10e + 7e^2]$.

If the ball is not homogeneous, the criterion for Case I and Case II is

$$\mu > \text{or} < \frac{k^2}{a^2 + k^2} \cdot \frac{\tan \alpha_1}{1+e},$$

and K becomes $a^2/e(a^2 + k^2)$, where k is the radius of gyration for a diameter. The discussion is otherwise unchanged, so that in this more general case

$$v^2 = ag \operatorname{cosec} 2\beta \cdot \frac{2a^2e + (a^2 + k^2)(1 + e^2)}{a^2e + (a^2 + k^2)e^2}$$

or

$$v^2 = ag \sec^2 \beta \cdot \frac{1+e}{2e(\tan\beta - \mu)},$$

according as $\mu \cot \beta$ is greater or less than

$$\frac{k^2(1-e)}{2a^2e + (a^2 + k^2)(1 + e^2)}.$$

Also solved by J. A. Caparo, A. M. Harding, and Joseph B. Reynolds.

Note.—No solution of 300 has been received. H. S. Uhler should have received credit for solving 297 and 298. Editors.

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas.

NEW QUESTIONS.

30. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

Note.—In transmitting this question the proposer writes, "The mathematical courses of our colleges seem to be designed chiefly for two classes of students, those expecting to pursue the

subject either in the pure or the applied field. It is conceivable that these courses are not the best for high-school teachers of mathematics, especially for those who do not pursue their studies in course beyond the college. It is quite likely that there should be more survey courses even at the expense of intensive work over a narrow field. It is quite certain that the emphasis should be changed, giving more geometrical, historical and pedagogical courses. Hence I offer the above complement to Question No. 27."

Replies to Question No. 27 were published in the December and January issues of the MONTHLY.

31. In the light of questions 27 and 30, the Editors wish to propose that a symposium be called for on the question: What are the actual courses now offered in colleges and universities in this country for the preparation of teachers of mathematics (1) for secondary schools, (2) for colleges? The discussion may well lead to the consideration also of what courses *should be offered* for the preparation of teachers of mathematics (1) for secondary schools, and (2) for colleges.

DISCUSSIONS.

I. RELATING TO A SIMPLE PROOF BY INDUCTION OF AN INTERESTING NUMBER RELATION.

BY CHARLES R. DINES, Dartmouth College.

THEOREM: *For any set* $(a_i \mid i = 1, 2, \cdots, n)$ of n *distinct numbers, real or imaginary* $(n > 1)$, *we have*

$$\text{(I)} \qquad \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{a_i - a_j} = 0.$$

Proof: (A) Relation (I) holds for the smallest admissible value of n, viz., $n = 2$.

(B) Assume that relation (I) holds when $n = r$; that is, for any set of r distinct integers,

$$(1) \qquad \sum_{i=1}^{r} \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{a_i - a_j} = 0.$$

Let

$$(2) \qquad \sum_{i=1}^{r+1} \prod_{\substack{j=1 \\ j \neq i}}^{r+1} \frac{1}{a_i - a_j} = N.$$

We wish to prove that N is zero. Multiply both sides of (1) by $1/(a_{r+1} - a_1)$ and add to (2). Then

$$(3) \qquad N = \sum_{i=1}^{r+1} \prod_{\substack{j=1 \\ j \neq i}}^{r+1} \frac{1}{a_i - a_j} + \frac{1}{a_{r+1} - a_1} \sum_{i=1}^{r} \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{a_i - a_j}.$$

But

$$\prod_{j=2}^{r+1} \frac{1}{a_1 - a_j} + \frac{1}{a_{r+1} - a_1} \prod_{j=2}^{r} \frac{1}{a_1 - a_j} = 0$$

and when $1 < i < r + 1$

$$\prod_{\substack{j=1 \\ j \neq i}}^{r+1} \frac{1}{a_i - a_j} + \frac{1}{a_{r+1} - a_1} \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{a_i - a_j} = \frac{a_{r+1} - a_1 + a_i - a_{r+1}}{(a_{r+1} - a_1)(a_i - a_{r+1})} \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{a_i - a_j},$$

which finally reduces to

$$\frac{1}{a_{r+1}-a_1}\prod_{\substack{j=2\\j\neq i}}^{r+1}\frac{1}{a_i-a_j}.$$

Thus (3) may be written

$$\begin{aligned} N &= \prod_{j=1}^{r}\frac{1}{a_{r+1}-a_j}+\frac{1}{a_{r+1}-a_1}\sum_{i=2}^{r}\prod_{\substack{j=2\\j\neq i}}^{r+1}\frac{1}{a_i-a_j} \\ &= \frac{1}{a_{r+1}-a_1}\left[\prod_{j=2}^{r}\frac{1}{a_{r+1}-a_j}+\sum_{i=2}^{r}\prod_{\substack{j=2\\j\neq i}}^{r+1}\frac{1}{a_i-a_j}\right] \\ &= \frac{1}{a_{r+1}-a_1}\sum_{i=2}^{r+1}\prod_{\substack{j=2\\j\neq i}}^{r+1}\frac{1}{a_i-a_j}. \end{aligned}$$

Then using relation (1) in form applicable to the set $(a_i \mid i = 2, 3, \cdots, r+1)$ we see that N is zero and, by induction, relation (I) holds as stated.

This proof of a simple relation between numbers is of interest in an elementary way in algebra. While the theorem may be of no particular value in applications, its proof by induction brings in certain elements which are not often found in a proof by that method. The ordinary assumption is used twice in the proof, whereas, in most, if not all, other induction proofs it is used only once.

A simple proof of the theorem may be given if we presuppose the complex variable theory. For, if we have a set of distinct numbers $(a_1, a_2, \cdots, a_n)$, $n > 1$, the quantity

$$\prod_{\substack{j=1\\j\neq i}}^{n}\frac{1}{a_i-a_j}$$

is, for every i, just the residue at the point a_i of

$$\prod_{j=1}^{n}\frac{1}{z-a_j}$$

and the sum of these residues is zero, so that the theorem is immediate.

II. RELATING TO THE FOLIUM OF DESCARTES.

By Maurice C. Baudin, Student at The University of Chicago.

The current methods of tracing the cubic

$$x^3 + y^3 - \mu xy = 0$$

are very complicated; so complicated indeed that their authors have found it too laborious to apply them thoroughly and draw a correct figure.[1] Dr. G. Teixeira in his *Obras sobre Mathematica* proposes a process by which, with a sensible amount

[1] In Dowling and Turneaure's *Analytic Geometry* the curve seems to have besides the double point at the origin, a cusp at P.

of algebraic work, we can obtain one point on the curve; and the same analysis must be repeated for every point. Other examples found in textbooks are of the same character. To remedy this I offer the following somewhat simpler and shorter method for plotting the Folium.

Let there be any right angle with its vertex at the origin of the cubic. Its sides

$$y - \lambda x = 0, \quad \lambda y + x = 0$$

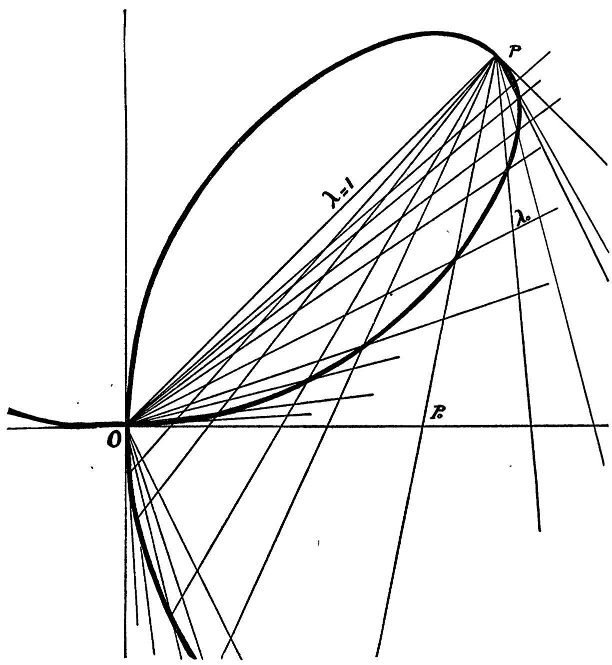
intersect the curve in two distinct points,

$$P_1\left(\frac{\mu\lambda}{1+\lambda^3}, \frac{\mu\lambda^2}{1+\lambda^3}\right), \quad P_2\left(\frac{\mu\lambda^2}{1-\lambda^3}, \frac{\mu\lambda}{\lambda^3-1}\right),$$

which determine the line

(1) $$f(x, y, \lambda) = (\lambda^2 + \lambda - 1)y - (\lambda^2 - \lambda - 1)x - \mu\lambda = 0,$$

where λ is a constant. If this right angle rotates about O, then $f(x, y, \lambda) = 0$, where λ now is the independent variable, is the equation of the family of straight lines P_1P_2 corresponding to the different values of λ. Let us determine the



envelope of these lines. Setting

(2) $$\frac{\partial f}{\partial \lambda} = 0$$

and eliminating λ from (1) and (2), we get

$$5x^2 + 5y^2 - 6xy - 2\mu x - 2\mu y + \mu^2 = 0,$$

which is the equation of a null ellipse centered at P $(\mu/2, \mu/2)$. Indeed this can be seen from equation (1), which may be written

$$y - x + \rho(2x - \mu) = 0,$$

where

$$\rho = \frac{\lambda}{\lambda^2 + \lambda - 1},$$

and is evidently satisfied for

$$x = y = \frac{\mu}{2}.$$

Therefore, $f(x, y, \lambda) = 0$ represents a pencil of lines through P. We can determine completely any one of these lines, $f(x, y, \lambda_0) = 0$, by its intercept

$$x = \frac{\mu\lambda_0}{1 + \lambda_0 - \lambda_0^2}.$$

Hence, we draw the right angle $(y - \lambda_0 x)(\lambda_0 y + x) = 0$, and plot

$$P_0 \left(\frac{\mu\lambda_0}{1 + \lambda_0 - \lambda_0^2}, 0\right),$$

and through P and P_0 we draw the line $f(x, y, \lambda_0) = 0$. The intersections of this line with $(y - \lambda_0 x)(\lambda_0 y + x) = 0$ are two points on the curve. So are the intersections of every line $f(x, y, \lambda_k) = 0$ through P_k $[\mu\lambda_k/(1 + \lambda_k - \lambda_k^2), 0]$ and P with $(y - \lambda_k x)(\lambda_k y + x) = 0$. P itself evidently is a point on the curve. Since the equation of the cubic is symmetrical in x and y the curve has the bisector $y - x = 0$ of the coördinate axes for its axis of symmetry; and we may confine ourselves to values of λ between 0 and 1.

A CORRECTION.

A correspondent has kindly pointed out that my recent Simple Proof of Hart's Theorem which appeared in the February number of the MONTHLY was wrongly named because though it was simple and had to do with Hart's theorem, it was not a proof. The correspondence of great circle to pole is, of course, one to two. I naïvely assumed that the pairs of circles corresponding to given circles would fall into two distinct groups and that one could confine one's attention to one of them. Unfortunately, in many cases, the two are hopelessly twisted together. If any other reader of the MONTHLY beside my correspondent has read the article, I hereby offer him my apology.

J. L. COOLIDGE.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

PROGRESS OF THE ASSOCIATION.

The Charter Membership. As these words are going into type, the membership list is growing at an average rate of eighteen per day. If this rate is maintained until April first, the charter membership will have reached 850. Unfortunately this March issue cannot be mailed until after April first (due to conditions beyond our control, which conditions also delayed the January and February issues and cannot be entirely overcome before June). Nevertheless, the printers' forms will be closed about March 25, and hence it will be impossible to announce the final results. Suffice it to say, the figures are already phenomenal and indicate that the interest in the new Association is neither sporadic nor half-hearted.

No adequate analysis of the membership has been compiled as yet, but a hasty glance shows: (1) that every state in the Union, the District of Columbia, Canada, England, and China are represented; (2) that New York is leading with 89 members, Illinois is second with 74, Ohio third with 69, and Massachusetts fourth with 57; (3) that not only those who signed the "Call" and attended the Columbus meeting, but large numbers of others have cast their influence with the new organization; and (4) that the membership is thoroughly representative, including many who are distinguished for research as well as those who are distinguished as teachers in both universities and colleges, including also many in normal schools and the larger high schools, and finally, engineers, actuaries and others not in the teaching profession.

An address list of the charter members, that is, those who joined before April first, will be prepared as soon as practicable and distributed to all members of the Association.

Institutional Membership. The growth of the institutional membership list has been as rapid as could have been expected. The idea is new and it requires time for the advantages to be realized and the arrangements to be made. Moreover, as there is no initiation fee in the case of institutions, there was not the same incentive, as in the case of individuals, to come in before a certain date. It is most gratifying, therefore, to note that some forty institutions have already become members and that the number is gradually increasing.

Lest some confusion may still exist with respect to whether an institutional membership includes its mathematical staff as individual members, the following paragraph on this point is reprinted from the February issue (also because a misprint as it there appeared may have added to the confusion).

A correspondent asks whether a delegate sent by an institutional member to a meeting of the Association would be counted as an individual member. This is clearly not contemplated by the constitution. *Individual members are those who support the Association by the payment of the individual dues,* and only the names of such persons will appear in the address list of individual members, to be printed later. There will also be printed a list of institutional members, but this

list will not be accompanied by the names of delegates since these will vary from time to time. The delegates actually in attendance at any meeting will be given with the report of the meeting. An individual member may, of course, be delegated by an institution to represent it at any meeting. The two copies of the Monthly to be sent to institutional members will, in all cases, be mailed directly to the library of the institution, and not to individual members of the faculty.

Sections of the Association. Evidence of activity in many directions indicates that the opportunities offered through the sections of the Association will be appreciated and that, in time, these will be widely developed. Therefore, the committee of the Council having this matter in charge has lost no time in formulating the regulations under which such sections may be organized. These have been agreed upon as follows:

REGULATIONS FOR SECTIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

1. Sections of this Association are governed by the provisions of the constitution of the Association, articles V, 1 and V, 2, and also by the following administrative rules, adopted by a committee of the Council and confirmed by the Council.

2. The membership of any section shall be limited to members of the national organization, and members of the national organization residing within the district covered by the section shall be *eo ipso* members of the section. This shall not operate, however, to prevent other persons from attending meetings of such sections, with the approval of the section.

3. A report of all meetings held during any year shall be sent to the secretary of the national association by the secretary of each section. The failure to present a report for two consecutive years shall be equivalent to notice that the section has been discontinued, and it shall then be omitted from the list of sections after due notice.

4. It is presumed that each section will have at least one meeting during each calendar year. Failure to have a meeting during each of two consecutive calendar years shall be held to be equivalent to notice of discontinuance of the section with the action provided for in the preceding paragraph.

5. The Constitution provides that expenses of sections shall not be payable by the parent organization. This is held to mean that the payment of any expense by the parent organization is optional with it. For the current year and until further action is taken by the Council, the national association will publish gratis the program of each meeting of any section and will furnish a sufficient number of copies for use before and during the meeting. All other expenses shall be borne by the section.

6. Any section may request the payment of a small supplementary amount to cover its expenses from the members of the Association in its district, but the non-payment of such an assessment shall not operate to change the standing of a member in the national association.

The Committee of the Council having this matter in charge, consisting of K. D. SWARTZEL, ALEXANDER ZIWET, and E. R. HEDRICK, Chairman, was appointed at the meeting of the Council in Columbus and was authorized to act with power in framing regulations for sections of the Association and in acting on petitions for the creation of such sections. See the January issue of the MONTHLY, page 6.

The first body to make application for admission as a section was in Kansas. A meeting was held in the autumn of 1915 at which the Kansas teachers of collegiate mathematics organized and appointed Professor U. G. Mitchell, of the University of Kansas, as their delegate to present their application at the Columbus meeting as soon as the national Association should give them an opportunity. They held their first meeting as a Section of the Mathematical Association of America at the University of Kansas on March 18, 1916. The program included (1) a report of the Columbus meeting by Professor U. G. Mitchell, (2) a paper on "Geometry for College Juniors and Seniors" by Professor J. W. Van der Vries of the University of Kansas, and (3) a discussion led by Professor Mary W. Newson of Washburn College.

It is not too much to hope that this is the beginning of a long series of section meetings to be held in every part of the country and that in this way the whole membership of the Association may come into direct contact with its activities.

Correspondence. Out of the mass of correspondence, an occasional letter is too good to withhold when we can secure permission to make it public. The following are of this character:

TO SECRETARY CAIRNS:

Please find enclosed dues and application for membership in what promises to be the "livest" mathematical association on the continent.

I believe some law of action at a distance will bring the benefits to us who are at aphelion and I hope that when the association is in proper working order a way will be found of accelerating our progress towards some center of energy and enthusiasm.

Yours sincerely,

NORMAN ANNING.

CHILLIWACK,
BRITISH COLUMBIA.

TO PRESIDENT HEDRICK:

My attention has recently been called to the fact that a new mathematical society has been formed under the title "The Mathematical Association of America," and I write to congratulate you and your colleagues on your successful inauguration of this new manifestation of interest in and capacity for mathematical work in America. When one looks back, as I am able to, a full half century over the development of mathematical studies in America, and when one reflects on the admirable progress of the past quarter of a century, he is stimulated to entertain hopes, if not glorious visions, for the future.

Although I am poor from membership in numerous societies and associations, it looks as if it will be necessary to succumb to the "one more" suggested by a circular just received; but whether it will turn out to be practicable or not, I desire to throw up my hat in admiration for the fine progressive spirit you and your colleagues in the Middle West are showing in these matters. In the future, therefore, when I am spending the summer on Sirius, or perhaps some hotter star, I shall hope to look in on such activities and help if practicable to encourage the workers who may be expected during the next hundred years to achieve still greater progress than has been attained during the past hundred years.

In the meantime, I have sent on my membership dues and subscription and am prepared to give you the most cordial support. There is great need for such an intermediate journal as you and your colleagues have evolved. This must help greatly to promote and diffuse interest and activity in mathematical science.

With all best wishes for the new organization, for the American Mathematical Monthly, and with personal regards,

Faithfully yours,

R. S. Woodward.

Carnegie Institution
of Washington, D. C.

To President Hedrick:

Two independent but converging tendencies in high school education have created a situation in which it seems possible for the new Mathematical Association of America to render a considerable service both to the general public and to the welfare of school mathematics. On the one hand there is a widespread movement for the adoption of the six-year high school program; on the other hand, there is a more or less prevalent sentiment that geometry, and, still more, algebra, are not planned and taught as they should be for the majority of high school boys and girls—who are of course not to enter college. More or less typical of such sentiment in its less rational forms are such drastic expressions—scarcely to be termed criticism—as, for example, the invocation of blessings on the girl who rejects algebra. Typical in a different sense are the confident dicta of the more psychologically minded, who assail cherished notions as to "mental discipline" by assuring the mathematicians that they are entangled in a "faulty and outworn psychology." Confronted with this bold accusation from one in authority, the simple-hearted teacher may easily plead guilty, or at least *nolo contendere*.

In the profession itself there are those who believe in "fusion," or in "composite" courses, those who feel that time is misspent in the seventh and eighth grades on arithmetical technicalities, those who lament the various limitations of our high school mathematics in comparison with that of secondary schools in other progressive nations.

For all of these reasons it seems to the writer opportune, not to say urgent, that the mathematical curriculum of the high school and of the seventh and eighth grades should be promptly and thoroughly revised—by mathematicians.

Other agencies are working on other subjects and if mathematicians do not undertake this task there is danger that it will be rashly attempted by the unskilled. Committees of the New England and Middle States Associations of mathematical teachers have been working on related problems for several years. For best results it is important that a national joint committee should be organized representing these and other bodies, unified by the support of the MATHEMATICAL ASSOCIATION OF AMERICA. Will the Association do its part?

Yours very sincerely,

H. W. TYLER.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY, March 15, 1916.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University.

Assistant Professor J. M. DAVIS has been promoted to an associate professorship of mathematics at the University of Kentucky.

MISS A. D. LEWIS, of Mt. Holyoke College, has been appointed professor of mathematics at the Kentucky College for Women.

At Rockford College, Ill., DR. BESSIE I. MILLER has been appointed professor of mathematics and physics.

DR. H. L. AGARD has been promoted to an assistant professorship of mathematics at Williams College.

At the State College of Washington, DR. E. C. COLPITTS has been promoted from an assistant professorship to an associate professorship of mathematics.

Longmans, Green and Company announce "A text-book on practical mathematics," by H. LESLIE MANN.

The publishing house of B. G. Teubner in Leipzig has in press "Partial differential equations of mathematical physics," by A. G. WEBSTER, of Clark University.

In the University of Washington Publications in mathematical and physical sciences for August, 1915, appears a paper by E. T. BELL, on "An arithmetical theory of certain numerical functions."

The Macmillan Company has recently issued a new edition of an "Elementary synthetic geometry of the point, line and circle," and an "Elementary synthetic solid geometry," by N. F. DUPUIS.

At the meeting of the principals and teachers of secondary schools in affiliation with the University of Cincinnati, February 19, 1916, Professor L. C. KARPINSKI, of the University of Michigan, presented an illustrated lecture on the "Story of algebra," which was recently given at the organization meeting of The Mathematical Association of America.

The Euclidean Circle is the name of the mathematical club at Indiana University. The Circle has existed for about twelve years, and is composed of students whose majors are in mathematics, juniors, seniors, postgraduates, and members of the mathematical faculty. The Club meets bi-weekly and the programs include reports by the members upon interesting features of the history and development of mathematics. The membership during the present year is about forty.

John Wiley & Sons recently published a volume entitled "Theory and applications of finite groups," consisting of three parts. Part I, written by Professor G. A. MILLER, consists of 192 pages and is entitled "Substitution and abstract groups"; Part II, written by Professor H. F. BLICHFELDT, consists of 86 pages and is entitled "Finite groups of linear homogeneous transformations"; Part III, written by Professor L. E. DICKSON, consists of 103 pages and is entitled "Applications of finite groups." The work is dedicated to CAMILLE JORDAN, and is the first treatise on group theory written by American mathematicians.

On November 7, 1915, in connection with the celebration of the one hundredth anniversary of his birth, a memorial tablet to KARL WEIERSTRASS (born October 31, 1815) was unveiled at his birthplace, Osterfelde, near Warendorf in Westphalia. On the tablet are these words: "An dieser Stätte wurde am 31·X·1815 Karl Weierstrass, der grosse Mathematiker, eine Leuchte der Berliner Universität, geboren." Weierstrass lived to be 81 years old and one of his most famous pupils was Sophie Kowalewski (1850–1891).

The annual meeting of the Pittsburgh Section of the Association of the Teachers of Mathematics in the Middle States and Maryland was held at the Carnegie Institute, Pittsburgh, January 29, under the presidency of Professor CLYDE S. ATCHISON, of Washington and Jefferson College. Papers were read as follows: "Recent advances in the teaching of mathematics," by R. H. HENDERSON, of the Woolslair High School; and "Entrance requirements in mathematics to a technical school," by Professor S. S. KELLER, of the Carnegie Technical Schools.

The American Mathematical Society will hold its sixth regular meeting at Chicago on Friday and Saturday, April 21–22, 1916, the first session opening at ten o'clock, A. M., in Ryerson Physical Laboratory of the University of Chicago. This will be the thirty-seventh regular meeting of the Chicago Section of the Society. The next regular meeting of the Society in New York will be held on Saturday, April 29th, at Columbia University.

In *Science* of January 21, 1916, is a summary of the report of the committee on academic freedom and academic tenure of the American Association of University Professors, presented at the annual meeting on January 1. The report, among other important recommendations, suggests the adoption by universities of four measures: (1) Action by faculty committees on reappointments; (2) definition of tenure of office; (3) formulation of grounds for dismissal; (4) judicial hearings before dismissal.

In further elucidation of the platform of the Association, attention is called to the article by President Hedrick in School and Society for March 11, 1916, in which he sets forth in greater detail for the general public the underlying causes which led to the formation of the Mathematical Association of America, and shows most emphatically that there can be no rivalry between this Association and the American Mathematical Society, but that both are needed and each has a distinct field in which to make its contribution to the advancement of mathematics in America.

An interesting article on "University Registration and Statistics" appears in *Science* of January 21. The registrations from thirty of the larger universities, including the large endowed universities and most of the state universities of the middle west, are compiled. These tables show a total registration in September, 1915, of 100,514 students, or approximately one student from each thousand of population in the United States. This student body is governed and instructed by more than 12,000 officers and instructors, or about one officer or instructor to every eight students. During the summer sessions of 1915, the thirty institutions report registrations of 35,652 students. For the year 1915–16, the following are the eight universities with largest registrations: Columbia (7,042); Pennsylvania (6,655); California (5,977); New York University (5,853); Michigan (5,821); Illinois (5,511); Harvard (5,435); Cornell (5,392).

An important item of news which will be appreciated by very many institutions is concerning successful mathematical clubs in colleges. It is well known that research clubs flourish in the large universities, but it may be thought impracticable to find a satisfactory basis for clubs in colleges, and especially in the smaller colleges. The fact is, however, that mathematical clubs do flourish not only in colleges but also in high schools, and that so-called "Junior Clubs," not primarily for research, are very successful in many universities, along side of the strictly research clubs. It will, therefore, be a great stimulus to all concerned to know how some of these successful clubs are organized and what is the character of their meetings. It is desirable to have reports from various kinds of institutions in many parts of the country, for instance, from colleges for men, colleges for women, coeducational institutions, and so on. Please send contributions of this character, as well as all other news items of general interest, to Professor D. A. Rothrock, Indiana University, Bloomington, Indiana.

The war has caused a change in the form of the *Educational Times* which has been published in London as a monthly since 1848. In the future it is to appear as a quarterly, but without the mathematical columns so long a feature of the monthly. Up to July, 1849, only occasional mathematical questions, solutions and papers were published. But with the August issue of that year, a beginning was made in numbering the questions. These have rapidly accumulated through the years until, with those in the issue for December, 1915, they now total 18,139. After about 1,400 questions and many solutions had been published, it was decided that some of the mathematical material published in the future should be reprinted. Thus Volume 1 of the "Reprint" appeared in July, 1864, and contained material taken from the *Times* for the year July, 1863–June, 1864. Latterly these "Reprints" have been issued semi-annually instead of annually. They total 103 volumes, 75 in the first series and 28 in the second.

It is now proposed to start another series entitled: *Mathematical Questions and Solutions*. This is to be issued as a monthly of 20 pages, 4 of questions and 16 of solutions, about twice as much as in the former average monthly issue of the *Times*. The first number of this new series was published on January 7, 1916.

The *Questions* are available at the subscription price of five shillings the half-year, or ten shillings the year. Subscribers will receive their copies post free, and, if desired, at the end of each half-year, a case for binding the same in a volume. Subscribers who do not desire the numbers monthly can receive the bound volumes at the end of each half-year at the same price. To non-subscribers the price of separate numbers will be one shilling net, and the complete volume six shillings, six pence.

Because of the interesting results stated in many of the questions proposed, it is manifestly more desirable to subscribe to the monthly issues. For this reason, too, a file of the *Educational Times* is valuable in a library for the mathematician, because of the hundreds of questions it contains, even since 1863, which have not appeared in the "Reprints"—as solutions were not forthcoming.

A portrait and biographical sketch of Mr. W. J. C. Miller, editor of the first 66 volumes of the "Reprints," appeared in this MONTHLY, volume 3, 1896, pages 159–163.

IMPORTANT NOTICE TO PRESENT MONTHLY SUBSCRIBERS.

Henceforth, the subscription price of the MONTHLY will be *three dollars net to all non-members of the* ASSOCIATION. The following adjustments for prospective members are proposed:

(1) Those who have already paid their subscriptions for the entire year 1916 are asked to send *one dollar* additional, which will entitle them to membership in the ASSOCIATION.

(2) Those who have not paid for 1916 are asked to send *three dollars,* which will entitle them to membership and include the MONTHLY. *No further subscriptions for 1916 will be received at the old rate of two dollars.*

In the case of subscriptions under (1) or (2) which expire *before* the end of 1916, please add *thirty cents extra* for each copy needed to complete the year. *Hereafter all subscriptions will date from January of each year.*

(3) An institution in which the Calculus is taught may become an *institutional member* of the ASSOCIATION by the payment of *five dollars* annually, which will entitle the library to receive *two copies of the* MONTHLY and the institution to send a voting delegate to all meetings of the ASSOCIATION. Institutions in which the Calculus is taught, whose libraries have already renewed their subscriptions for 1916, are asked to send *three dollars additional* and thus become institutional members of the ASSOCIATION.

Other institutions, and those not wishing to become institutional members, whose library subscriptions have already been renewed for 1916, are asked to send *one dollar additional* to complete the advanced price of the MONTHLY. No further subscriptions will be received at the old rate of two dollars, *and no discount from the advanced rate of three dollars will be allowed on subscriptions made through agencies.*

(4) The obligations of the MONTHLY for 1916 will, of course, be fulfilled on the former basis in the case of any individual or institution whose subscription has already been paid, and who may decline to make the adjustment on the new basis.

(5) Please note that all subscriptions to the MONTHLY and dues in the ASSOCIATION are to be paid to the SECRETARY-TREASURER, Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

If you have not already returned the membership blank, please do so at once. Delay may make it impossible to secure the back issues of the MONTHLY.

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.
2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.
2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.
2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.
3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.
4. The Council shall have authority to fill vacancies *ad interim*.

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.
2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.
2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.
2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

ARTICLE VII—DUES.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

The initiation fee shall be waived in case of those who join the Association before April 1, 1916, and this clause shall be dropped after its provisions have been fulfilled.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

ARTICLE VIII—AMENDMENTS.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list or charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the othef being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

Twelve members of the Council shall be elected at the first meeting of the Association, and the secretary shall draw lots to determine which four of those elected shall serve for one, for two, and for three years respectively. (This clause shall be dropped after its provisions have been fulfilled.)

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution

This shall not be construed to affect existing contracts with any subscribers or news agencies for the year 1916, who may decline to readjust on the new basis. (This clause shall be dropped after its provisions have been fulfilled.)

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

The American Mathematical Monthly

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VOLUME XXIII APRIL, 1916 NUMBER 4

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THE UNIFICATION OF FRESHMAN MATHEMATICS.

By JOSEF A. NYBERG.[1]

In a recent article,[2] Professor F. L. Griffin has presented a method for introducing calculus into the freshman year. A course as there outlined would, perhaps, be successful only in a college which is able to command an exceptionally good student body, and might possibly be a failure in a state university where the preparation of the entering classes is decidedly varied and poorer. In these schools, which must handle students of widely different preparations and abilities, it is not of so much importance to teach the calculus to the extent outlined by Mr. Griffin as to correlate the freshman work and to introduce enough differential calculus to be of service in the study of physics during the first months of the sophomore year. In this article I wish to outline a possible method for such schools. In the discussion following the outline, I consider its essential elements.

Part I is the usual introduction explaining such terms as scales, graphs, functions, ...

Part II is a study of the line and the elementary elliptic, parabolic, and hyperbolic curves, together with theorems on the symmetry and the translation of curves so that the general equation of the second degree without the xy term can be discussed.

Part III, divided into two sections, is first a study of trigonometry under five heads:

1. Definitions and fundamental relations between the six functions;
2. The radian measure of angles and polar coördinates;
3. The addition formulæ;
4. The relations between the six elements of a triangle;
5. The inverse trigonometric functions;

[1] Fellow at Princeton University. Formerly instructor at the University of Wisconsin.

[2] *An Experiment in Correlating Freshman Mathematics*, THE AMERICAN MATHEMATICAL MONTHLY, December, 1915.

and, secondly, a study of the applications:

1. The straight line (angles between lines, polar and normal forms of the equations, distance from a point to a line, . . .);
2. The solution of cubics with real roots;
3. Kinematics (uniform circular motion, simple harmonic motion, progressive waves).

Part IV is a presentation of logarithms:

1. The explanation of the exponential function a^x, and the significance of the constant e;
2. The use of logarithms in calculations (equations from scientific applications, compound interest, oblique triangles);
3. Logarithmic coördinates;
4. Hyperbolic functions and the catenary.

Part V is the introduction to differential calculus preceded by parts of college algebra:

1. Illustrations and review of periodic, single and multiple valued, odd and even, continuous and discontinuous functions;
2. Zeros of functions and simultaneous equations;
3. The remainder theorem and synthetic division;
4. The rate of increase of functions (derivatives of x^n, $\sin x$, $\cos x$, a^x);
5. The technique of differentiation with the usual elementary applications.

Part VI is the customary work on conics and other plane curves.

Part VII is vectors in their relation to forces and complex numbers.

The basis of this arrangement has been not so much the attempt to *correlate* the different subjects, as to take advantage of the *unity* which is present in mathematics and which we have hitherto overlooked. The best correlation will mean a fluctuation from one subject to another whenever two problems in distinct fields can be solved by similar methods or when a fundamental concept underlies different subjects, or when a result from one subject can be used to extend our knowledge of another. In this sense the above outline is itself a correlation, for some problems of lines are solved after the introduction of trigonometry, and graphs can be used in deriving trigonometric relations, etc. Unification, on the other hand, means the presentation of the different fields as parts of a single whole so that no part can be omitted without disturbing the completeness of the structure. The unity of the above outline will be more evident if I give the following titles to the divisions:

I. The aims and methods of mathematics.
II. The simple algebraic functions.
III. The trigonometric functions.
IV. The exponential and logarithmic functions.
V. General properties of functions (periodicity, zeros, derivatives).
VI. Irrational algebraic functions.
VII. Functions of two variables.

By an elaboration of these titles I shall try to show how the course is unified

by keeping constantly in mind that mathematics is a study of relations between variables, or the properties of functions. Each division will then involve a study of a particular class of functions and we may jump from one subject to another only when the discovery of some relation enables us to extend our knowledge of a previous subject.

The first division, being the introduction, brings in the notion of variable and dependence of one variable upon another, shows how numbers arranged on a line (a scale) enable us to exhibit the correspondence between different values. The graph obtained by placing the scales at right angles should appear to the student only as an easy way of exhibiting this correspondence; and the teacher should emphasize the relation between the points on the scales rather than the points on the graph. The language of coördinates and axes is introduced apparently only as an excuse to simplify the terminology. By emphasizing the fact that a curve is an exhibition of the restriction on, or the relation between, the values of the variables, we can get away from the graphic algebra which teaches the plotting of curves and mistakes the means for the end.

The ideas of such an introduction comprise the first chapter of any text on coördinate geometry, and are presented sooner or later in every course on college algebra, and in all the recent texts on trigonometry. But in none of these places can the emphasis be properly placed on the functional relation. Geometry places the emphasis on coördinates and curves; trigonometry introduces the ideas merely to graph the functions and then stops; the algebras use the curves as geometric means of solving equations. Irrespective of what course the student is about to pursue there is no reason why the year should not begin with a thorough explanation of the notion of function.

The second division then begins with the simplest function, the linear. By presenting the line in its slope form we can introduce the notion of the rate of increase of a function and illustrate how a function is determined by its rate of increase and its value at one point. While this idea will not recur until much later there are nevertheless problems to which it can be applied, for example, the coefficient of linear expansion. We consider briefly the functions of the type x^n, and then formulate the general methods of curve tracing, for example, the change in a curve when in its equation x is replaced by kx or by $x - a$. These methods enable us to study the parabola, ellipse, and hyperbola (without the xy term) and incidentally can furnish a review of quadratics and exponents.

The third division considers what I prefer to call the trigonometric functions rather than trigonometry, for the latter term might not, for example, include polar coördinates except as a side-issue, whereas as in studying functions we would assuredly draw their graphs as an aid in finding their properties, and the polar graph is as useful as the cartesian. While in all teaching theory and application are blended, they are separated in the outline to conform with the idea that we seek the properties of functions first and then use these properties to enlarge our ideas as to previous problems. The applications carry us from field to field but we are thoroughly conscious of our travels and have a legitimate

excuse for the journey. It is this consciousness of change and the deliberateness of it which will remove the difficulties pointed out by E. B. Wilson:[1] ". . . most problems . . . are really problems in algebra, in trigonometry, or in analytics, and the training in classification according to these topics is highly valuable;" for we do classify the problem and then focus on it the nature of the functions involved.

The treatment of the fourth division is along similar lines. We are not interested in logarithms alone, but in the logarithmic function. The subject can be well introduced by a problem involving a function whose rate of increase is proportional to itself, for example, that of determining the number of bacteria in a solution. Any approximate solution leads to an exponential function; the correct solution leads to a splendid explanation of the significance of the constant e and unquestionably the very best explanation a student can have. This has always been one of the weak spots in calculus; merely calling e the natural base does not reveal its significance. Logarithms may then be introduced as answering the question of the time required for the bacteria to reach a specified number. After the customary formal treatment of the uses of logarithms we consider the various exponential relations and the use of logarithmic coördinates. The exponential curve, the catenary, and the hyperbolic functions close the division.

To this point we have consistently presented the fundamental functions, x^n, the trigonometric, and the exponential. The fifth division shows again the unity of the subjects by treating the properties common to all the functions. At this stage the student has worked with a sufficient number of functions to profit by classifying them as odd, periodic, continuous, etc. This is also a suitable place for parts of algebra such as the zeros of functions, sets of zeros of two functions, the remainder theorem, and synthetic division. These subjects now enter as a natural step in finding the properties of functions.

The next part needs perhaps more careful attention than the others because we must make sure that the calculus is not introduced at the expense of other subjects and that the work will not need repetition during the following year. That no subject is slighted will be evident before the end of the article. Any method of correlation will, in fact, enable us to treat more subjects during the year because we have at our disposal the time gained through the avoidance of such repetition as is always unavoidable when algebra, trigonometry and analytics are taught in three distinct courses. Each of these courses, for example, must separately introduce the notions of functions and of coördinates; under the present outline this is done thoroughly once. Again, many of the results of analytics can be derived by the shorter methods of calculus, and this in itself is desirable unless we agree to prefer old methods to new ones merely because they are old.

If less than fifteen or even twenty hours are devoted to the calculus we may feel sure that the student will forget all about it before the following year.

[1] A review of C. S. Slichter's *Elementary Mathematical Analysis* in the MONTHLY for February, 1915, page 56.

For a treatment in twenty-five hours the writer suggests a complete development of the technique of differentiation together with such applications as are found in curve tracing (maxima, minima, and inflexion points), maxima and minima of functions, and problems dealing with velocity and rates. No reference should be made to integral calculus except of course the customary exercises in finding $f(x)$ when its derivative is given and even these should be used only to impress on the memory the standard formulæ of differentiation. With this accomplished the sophomore will have a definite background and the time will not be wasted.

In the course outlined by Mr. Griffin the analytic geometry is postponed until the junior year. Unless, however, the student follows mathematics for three years, analytics can be properly used to fill in the remainder of the freshman year. It will be noticed in the present outline that the line, polar coördinates, and the simple conics have been treated before the sixth division is reached. The conics are now defined by their focal definitions ($r_1 + r_2 = 2a$, etc.), the existence of directrices proved, the eccentricities defined and compared. Then the latus rectum is introduced preliminary to the polar equation, $\rho(1 + e \cos \theta) = l$, which holds for all the conics. The change in the equation of a curve under rotations and the analysis of the general equation of the second degree follows. A study of equations involving arbitrary constants leads to the discussion of lines tangent to a conic where, if the teacher so chooses, the methods of calculus may be used instead of handling the discriminant of the quadratic. By using both methods or by treating these same problems in the previous division the student will see how the new theory improves old methods. The discussion of locus problems leads to such other curves as the witch, cissoid, cycloids, conchoids, etc., and the position of these subjects here is advantageous in that the end of the year will be approaching and so these subjects can be treated as briefly or as thoroughly as time permits.

The student in an engineering school, as a preparation for the sophomore year, needs a knowledge of vectors and this subject ends the year's work. As the theory involves always the use of the trigonometric functions, vectors might be studied as one of the applications of these functions. But the present arrangement preserves the unity of the course, inasmuch as the last division then deals with variables whose values can not be represented by points on a line. The position of vectors and complex numbers at the end of the year again offers possibilities for variation in treatment.

The various divisions of this outline constitute the essential parts of "*An Introduction to the Elementary Functions*," a fit title for this course. The fact that such a title is possible and appropriate is a last proof of the unity of the material. Of course a mere title is of slight value in teaching and any outline can fail unless the presentation has coherence, a thing which must be left to the textbook writer. Also, unless the teacher consistently adheres to the notion that every chapter is a study of the relations between variables, properties of functions and the applications of these properties, all unity is lost.

There are three types of students for whom a course as here outlined is

especially suitable. The non-specialist who wishes an introduction to mathematics can profit less by trigonometry or algebra than by an immediate insight into the theory of functions. Unity is to him of more importance than any other element. The second type is the engineering student who during his freshman year needs an introduction to calculus and to vectors, as a preparation for physics and mechanics, rather than most of the work in college algebra. Partial fractions are best considered preliminary to certain integrations; the binomial theorem fits into the study of series; permutations and combinations are only of the most indirect value. The third type is the student who would later specialize in mathematics, and for him the present outline includes everything that he now gets in his freshman courses of trigonometry, algebra and analytics with the exception of permutations, a loss more than counterbalanced by his knowledge of calculus.

THE DUPLICATION PROBLEM.[1]

By JAMES H. WEAVER, West Chester High School.

There have come down to us from the remote past three problems of perennial interest, namely, the duplication problem, the trisection problem, and the quadrature problem. The first of these has for its object the finding of the edge of a cube that is double a given cube, the second the trisection of any angle, and the third the finding of a square equivalent to a given circle. It is the object of this paper to give:

I. A short historical sketch of the duplication problem, calling attention to the various methods of attacking the problem that were made possible by the advancement of mathematics.

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[1] The following authorities may be consulted for a fuller discussion of the problem:

(*a*) *Collections of Pappus*, ed. Hultsch, Berlin, 1876, page 31 and ff. In addition to the approximate solution mentioned in the text, this contains the principal solutions of the Greeks of the Alexandrian School.

(*b*) *Historia Problematis de Cubi Duplicatione*, by N. T. Reimer, Göttingen, 1798, 8vo, 238 pages. A very full account of the history of the problem.

(*c*) *Historia Problematis Cubi Duplicandi*, by C. H. Biering, Copenhagen, 1844, 4to, 64 pages. This was largely stolen from (*b*) according to S. Günther. See Cantor, *Geschichte der Mathematik*, Volume 4, pages 28–9.

(*d*) W. W. R. Ball, *Mathematical Recreations and Essays*, New York, Sixth Edition, 1914, pages 285–291. The discussion here is merely an outline of a few of the most noted solutions with references to the original sources.

(*e*) *Das Delische Problem*, A. Sturm, Linz, 1895–7. 4to, 140 pages. A full discussion of the problem in all its stages of development.

(*f*) Cantor, *Geschichte der Mathematik*, Vol. I, first edition, page 349. We have here an authoritative historical account of some of the solutions of the Greeks.

(*g*) Article by A. Conti, found in the following books: *Questioni Reguardanti le Matematiche Elementari*, Volume 2, Bologna, 1914, pages 185–231, and *Fragen der Elementar Geometrie*, Theil 2, Leipzig, 1907. Both these books were edited by Enriques. This article gives a very good discussion of some approximate solutions and takes up the question of the impossibility of a solution by means of ruler and compasses.

(*h*) *Famous Problems of Elementary Geometry*, F. Klein, translated by Beman and Smith, Boston, 1897. This volume includes a discussion of the impossibility of the problem by means of ruler and compasses. We will refer to the above mentioned accounts by the names of the authors; thus, Sturm, p. . . ., etc.

II. Solutions by means of the conchoid and the cissoid.

III. An approximate solution, recorded by Pappus, illustrating the difficulty under which the Greeks labored when dealing with problems of this type.

I. Historical Sketch.

Just how the duplication problem originated is not known. It probably dates back to the early Pythagoreans (about 530 B.C.) who had succeeded in finding the side of a square that was double a given square, and proving that the sides of the two squares are incommensurable. After this accomplishment it would be the natural thing to attempt to find the edge of a cube that is double a given cube. But tradition has given the origin of the problem a romantic setting. Eutocius (about 480 B.C.)[1] has preserved for us one version of the story which was related by Eratosthenes (276–194 B.C.) in a letter to king Ptolemy. It is as follows:

"It is related that one of the old tragic poets whom Minos had imported says that when Minos wished to erect for his son Glaucus a tomb and noticed that its dimensions were one hundred feet (ἑκατομ πεδος) on all sides he exclaimed: 'You have enclosed too small a space for a royal tomb. Double it, but forget not the beautiful form. Therefore double each edge of the monument.'

"But he had clearly erred. For by doubling the edges, the surface is made four times as great and the volume is increased eight fold. Nevertheless it made the question of how a body could be doubled without changing its form an object of investigation among geometers, and this is called the duplication of the cube. That is, they set up a given cube and sought to double it."

The first real progress in the solution of the problem was made by Hippocrates of Chios (about 420 B.C.). He reduced it to the one of finding two mean proportionals to two given lines, a form in which it has since been stated.[2] He did not however succeed in finding the mean proportionals. This task was left to Archytus of Tarentum who (about 400 B.C.) accomplished it by means of the intersections of solids.[3] Soon after this Menaechmus (about 340 B.C.), probably

[1] Archimedes, *Opera Omnia cum Commentariis Eutocii*, ed. Heiberg, Leipzig, 1880–1, volume 3, page 102 ff.

[2] Hippocrates noted that if two lines a and $2a$ were given and if two mean proportionals could be inserted between them such that

$$a : x = x : y = y : 2a,$$

then $x^2 = ay$ and $y^2 = 2ax$, from which it readily appears that $x^3 = 2a^3$.

[3] The following is an outline of the solution of Archytus. Let AD be the larger of the two given lines. Then on AD as diameter describe a circle and on this circle as base erect a cylinder. From A in the circle draw the chord AB equal to the smaller of the two given lines and extend this chord until it meets the tangent drawn from the point D. Let this point be P. Then let the triangle APD be revolved about AD as axis generating a cone. Then suppose a semicircle drawn on AD as diameter and perpendicular to the circle ADB, and let this semicircle be so moved that A remains fixed and the semicircle remains perpendicular to the circle ABD. It will then generate a solid. The intersection of the three solids just described will determine a point. From this point draw an element of the cylinder, and let this cut the circle ABD in C. Draw AC. It can then be easily proved that $\overline{AC}^3 = 2\overline{AB}^3$.

following the suggestion of Archytus,[1] discovered the conic sections and used them to give two solutions of the duplication problem.[2] Plato (about 340 B.C.) contrary to his usual custom of dealing with such problems produced a mechanical solution.[3] Nicomedes invented for this purpose the conchoid[4] and Diocles at about the same time (second century B.C.) produced the cissoid.[5] All the other solutions given by the Greeks were either mechanical or depended on the conic sections.[6]

After the decline of Greek geometry nothing was done to advance the problem until the sixteenth century A.D. when Stifel (1486–1567) attacked it from the side of number theory.[7] Then Viète in his geometry (1593) pointed out the fact that every cubic or biquadratic that is not otherwise reducible leads to either the duplication or the trisection problem when solved.[8] Shortly after this Claud Richard records (1645) some solutions that he attributes to Christopher Grienberger, a contemporary, and which employ a new curve called the proportionatrix.[9] Other solutions of a geometric character were given by Grégoire de Saint-Vincent (1647), Newton (1642–1727) and Huygens (1625–1695).[10]

But the answer to the question of the impossibility of a solution of the problem by means of straight lines and circles was finally given by Descartes (1596–1650) in his geometry.[11] In Book II of this work Descartes classified problems as did the Greeks, into plane, solid and linear[12] and proved that, algebraically considered, plane problems correspond to equations of the first degree and of the second degree, solid problems to equations of the third degree, and linear problems to equations of the fourth and higher degrees.

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[1] In his solution Archytus made use of a point determined by the intersections of solids, one of which was a cone. The locus of such a point could be considered as a curve on a cone. This would naturally lead to the investigation of the properties of curves on a cone.

[2] The two solutions of Menaechmus consist analytically in finding (1) the intersection of the two parabolas $x^2 = ay$ and $y^2 = 2ax$ other than the one at the origin, and (2) the intersection of the hyperbola $xy = 2a^2$ and the parabola $x^2 = ay$. For a full discussion of these solutions and the development of conic sections at this period see Heath's introduction to his edition of the *Conics of Apollonius*, Cambridge, 1896.

[3] For a description of the mechanical device of Plato see Sturm, page 49 and ff.

[4] Nichomedes also used this curve to trisect an angle, Pappus, page 57.

[5] A description of these curves and the solutions by means of their aid is given in part II of this paper.

[6] For a description of the other solutions of the Greeks see Sturm, pages 17–97.

[7] Stifel used the theory of irrationals given in Book X of the *Elements*. See Sturm, page 113.

[8] See Sturm, pages 125–7 and Ball, page 290.

[9] A proportionatrix is defined as follows: Describe about C as center the semicircle ADB on the diameter AB, then with CB as diameter draw the semicircle CEB on the same side of the diameter as ADB. Then draw any line BED cutting these semicircles in the points E and D. Then on AB take $BF = BE$. Then describe about F as center with a radius FB a circle cutting DB in G. Then the locus of G for different positions of BED is a proportionatrix.

[10] For an outline of these solutions and for references to the original sources, see Ball, page 290.

[11] The geometry appeared in 1637. In 1649 a Latin edition was put out by Van Schooten. A modernized French edition was published at Paris in 1886 under the title La Géométrie de René Descartes.

[12] Plane problems were those that could be solved by straight lines and circles; solid problems those that required conic sections, and linear problems included all others. See Pappus, pages 54 and 270.

Then in the third book Descartes takes up a discussion of the duplication and trisection problems and shows that the solution of all irreducible equations of the third degree leads to the solution of one or the other of these problems. His procedure is as follows: The intersection (x, y) of the parabola $x^2 = ay$ and the circle $x^2 + y^2 = ax + by$ is such that $a : x = x : y = y : b$. Thus the equation $x^3 = a^2b$ is given by the intersection of a parabola and a circle. But the parabola $y^2 = \frac{1}{4}x$ and the circle $x^2 + y^2 - \frac{13}{4}x + 4ay = 0$ intersect in a point which solves the trisection problem. The ordinate of this point of intersection is given by the equation $4y^3 = 3y - a$. Therefore a parabola and a circle will solve an equation of the third degree, provided the squared term is missing. But since every cubic may be reduced to this form, every cubic may be solved by the intersection of a parabola and a circle, which is the same thing as solving either the duplication or the trisection problem.[1]

II. Solutions of Nicomedes and Diocles.

Solution of Nicomedes.[2] Nicomedes first considers the problem of constructing two mean proportionals between any two lines of given length. He proceeds as follows. Let CD and DA be two straight lines (Fig. 1). Complete the rectangle $ABCD$, bisect the lines AB and BC in the points L and E. Produce

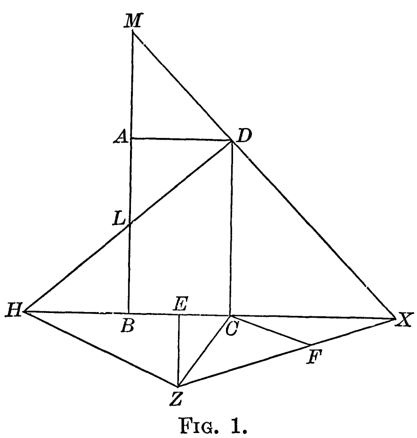


Fig. 1.

the lines DL and CB until they intersect in H. Draw EZ perpendicular to BC so that $ZC = AL$. Then draw ZH and CF parallel to ZH (F being indeterminate). Extend BC to X so that when a straight line is drawn from Z to X, $FX = AL = ZC$. This can be done by means of the conchoid.[3] Then draw

[1] See Sturm, pages 127–131 and Ball, pages 290 and 292.

[2] Pappus, Vol. I, pages 59–63 and pages 249–251.

[3] Z is the fixed point and CF is the fixed line of the conchoid, which is constructed as follows. Let there be a point Z and a line CF each given in position, and a line AL given in magnitude. Then let there be drawn through Z any line as ZX cutting CF in F and let $FX = AL$. The locus of X will be the curve called the conchoid of Nichomedes. There will of course be two branches to the curve, one on each side of CF.

XD and extend it until it meets *AB* produced in *M*. Then

$$DC : CX = CX : MA = MA : AD.$$

Proof: Since *BC* is bisected in *E* and *CX* is an extension of *BC*, then $BX \cdot XC + CE^2 = EX^2$ (Elements, II, 6).

Then by adding EZ^2, $BX \cdot XC + CE^2 + EZ^2 = EX^2 + EZ^2$, or $BX \cdot XC + CZ^2 = ZX^2$. Then by similar triangles, $MA : AB = MD : DX$ and $MD : DX = BC : CX$.

Therefore, $MA : AB = BC : CX$. But $AB = 2AL$ and $BC = \frac{1}{2}HC$, so that $MA : AL = HC : CX$.

Therefore, $MA : AL = ZF : FX$, and by composition $ML : AL = ZX : FX$. But by hypothesis, $AL = FX$, therefore $ML = ZX$, and $ML^2 = ZX^2$. Also (*Elements*, II, 6) $ML^2 = BM \cdot MA + AL^2$, and we have proved above that $ZX^2 = BX \cdot XC + CZ^2$ and $AL^2 = CZ^2$.

Therefore, $BM \cdot MA = BX \cdot XC$, or $BM : BX = CX : MA$.

But on account of parallels *MB* and *DC*, $BM : BX = DC : CX$.

Therefore, $DC : CX = CX : MA$.

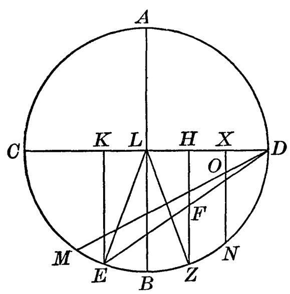
Also on account of parallels *BX* and *AD*, $MB : BX = MA : AD$.

But from above $BM : BX = CX : MA$. Hence $CX : MA = MA : AD$.

Therefore, $DC : CX = CX : MA = MA : AD$. Thus *CX* and *MA* are the two required mean proportionals between *DC* and *AD*.

From this the duplication problem follows readily, as in note 2, page 107. Because if we have two given lines a and b and if c and d are the two mean proportionals to these lines, then by *Elements, V, Def. 11*; VIII, 12; XI, 33, $a : b = a^3 : c^3$. And if b is double a then c^3 is double a^3.

Solution of Diocles.[1] This solution is different from others because it does not find directly two mean proportionals to two given lines, but, after setting up a cissoid, finds two mean proportionals to two lines that have to each other a ratio equal to the ratio of the two given lines to each other, and from these auxiliary mean proportionals finds those required.



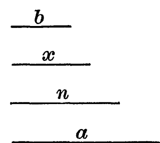


FIG. 2.

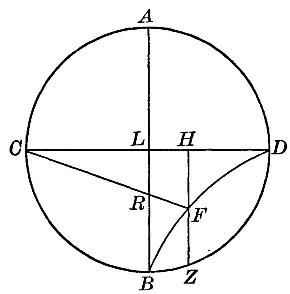


FIG. 3.

Let there be drawn in a circle two diameters perpendicular to each other as *AB* and *CD* (Fig. 2) and on either side of *B* cut off equal arcs *EB* and *BZ*, and draw *ZH* parallel to *AB*, and draw *DE* cutting *ZH* at *F*. Then *ZH* and *HD* will be two mean proportionals between *CH* and *HF*.

Proof: Let *EK* be drawn parallel to *AB*. Then $EK = ZH$ and $KC = HD$, as is seen if we draw *EL* and *ZL*.

For $\angle CLE = \angle ZLD$ and the angles at *K* and *H* are right angles. Consequently the triangles *ELK* and *ZLH* are congruent, since $LE = LZ$.

[1] This is on the authority of Eutocius, Archimedes, *Opera*, Vol. III, p. 78 ff.

Therefore, $KL = LH$ and hence $CK = HD$.

Since $DK : KE = DH : HF$, on account of similar triangles, and since $DK : KE = KE : KC$, for KE is the mean proportional between DK and KC, then $DK : KE = KE : KC = DH : HF$.

Then, since $DK = CH$, $KE = ZH$ and $KC = HD$, $CH : HZ = HZ : HD = HD : HF$.

This proves that ZH and HD are two mean proportionals between CH and HF.

Again if any two other equal arcs BM and BN are cut off and NX drawn parallel to AB, and DM is drawn cutting NX in O, then by reasoning similar to the above, NX and XD are the two mean proportionals between CX and XO. (The locus of all points determined, such as F and O, is called a cissoid.)[1]

Let there be now two given lines a and b, between which it is desired to find two mean proportionals (Fig. 3). Let there be constructed in a circle the two diameters CD and AB perpendicular to each other, and let the cissoid BFD be described. Let L be the center of the circle and R a point on the radius LB such that $a : b = CL : LR$. Draw CR and produce it until it cuts the cissoid in the point F, and draw through F the line HZ parallel to AB, cutting CD in H and the circle in Z. Then HZ and HD are two mean proportionals between CH and HF. (This has been proved above.)

Since $CH : HF = CL : LR$ and $CL : LR :: a : b$, it is only necessary to insert between a and b two lines, as n and x, so that these four will be in the same ratio as CH, HZ, HD and HF. Then n and x are the two mean proportionals between a and b.[2]

Then, as was shown above in the solution of Nicomedes, $a : b = a^3 : n^3$.

Pappus[3] also gives a solution which is essentially the same as that of Diocles given above. The only difference between the two solutions is this: Pappus, instead of using a curve to determine F, revolves a ruler about D until the portion between CR and RB equals the segment cut off by the line RB and the arc BC. But this is equivalent to laying off equal arcs on either side of AB.

III. Solution of Pappus.

The following is a discussion of an approximate solution proposed to Pappus (end of the third century A.D.) by an unknown geometer. It is recorded in Book III of the *Collections*[4] and is here reproduced with slight modifications of a literal translation. The construction without proof was handed over to Pappus, who shows in his discussion of it, that, although rather long, the construction does nothing more than assume that two mean proportionals are inserted between two given lines (a thing considered impossible by all preceding mathematicians provided only ruler and compasses are allowed in the construction) and that, considered geometrically, it is inexact. Pappus did not seem to realize the fact that the construction affords a method for obtaining an approximate solution of the problem in question. The construction and criticism proceed as follows:

Proposed Construction. Let there be two straight lines AB and AC perpendicular to each other (Fig. 4), and let there be drawn from B a line BD parallel to AC and let $BD = AB$. Join DC and produce DC until it meets AB produced in E, and from E draw EF parallel to BD.

Produce BD, and from D draw a line parallel to BE and intersecting EF in H; and in BD produced take $DN = NL = LX = XK = BD$, and through the points N, L, X, K draw NO, LM, XP, KF intersecting EF in the points O, M, P, F.

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[1] Eutocius however does not give any name to the curve. On the other hand Pappus and Proclus both mention the name cissoid but do not mention the name of the inventor of the curve, Nevertheless, the properties of the curve described by Eutocius seem to coincide with those given by Proclus in connection with the curve which he calls the cissoid. See Sturm, page 87.

[2] The determination of n and x may be made in the following manner. $CH : HZ = a : n$, and $HZ : HD = n : x$, and from this it is evident that $HD : HF = x : b$.

[3] Pappus, pages 64, 167 and 1070.

[4] Pappus, p. 30 and ff.

On KF take a point R so that $KR = AB$, and let KR be bisected in S, and on KF let T and Q be points such that $KF : FS = FS : FT = FT : FQ$.

Then on the line XP take $XX' = AB$, and join $X'K$ and $X'Q$ and draw from the point S a line SY parallel to $X'Q$ and cutting $X'K$ in Y, and from Y the line YW parallel to XK cutting ML in the point W.

Then let there be on ML the points A' and B' such that $LM : MW = MW : MA' = MA' : MB'$. And on ON take $NC' = AB$ and draw $C'L$, $C'B'$, and WD' parallel to $C'B'$ and cutting $C'L$ in D'.

Then from D' draw $D'E'$ parallel to LN and cutting DH in E'. Let Z' and F' be two points on DH such that $DH : HE' = HE' : HZ' = HZ' : HF'$.

Now draw $F'C$ and draw $Z'K'$ and $E'L'$ parallel to $F'C$ and cutting DC in the points K' and L', respectively, and from the points K' and L' draw the lines $K'M'$ and $L'N'$ parallel to AC and intersecting AB in M' and N', respectively.

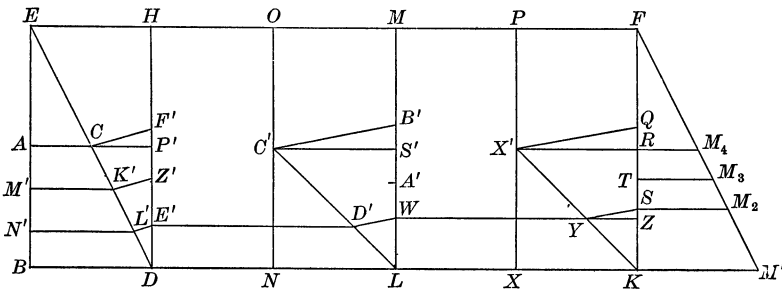


FIG. 4.

Then $M'K'$ and $N'L'$ are two mean proportionals between AC and BD.

Discussion of Pappus. Now the position of the point Q depends upon the ratio $KF : FR = BE : EA$ and hence Q may fall between F and R or between R and T, as we shall show.[1]

First let the ratio be double, that is, $KF : FR = 2 : 1 = 4 : 2$. Then $KF : FS = 4 : 3$ (for by construction $RS = \frac{1}{2}RK$), and from hypothesis $FS : FT = KF : FS$, that is, $KF : FS = FS : FT$ or $4 : 3 = 3 : 2\frac{1}{4}$. But $FS : FT = FT : FQ$, that is, $3 : 2\frac{1}{4} = 2\frac{1}{4}$: a number less than 2. If, however, we take $FR = 2$, then $FQ < FR$, and Q falls between the points F and R.

Again, let the ratio be quadruple, then $KF : FR = 8 : 2$, and so $KF : FS = 8 : 5$. For, if we suppose that KF contains 8 units and FR 2 units, then $RK = 6$ units and $RS = 3$ units, and, therefore, $FR + RS = FS = 5$ units. But $8 : 5 = 5 : 3\frac{1}{8}$, and $5 : 3\frac{1}{8} = 3\frac{1}{8}$: a number less than 2, so that, as above, Q will fall between F and R.

Finally, let the ratio be quintuple, that is, $KF : FR = 10 : 2$. Then $KF : FS = 10 : 6$. And $10 : 6 = 6 : 3\frac{3}{5}$, and $6 : 3\frac{3}{5} = 3\frac{3}{5}$: number greater than 2. Therefore Q falls between the points R and T. (For $FQ > FR$. But the same $FQ < FT$, since by hypothesis $KF : FS = FT : FQ$. Therefore Q falls between R and T.)

Now, since we have proved that Q may fall between F and R or between R and T, we will consider first that, no matter where Q may be assumed, $SF : FT = FT : FR$ is not possible, for that is assuming the thing sought. For let the line XK be produced to M'' so that $KM'' = XK$ and let $M''F$ be drawn and lines parallel to KM'' through the points S, T, and R intersecting FM'' in the points M_2, M_3, M_4, and the problem will be completed as is manifest. Because $KM'' : SM_2 = SM_2 : TM_3 = TM_3 : RM_4$, and $KM'' = BD$ and $RM_4 = AC$. Therefore, SM_2, and TM_3 are two mean proportionals between BD and AC, a thing which is impossible. For since FK is a straight line and in it is a point R, it is not possible by means of the ratios of plane figures to assume two points T and S in FK such that $KF : FS = FS : FT = FT : FR$; for the problem is by nature solid.[2]

Therefore, the proposer did not dare to say that the point Q would fall on R, for that would assume the whole problem, but he has chosen Q between F and R and then finished the construc-

[1] This is shown by special numbers. The method is of interest because it shows how the Greeks were forced to attack problems of this nature because of a lack of algebra.

[2] See note 12, p. 108. Pappus here as elsewhere makes the statement that this problem, as well as the trisection problem, is not possible by means of straight lines and circles. It is in no wise probable that the Greeks ever proved this statement. See Pappus, p. 271.

tion arbitrarily, and gradually fallen back into the first difficulty. He has produced a long construction, not in order to deceive his readers, but he has led himself into error, as I shall show after I have considered the problem in a sensible manner.

Now since the ratio $KF : FR$ is given and FK is given (for it is necessary that each be assumed), then FR is given (*cf.* Euclid's work, *The Data*, proposition 2), therefore the remainder KR is given (*Data*, 4). But SR is given because it is half of RK, and also FR is given, therefore FS is given (*Data*, 3). Therefore the ratio $KF : FS$ is given (*Data*, 1). And by hypothesis $KF : FS = FS : FT$, and we have proved that FS is given, therefore FT will be given. And in the same way FQ will be given. Therefore the difference $FR - FQ$ will be given.

Now suppose the point Q to be between F and R. We have proved this to be possible. Then since the difference $FR - FQ = QR$ is given, and the line joining X' and R is given, because it equals XK, the right triangle $QX'R$ is given in species and magnitude.[1] Therefore the angle RQX' is given, and because of the parallels $X'Q$ and YS we have $\angle RQX' = \angle KSY$.

Now let the line WY be produced till it intersects FK in Z. Then the triangle SZY is given in species (*Data*, 40). But it is also given in magnitude.[2] Therefore YZ is given. Therefore YZ, which is parallel to XK, is in the same straight line with WY. Therefore $WL = ZK$ is given (opposite sides of a parallelogram). Then, because $KF = LM$, and $SK > WL$, for $WL = ZK$, $FS < MW$. Also $KF : FS = FS : FT = FT : FQ$ and $LM : MW = MW : MA' = MA' : MB'$. Therefore $FQ < MB'$.[3] Hence, $KF - FQ > LM - MB'$, that is $B'L < QK$.

Then, again, because WL is given (proved above) and LM is given (since it equals KF which was given), MW is given, and from this the ratio $LM : MW$ is given. And by hypothesis $LM : MW = MW : MA'$, and MW is given, therefore MA' is given.

By similar reasoning MB' may be proved to be given, and from this the point B' is given, and may be between the points S' and M or between S' and A' (S' being a point on ML such that $S'L = KR = AB$).

But if the point B' is assumed to fall on S' it is equivalent to assuming the problem. For, again, in the line ML in which there is a given point S', two points A' and B' are assumed such that $LM : MW = MW : MA' = MA' : MS'$, a thing that no one will grant to be possible. Then if $C'S'$ is drawn, as above, the triangle $S'B'C'$ will be given in species and magnitude and finally DE' will be given. And DH is given and DE' is given, therefore HE' is given, and the ratio $DH : HE'$ is given, and $DH : HE' = HE' : HZ' = HZ' : HF'$. Then (if $DP' = KR$) F' may fall between P' and Z' or P' and H, but not on P' for that is equivalent to assuming the whole problem. But wherever F' may be, let it be supposed to be above P'.

Then when $F'C$ is drawn and parallel to $F'C$ the two lines $Z'K'$ and $E'L'$, and through the points K' and L' parallel to AC the two lines $K'M'$ and $L'N'$ are drawn, it is clear that the problem is not solved. For since $F'C$ is not parallel to EH the angle $CF'H$ will be obtuse if F' falls between P' and H, and acute if F' falls between P' and Z'. But if CP' is drawn, the angle at P' is a right angle. Therefore, the problem will be solved only if F' falls on P', so that in the line DH there are two points E' and Z' such that $DH : HE' = HE' : HZ' = HZ' : HP'$. But if this is not granted the problem can not be solved by means of planes.[4] And this will be persuasive to all by means of numbers, if they admit the table of Ptolemy of lines in a circle.[5] In fact it would have been more satisfactory if he, like the rest, had left the solution in doubt instead of producing it in this manner.

Such is the comment of Pappus. It remained for Günther and Pendlebury, in comparatively recent times, to exhibit the construction of the unknown geometer as one of a series of approximations leading to an accurate solution.[6]

[1] A figure is said to be given in species if its angles and the ratios of its sides are given.

[2] For YS is given because $X'Q : YS = KQ : KS$. From which it readily follows that the triangle YZS is given in magnitude.

[3] For proof of this statement see Pappus, p. 51.

[4] By a "solution by planes" is meant a solution by means of straight lines and circles.

[5] This is the table of chords of Ptolemy given in the *Almagest*. This table uses the measure of the chords to represent the measure of the arcs subtended by the chords. The lengths of the chords of certain arcs in a given circle were known, and, using these to start with, the lengths of the chords of the sum or difference of two arcs or of the half arcs were found. For the method of doing this, see Gow, *Short History of Greek Mathematics*, page 294 f.

[6] For a further discussion of this construction see Sturm, page 94 ff.

FRANK ASBURY SHERMAN.

Frank Asbury Sherman, at the time of his death, February 26, 1915, was professor emeritus on the Chandler foundation in Dartmouth College, having retired from active service in 1911, after a term of 40 years of uninterrupted teaching. He was born in 1841 at Knox, Maine, the son of Harvey Hatch and Elizabeth (Daly) Sherman, and was a direct descendant of William Sherman of the original Plymouth colony, who afterwards settled at Marshfield, Mass. Of stalwart form and sturdy character he was a typical son of the Pine Tree state. He received his early education in her common schools. At the breaking out of the Civil War he was fitting himself for college at the East Maine Conference Seminary. Imbued with a patriotic spirit, possessing the strength and ambition of opening manhood, he heard his country's call to arms and enlisted for three years in the fourth Maine volunteer infantry, in which service he was soon followed by his two brothers, Frederick and Augustus. He was severely wounded in the battle of Fredericksburg and later in the battle of the Wilderness, the latter wound causing the loss of his left arm near the shoulder. He quickly acquired the ability to use his remaining arm for all purposes and was wonderfully independent of the help of others in all the work he undertook. He was in active service through nearly the entire war and was mustered out in March, 1865. He at once resumed his work preparatory to entering college and in 1866 entered the Chandler Scientific Department of Dartmouth College, graduating therefrom in 1870 at the head of his class. He particularly enjoyed mathematics and the sciences. His high scholarship in the former and his early experience in teaching, by which means he had worked his way through college, led to his being invited to teach this subject in the Worcester Polytechnic Institute immediately after his graduation. The next year, his alma mater, needing an instructor in mathematics, tendered him a call as associate professor of mathematics and he was promoted to a full professorship the following year, namely, in 1872. This position he held till 1893, when the Chandler Scientific Department was merged into the College and he was made professor of mathematics on the Chandler foundation. The degree of Master of Science was conferred upon him in 1875. While nominally a professor of mathematics during his entire forty years of active service, for the first twenty years or more he was always ready to fill any gap; and taught various allied subjects, such as descriptive geometry, mechanics, physics, and free-hand drawing, always giving long hours in this service with the same conscientious purpose as had actuated his military career. His willingness to work and to do his part was by no means limited to collegiate lines. He will long be remembered by most of the older alumni as the secretary of their association, serving as he did in that capacity for fourteen years. He also generously served the public in various civic capacities. For many years he was a member of the local board of education and in that position designed and largely superintended the construction of their first high-school building. For several years he was a member of the village governing board. He was not a man who sought popularity or public office, but when called upon to serve, he responded quickly and did his work

promptly, carefully and with fidelity, and in most cases without remuneration. His work as a mathematician was more conspicuous in the class room than in the production of mathematical literature. He was an admirable instructor, a strict disciplinarian, though never harsh. Every student who took his courses felt that he had received full reward for his time and efforts. He took a deep personal interest in the work of each of his students and was beloved, respected, and admired by the alumni, as was shown at every succeeding Commencement, and during his last illness and up to the time of his death.

JOHN VOSE HAZEN.

DARTMOUTH COLLEGE.

NEW BOOKS RECEIVED.

A BUDGET OF PARADOXES. By Augustus De Morgan. Reprinted, with the author's additions, from the Athenæum. Second edition. Edited by David Eugene Smith. The Open Court Publishing Co., Chicago, 1915. Volume I, viii+402 pages. Volume II, 387 pages. $3.50 per volume.

FUNCTIONS OF A COMPLEX VARIABLE. By E. J. Townsend. Henry Holt and Co., New York, 1915. vii+378 pages. $4.00.

DIOPHANTINE ANALYSIS. By R. D. Carmichael. Mathematical Monographs, No. 16. John Wiley and Sons, New York, 1915. vi+118 pages. $1.25.

ANALYTIC GEOMETRY. By H. B. Phillips. John Wiley and Sons, New York, 1915. vii+193 pages. $1.50.

ROBERT OF CHESTER'S LATIN TRANSLATION OF THE ALGEBRA OF AL-KHOWARIZMI, WITH AN INTRODUCTION, CRITICAL NOTES AND AN ENGLISH VERSION. By Louis Charles Karpinski. The Macmillan Co., New York, 1915. 164 pages.

THEORY AND APPLICATIONS OF FINITE GROUPS. By G. A. Miller, H. F. Blichfeldt and L. E. Dickson. John Wiley and Sons, New York, 1916. xvii+390 pages. $4.00.

HISTORICAL INTRODUCTION TO MATHEMATICAL LITERATURE. By G. A. Miller. The Macmillan Co., New York, 1916. xiii+302 pages. $1.60.

FUNDAMENTAL CONCEPTIONS OF MODERN MATHEMATICS. Variables and quantities with a discussion of the general conception of functional relation. By Robert P. Richardson and Edward H. Landis. The Open Court Publishing Co., Chicago, 1916. xxi+216 pages. $1.25.

CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS. By George Cantor. Translated and provided with an introduction and notes by Philip E. B. Jourdain. The Open Court Publishing Company, Chicago, 1915. ix+211 pages. $1.25.

ANALYTIC MECHANICS. By John Anthony Miller and Scott Barrett Lilly. D. C. Heath and Co., Boston, 1915. xv+297 pages. $2.00.

THE ELEMENTS OF SURVEYING AND GEODESY. By W. C. Popplewell. Longmans, Green and Co., London, 1915. xi+244 pages. $2.25.

PLANE AND SPHERICAL TRIGONOMETRY WITH TABLES. By George Wentworth and David Eugene Smith. Ginn and Co., Boston, 1915. iv+230+26+104 pages. $1.35.

PLANE AND SOLID GEOMETRY. By Webster Wells and Walter W. Hart. D. C. Heath and Co., Boston, 1916. vi+467 pages.

SCHOOL ALGEBRA. First Course. By H. L. Rietz, A. R. Crathorne, and E. H. Taylor. Henry Holt and Co., New York, 1915. v+271 pages. $1.00.

SCHOOL ALGEBRA. Second Course. By H. L. Rietz, A. R. Crathorne, and E. H. Taylor. Henry Holt and Co., New York, 1915. x+235 pages. $.75.

FIRST YEAR MATHEMATICS FOR SECONDARY SCHOOLS. 4th edition. By Ernst R. Breslich. University of Chicago Press, Chicago, 1915. xxiv+345 pages. $1.00.

A TWENTIETH CENTURY ARITHMETIC. By C. S. Jackson, F. J. W. Whipple, and Lucy Roberts. J. M. Dent and Sons, London, 1915. viii+495 pages.

PLANE GEOMETRY. By John W. Young and Albert J. Schwartz. Henry Holt and Co., New York, 1915. v+223 pages. $1.00.

HOW TO STUDY AND WHAT TO STUDY. By Richard L. Sandwick. D. C. Heath and Co., Boston, 1915. v+170 pages. $.60.

FUNDAMENTAL SOURCES OF EFFICIENCY. By Fletcher Durell. J. B. Lippincott and Co., Phila., 1914. 368 pages. $2.50.

BOOK REVIEWS.

Send all communications to W. H. BUSSEY, University of Minnesota.

Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi with an Introduction, Critical Notes and an English Version. By LOUIS CHARLES KARPINSKI, University of Michigan. The Macmillan Co., New York, 1915. 164 pages.

This monograph impresses the reader as being an exceedingly thorough study. It exhibits the modern trend toward a more searching and more critical study of historic material. Its scope is comprehensive. It is much more than a reprint of Robert of Chester's Latin translation from the Arabic, accompanied by a translation into English. It contains in outline the development of algebra before the time of Al-Khowarizmi, an account of Al-Khowarizmi's Algebra and Arithmetic, and their bearing upon the development of mathematics. Much biographical detail is gleaned from out-of-the-way sources, pertaining to Al-Khowarizmi, Robert of Chester and other medieval writers who prepared translations of, or were directly influenced by, the algebra of the great Mohammedan. Robert of Chester's Latin translation was made in the twelfth century. The different extant manuscripts are compared and the deviations from the text of Scheybl, which is followed in this edition, are given in foot-notes with a minutia that seems almost excessive. A Latin glossary assists the reader in making out the medieval

Latin. Added interest is secured by photographic reproductions of pages from the three most complete manuscripts of Robert of Chester's translation. Neither time nor expense has been spared in making the monograph a minute, yet attractive study of the earliest translation into Latin of the famous Arabic text. More profoundly than any other work on algebra that was brought out during the twelve centuries intervening between Diophantus and the Italians, Tartaglia and Cardan has that Arabic text influenced the progress of algebra in the Occident.

FLORIAN CAJORI.

COLORADO COLLEGE,
COLORADO SPRINGS, COLO.

Analytic Geometry. By H. B. PHILLIPS. John Wiley and Sons, New York, 1915. With answers to exercises. vii+197 pages.

The author tells us in his preface that "he has written this text to supply a course that will equip the student for work in calculus and engineering without burdening him with a mass of detail useful only to the student of mathematics for its own sake. . . . If more than the briefest course is given, the best way to spend the time is in working a large number of varied examples based upon the few fundamental principles which occur constantly in practice."

A wise innovation is a brief discussion of the vector, its use illustrated by a few simple problems such as the division of a line in given ratio, the area of a triangle in terms of the coördinates of its vertices, and the location of the center of gravity of a system of weights. But our author straightway abandons this excellent line of procedure and in the sequel has no regard for the directions on any lines other than the coördinate axes. The angle φ "from the positive direction of the x-axis to the line MN," as used here, is the least positive angle that can so be measured. If two lines L_1 and L_2 make with the x-axis the angles φ_1 and φ_2, then the angle β from L_1 to L_2, according to the text (p. 41), is $\varphi_2 - \varphi_1$ in every case. But if $\varphi_1 > \varphi_2$, this angle is obviously $180° + \varphi_2 - \varphi_1$, unless we regard β as then negative, contrary to custom and contrary to our author himself in the very first application he makes of the familiar formula for $\tan \beta$ (Ex. 3, p. 41. The angle from AB to AC is negative according to the definition of β on p. 41.) It seems to me a matter of regret that the teacher in his class room should accept without comment as to general validity a demonstration based merely on the most obvious geometrical construction. That this careless attitude should find place in our textbooks is a matter for serious concern.

In Art. 26 it seems to me that we have the distance from the line $Ax + By + C = 0$ to the point (x_1, y_1) rather than the distance from the point to the line, but since we are told that such a sign must be used with $\pm \sqrt{A^2 + B^2}$ that the result be positive, this is a matter of little moment.

The definitions of fundamental curves and their properties deserve mention. The slope of the line determined by the points (x_1, y_1) and (x_2, y_2) is *by definition* $m = (y_2 - y_1)/(x_2 - x_1)$, the term inclination is not used. As working tools, one uses only two forms for the rectangular equation of the straight line, $y - y_1 = m(x - x_1)$, and $y = mx + b$. The ellipse is obtained through deformation of

the circle, and it is noted that the ratios obtained by dividing the squares of the distances of any point on the ellipse from the axes by the squares of the parallel semi-axes, have a sum equal to 1. The parabola is defined directly as the locus of points the squares of whose distances from one of two perpendicular lines are proportional to their distances from the other, while the hyperbola is the locus of points the product of whose distances from two lines is constant. This makes possible, without any mention of transformation of coördinates, the derivation of the equations of these curves with any arbitrary pair of perpendicular lines as axes, and leads immediately to a simple classification of the types of curves represented by the general equation of the second degree.

Chapter 5 begins with an excellent discussion of the standard methods of sketching the graph for an algebraic equation. Article 40 on the direction of a curve near a point is notable, since it is the nearest approach to mention of a tangent line to be found in the book. Any discussion of tangent lines is left for the calculus. Such topics as diameters, poles and polars, and their ilk, are also absent. There is a brief discussion of trigonometric and exponential curves, and in the section on empirical equations there are examples of each of the four types $y = mx + b$, $y = ax^2 + bx + c$, $y = ax^n$, and $y = ab^x$.

In the chapter on polar coördinates we find the one time familiar definition of a conic as "the locus of a point moving in such a way that its distance from a fixed point is proportional to its distance from a fixed straight line." Its equation in polar coördinates is changed to rectangular coördinates and the ellipse, hyperbola and parabola are discovered as conics. In the section on the intersection of curves, care is taken to point out that a point may lie on a curve although its coördinates (as given) do not satisfy the equation of the curve. The illustrative example is correct if $a = 1$.

A chapter on parametric representation and one on transformation of coördinates close the portion of the book devoted to plane geometry. A single page is given to the discussion of the general equation of the second degree. As noted above, we already have working rules for the determination of the nature of the locus. The concluding three chapters (32 pages) are devoted to the geometry of space of three dimensions. The vector is again discussed and the equations and properties most frequently needed for the line, the plane; the conicoids and simple curves are brought out. The locus problem is introduced early in the text and is brought in again and again, fixing new principles by repeated use. The problem lists are frequent and extended.

The author seems to have put into this text about what he might offer to his own classes and to have added comparatively few topics merely for the sake of making a complete treatment. This should make the book usable from the point of view of the teacher. One who is looking for a drill course on "the few fundamental principles which occur constantly in practice" should consider this book.

G. R. Clements.

The University of Wisconsin,
Madison, Wis.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

455. Proposed by JOS. B. REYNOLDS, Lehigh University.

Solve for x_n (not in determinant form) the simultaneous equations,

$$\tfrac{4}{3}x_n + 2x_{n-1} + 2x_{n-2} \cdots 2x_4 + 2x_3 + 2x_2 + 2x_1 = g,$$

$$\tfrac{10}{3}x_n + \tfrac{22}{3}x_{n-1} + 8x_{n-2} \cdots 8x_4 + 8x_3 + 8x_2 + 8x_1 = 4g,$$

$$\tfrac{16}{3}x_n + \tfrac{40}{3}x_{n-1} + \tfrac{52}{3}x_{n-2} \cdots 18x_4 + 18x_3 + 18x_2 + 18x_1 = 9g,$$

$$\tfrac{22}{3}x_n + \tfrac{58}{3}x_{n-1} + \tfrac{82}{3}x_{n-2} + \tfrac{94}{3}x_{n-3} \cdots 32x_4 + 32x_3 + 32x_2 + 32x_1 = 16g,$$

$$\tfrac{28}{3}x_n + \tfrac{76}{3}x_{n-1} + \tfrac{112}{3}x_{n-2} + \tfrac{136}{3}x_{n-3} + \tfrac{148}{3}x_{n-4} \cdots = 25g,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\tfrac{1}{3} + 2n - 1)x_n + (\tfrac{1}{3} + 6n - 5)x_{n-1} + (\tfrac{1}{3} + 10n - 13)x_{n-2} + (\tfrac{1}{3} + 14n - 25)x_{n-3} \cdots (\tfrac{1}{3} + 2n^2 - 1)x_1 = n^2 g.$$

456. Proposed by PAUL CAPRON, U. S. Naval Academy.

If

$$S_{i,n} = \sum_{k=1}^{k=n-i+1} \frac{(i+k-1)!}{(k-1)!},$$

show that $S_{i,n}$ is equal to $1/(i+1)$ times the last term of $S_{i+1,n+1}$; as, for instance, that

$$S_{1,n} = 1 + 2 + \cdots + n = \frac{n}{2}(n+1),$$

that

$$S_{2,n} = 1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \tfrac{1}{3}(n-1)n(n+1),$$

etc.

GEOMETRY.

486. Proposed by ARON INGVALE, Brooklyn, N. Y.

Does the following construction trisect an angle? With the vertex, O, of the given angle as center and with a radius R, describe a circle intersecting the sides of the given angle in A and B. With a radius $\frac{3}{4}R$, and center on OA, describe a circle tangent to the other circle at A and cutting the other side of the angle at E. At E draw a tangent to the last circle and produce it to meet the first circle at F. Draw FO. Then is angle BOF one-third of the angle BOA?

REMARK. Though the construction does not, of course, lead to the trisection of an angle in general, yet as a first approximation it is very good. This fact together with the fact that the construction is very simple, and that the proposer's demonstration that it does trisect the angle is very illusive, are the reasons for giving the problem a place in the MONTHLY.

EDITORS.

487. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If segments from the vertices A and B of a triangle to the opposite sides are of equal length and divide the angles A and B (measured from AB) proportionally, the triangle is isosceles.

488. Proposed by ROGER A. JOHNSON, Western Reserve University.

If triangles are constructed on a given base, having the radii of the incircle and circumcircle in a constant ratio, determine the locus of the vertex. (Necessarily the constant ratio is not greater than $\frac{1}{2}$.)

CALCULUS.

405. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the greatest quadrilateral which can be formed with the four given sides a, b, c, and d taken in order.

406. Proposed by C. N. SCHMALL, New York City.

Given $f(x + h) + f(x - h) = f(x) \cdot f(h)$, determine by Taylor's theorem or otherwise the nature of the function f.

MECHANICS.

324. Proposed by H. S. UHLER, Yale University.

A rigid body of any shape is at rest in a neutral liquid which is also at rest and has an indefinitely great volume. The body is so situated that the free surface of the liquid is tangent to it at its highest point (or points). All the space above the liquid is filled with a neutral, stagnant fluid whose density is not greater than the density of the liquid. Show that the work done in raising (pure translation) the body very slowly until the interface of the two fluids is tangent to it at its lowest point (or points) is expressible by the formula $mgh - gV(\rho_1 h_1 + \rho_2 h_2)$, where $m \equiv$ mass of body, $V \equiv$ volume of body, $\rho_1 \equiv$ mean density of lower medium in the region involved, $\rho_2 \equiv$ density of upper medium, $h_1 \equiv$ distance of center of mass of displaced liquid below the free surface in the initial position of the body, $h_2 \equiv$ elevation of center of mass of displaced fluid above interface in final position of body, and $h = h_1 + h_2$. (Neglect surface-tension, adhesion, cavities in upper portion of body, etc. This problem arose in connection with a question concerning the raising of a dense object from the bottom of a harbor to the deck of a vessel.)

325. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

The lever of a testing-machine is c feet long, and is poised on a knife-edge a feet from one end and b feet from the other, and in a horizontal line, above which the beam is symmetrical. The beam is m inches deep at the knife-edge, and tapers uniformly to a depth of n inches at each end; the width of the beam is the same throughout its length. Find the distance of the center of gravity of the beam from the knife-edge.

NUMBER THEORY.

242. Proposed by NORMAN ANNING, Chilliwack, B. C.

Find a function of n which is equal to A_k when $n \equiv k \pmod p$, $k = 1, 2, 3, 4, \cdots, p$.

243. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the rational value of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.

Below are given problems in Number Theory proposed between January, 1913, and January, 1915, for which no solutions have been received. Ten problems in this subject were proposed during 1915. For some of these, solutions have been received and others are doubtless under consideration by those interested. They are Nos. 227–236. While not neglecting these more recent ones, may we also have coöperation in clearing up the older list?

191. (I...ctly numbered 187 in the June, 1913, issue.) **Proposed by L. E. DICKSON, University of Chicago.**

Find an amicable number triple by solving one of the equations (other than the last) in the MONTHLY, March, 1913, page 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

192. (Incorrectly numbered 188 in the June, 1913, issue.) **Proposed by ARTEMAS MARTIN, Washington, D. C.**

Find rational values for v, w, and x that will simultaneously satisfy the conditions:

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2v^2 + m^2n^2(m^2 + n^2) = \square, \tag{1}$$

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2w^2 + m^2n^2(m^2 + n^2) = \square, \tag{2}$$

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2x^2 + m^2n^2(m^2 + n^2) = \square, \tag{3}$$

and n being known quantities.

196. (Incorrectly numbered 192 in the September, 1913, issue.) **Proposed by CHARLES MACAULEY, Chicago, Ill.**

Combinations containing an even number of letters are formed with the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's stand in the same column; all the b's in the same column; all the c's in the same column, etc.

198. (November, 1913, issue.) **Proposed by ARTEMAS MARTIN, Washington, D. C.**

Prove that every even number is the sum of two prime numbers.

Note.—This problem has long been known and no proof has ever been given. EDITORS.

202. (December, 1913.) **Proposed by A. R. SCHWEITZER, Chicago, Ill.**

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, then $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element ξ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example, such a system is,

12,	23,	34,	45,	56,	67,	78,	89,	91
13,	24,	35,	46,	57,	68,	79,	81,	92
14,	25,	36,	47,	58,	69,	71,	82,	93
51,	62,	73,	84,	95,	16,	27,	38,	49

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set, (2) for each triad $\alpha\beta\gamma$ in the set there exists an element ξ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set, (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element ξ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating *n-ads* for $n > 4$ is obvious.

205. (February, 1914.) **Proposed by E. T. BELL, University of Washington.**

Show that in the usual arithmetical sense the form that follows admits of composition; give the requisite transformations; and indicate how several, if not all, solutions may be found:

$$x_0^2 + nrx_1^2 + mrx_2^2 + mnx_3^2 + mnrx_4^2 + mn^2r^2x_5^2 + nr^2m^2x_6^2 + rm^2n^2x_7^2.$$

208. (March, 1914.) **Proposed by E. T. BELL, University of Washington.**

If an odd number is perfect it cannot be the sum of two squares.

209. (March, 1914.) **Proposed by R. D. CARMICHAEL, University of Illinois.**

Prove that the difference of the sixth powers of an integer cannot be the square of an integer.

211. (**April, 1914.**) **Proposed by E. T. BELL, University of Washington.**

If an odd perfect number exists, the total number of its divisors is a multiple of 2, but not of 4; or, what is the same thing, an odd perfect number must be of the form $p^{2a-1}n^2$, where p is prime and a is odd.

214. (**April, 1914.**) **Proposed by A. J. KEMPNER, University of Illinois.**

Let a be a positive integer $\geqq 2$, and let $T(n)$ denote the number of distinct divisors of the positive integer n, including both 1 and n, so that $T(1) = 1$, $T(2) = 2$, $T(3) = 2$, $T(4) = 3, \cdots$. Show that

$$\Sigma_{n=1}^{n=\infty} T(n)/a^n = \Sigma_{n=1}^{n=\infty} 1/(a^n - 1).$$

The special case $a = 10$ gives, as is easily seen:

$$9 \sum_{n=1}^{n=\infty} \frac{T(n)}{10^n} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots.$$

217. (**May, 1914.**) **Proposed by E. T. BELL, University of Washington.**

(i) If r is a prime greater than 2, and $p \equiv 2^a r + 1$ is prime, the only solution, when n is greater than 2, of $x^n - y^n = p$, is $n = 3$, $x = 2$, $y = 1$.

(ii) The only primes that are simultaneously of the forms $4k + 1$ and $3^m - 2^m$ are 1 and 5.

(iii) Generalize (ii).

219. (**June, 1914.**) **Proposed by R. D. CARMICHAEL, University of Illinois.**

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

221. (**September, 1914.**) **Proposed by T. E. MASON, Purdue University.**

Find a number x such that the sum of the divisors of x is a perfect square. [Carmichael, *Theory of Numbers*, page 17.]

222. (**October, 1914.**) **Proposed by A. H. HOLMES, Brunswick, Me.**

Find rational values for m and n such that $(m^2 + 1)/m^2 + (n^2 + 1)/n^2$ may be the square of an integer.

223. (**October, 1914.**) **Proposed by T. E. MASON, Purdue University.**

Show that

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}}$$

is an integer, r, s, and t being positive integers. Generalize to the case of n integers, $r, s, t, u, \cdots$. [Carmichael, *Theory of Numbers*, page 28.]

SOLUTIONS OF PROBLEMS.

ALGEBRA.

443. Proposed by A. M. KENYON, Purdue University.

If p_r denote the sum of all the r-factor products that can be formed from the first n natural numbers ($p_r = 0$ for $r > n$), and if

$$D_s = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ 2p_2 & p_1 & 1 & \cdots & 0 \\ 3p_3 & p_2 & p_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ sp_s & p_{s-1} & p_{s-2} & \cdots & p_1 \end{vmatrix}$$

show that

$$\sum_{i=0}^{k} (-1)^i c_i \binom{k}{i} D_{2k-i} = 0, \qquad k, n = 1, 2, 3, \cdots,$$

where

$$c_i = \frac{2k+1-i}{1+i}$$

when i is even and $(2n+1)$ when i is odd; and $\binom{k}{i}$ is the coefficient of x^i in $(1+x)^k$.

Solution by the Proposer.

The roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \cdots + (-1)^n p_n = 0$$

are the natural numbers $1, 2, 3, \cdots n$.

Solving Newton's formulæ[1] for the sums of like powers of the roots, we obtain

$$1^s + 2^s + 3^s + \cdots + n^s = D_s, \qquad n, s = 1, 2, 3, \cdots,$$

A relation between the D's of odd subscript has been published[2] which is equivalent to

$$(1) \qquad \sum_{i=0}^{I(k/2)} \binom{k}{2i+1} D_{2k-1-2i} = 2^{k-1} D_1^k, \qquad k, n = 1, 2, 3, \cdots,$$

and the following relation[3] exists among the D's of even subscript,

$$(2) \qquad \sum_{i=0}^{I(k/2)} \frac{2k+1-2i}{1+2i} \binom{k}{2i} D_{2k-2i} = (2n+1) 2^{k-1} D_1^k, \qquad k, n = 1, 2, 3, \cdots.$$

These formulæ, in which $I(k/2)$ denotes the integral part of $k/2$, are readily established by induction. Multiplying (1) by $2n+1$ and subtracting the result from (2) we get the formula sought.

444. Proposed by J. E. ROWE, Pennsylvania State College.

Prove that the determinant

$$\begin{vmatrix} \cot A, & \cot B, & \cot C \\ 1, & 1, & 1 \\ \cos^2 A, & \cos^2 B, & \cos^2 C \end{vmatrix} = 0, \qquad \text{if} \qquad A + B + C = 180°.$$

Solution by S. E. Rasor, Ohio State University.

Transforming trigonometrically and rearranging, the determinant becomes

$$\frac{-1}{4 \sin A \sin B \sin C} \begin{vmatrix} 2 \cos A \sin B \sin C, & 2 \cos B \sin C \sin A, & 2 \cos C \sin A \sin B \\ 1, & 1, & 1 \\ 2 \sin^2 A, & 2 \sin^2 B, & 2 \sin^2 C \end{vmatrix}.$$

By the formula $2 \sin A \sin B \cos C = \sin^2 A + \sin^2 B - \sin^2 C$ when $A + B + C = 180°$, this reduces to

$$\frac{-1}{4 \sin A \sin B \sin C} \begin{vmatrix} \sin^2 B + \sin^2 C - \sin^2 A, & \sin^2 A + \sin^2 C - \sin^2 B, & \sin^2 A + \sin^2 B - \sin^2 C \\ 1, & 1, & 1 \\ 2 \sin^2 A, & 2 \sin^2 B, & 2 \sin^2 C \end{vmatrix},$$

[1] See, for example, Cajori's *Theory of Equations*, pages 85–86.

[2] Stern, *Crelle's Journal*, volume 84, pages 216–218.

[3] *Proceedings Indiana Academy of Sciences*, 1914, page 440.

or

$$\frac{-1}{4\sin A\sin B\sin C}\begin{vmatrix} \sin^2 A+\sin^2 B+\sin^2 C, & \sin^2 A+\sin^2 B+\sin^2 C, & \sin^2 A+\sin^2 B+\sin^2 C \\ 1, & 1, & 1 \\ 2\sin^2 A, & 2\sin^2 B, & 2\sin^2 C \end{vmatrix}.$$

But this is equal to zero since two rows are alike after dividing out $\sin^2 A + \sin^2 B + \sin^2 C$. It is to be noticed also that the above determinant is equal to zero for any values whatever of A, B, C provided only that two of them are alike.

Also solved by ELIJAH SWIFT, G. W. HARTWELL, H. POLISH, R. M. MATHEWS, H. L. AGARD, H. S. UHLER, CLIFFORD N. MILLS, W. W. BURTON, CARL A. W. STROM, A. M. KENYON, J. H. WEAVER, and A. H. WILSON.

CALCULUS.

387. Proposed by C. N. SCHMALL, New York City.

Show that the volume bounded by the cone $x^2 + y^2 = (a - z)^2$ and the planes $x = 0$, $x = z$, is $\frac{4}{9}a^3$.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

The projection on the XY-plane of the curve of intersection of the cone and the plane $z = x$ is $y^2 = a^2 - 2ax$.

If we change this equation to polar coördinates we obtain

$$\rho = \frac{a}{1+\cos\theta} = \frac{a}{2}\sec^2\frac{\theta}{2}.$$

Hence,

$$\frac{v}{2} = \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}\int_x^{a-\sqrt{x^2+y^2}} dz\cdot\rho d\rho d\theta = \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}(a-\sqrt{x^2+y^2}-x)\rho d\rho d\theta$$

$$= \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}(a\rho-\rho^2-\rho^2\cos\theta)d\rho d\theta = \frac{a^3}{24}\int_0^{\pi/2}\sec^4\frac{\theta}{2}d\theta = \frac{a^3}{9}.$$

Hence the entire volume is $v = \frac{2}{9}a^3$.

II. SOLUTION BY GEO. W. HARTWELL, Hamline University.

If this volume is sliced parallel to the xy plane, the sections between $z = 0$ and $z = a/2$ will be segments of circles and from $z = a/2$ to $z = a$ semicircles.

Integrating then we have

$$\int_0^{a/2}\left[\tfrac{1}{2}\pi(a-z)^2 + z\sqrt{a^2-2az} - (a-z)^2\cos^{-1}\frac{z}{a-z}\right]dz + \int_{a/2}^a \tfrac{1}{2}\pi(a-z)^2 dz$$

$$= \int_0^a \tfrac{1}{2}\pi(a-z)^2 dz + \int_0^{a/2}\left[z\sqrt{a^2-2az} - (a-z)^2\cos^{-1}\frac{z}{a-z}\right]dz.$$

Integrating we have

$$[-\tfrac{1}{6}\pi(a-z)^3]_{z=0}^{z=a} + \left[-\frac{(2a^2+6az)(a^2-2az)^{3/2}}{30a^2} + \frac{(a-z)^3}{3}\cos^{-1}\frac{z}{a-z} - \frac{a^2\sqrt{a^2-2az}}{3}\right.$$

$$\left. + \frac{(2a^2+2az)\sqrt{a^2-2az}}{9} - \frac{(2a^2+2az+3z^2)\sqrt{a^2-2az}}{45}\right]_{z=0}^{z=(a/2)} = \tfrac{2}{9}a^3.$$

Also solved by H. L. AGARD, FRANK R. MORRIS, NORMAN ANNING, H. S. UHLER, and the PROPOSER.

388. Proposed by PAUL CAPRON, U. S. Naval Academy.

If $f(x, y) = 0$ represents a curve having a simple tangency to the axis of x at the origin, the

value of $x^2/2y$, derived from $f(x, y) = 0$, and evaluated for $x = 0$, $y = 0$, will be the radius of curvature at the origin; or if the curve is similarly tangent to the y-axis at the origin, $y^2/2x$, evaluated for $x = 0$, $y = 0$, is the radius of curvature at the origin.

Solution by A. M. Harding, University of Arkansas.

The equation of any curve having simple tangency to the axis of x at the origin may be written in the form

$$f(x, y) = Ay + \tfrac{1}{2}Bx^2 + Cxy + \tfrac{1}{2}Dy^2 + \cdots = 0,$$

where $A, B, C, D, \cdots$ are constants and $A \neq 0$, $B \neq 0$.

The radius of curvature at any point is given by

$$r^2 = \frac{[f_x^2 + f_y^2]^3}{[f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2]^2}, \quad \text{where} \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \text{ etc}$$

When $x = 0$ and $y = 0$, we find

$$f_x = 0, \quad f_y = A, \quad f_{xx} = B, \quad f_{xy} = C, \quad f_{yy} = D.$$

Substituting, we obtain $r^2 = A^2/B^2$ or $r = A/B$. Dividing the given equation by B, we have

$$\frac{A}{B} + \frac{x^2}{2y} + Cx + \frac{Dy}{2B} + \cdots = 0.$$

Letting x and y approach zero,

$$\frac{A}{B} = -\left[\frac{x^2}{2y}\right]_{\substack{x=0\\y=0}}.$$

Hence,

$$r = -\left[\frac{x^2}{2y}\right]_{\substack{x=0\\y=0}}.$$

In the second case the method is the same and the equation has the form

$$f(x, y) = Ax + \tfrac{1}{2}Bx^2 + Cxy + \tfrac{1}{2}Dy^2 + \cdots = 0.$$

Also solved by Horace Olson in a special form and with incorrect interpretation of "simple tangency."

389. Proposed by FRANK R. MORRIS, Glendale, Calif.

A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it?

I. Solution by H. S. Uhler, Yale University.

Since the problem involves rectilinear motion at different speeds it is a question of refraction and can be solved at once by the methods of geometrical optics; for, by Fermat's principle, the time taken by light in going from the southeast to the northwest corner of the section will be either a minimum or a maximum (in this case, a minimum). The index of refraction of the medium outside the circle relative to the medium inside the circle is 4/3 (water and air, say). Hence, the "optical invariant" is

$$3 \sin i = 4 \sin r. \tag{1}$$

By hypothesis, (Fig. 1).

$\overline{OW} = \overline{WN} = 1$ mile. $\overline{CA} = \overline{CB} = \overline{CT} = 1/\pi$ miles, and $\overline{PC} = 120$ rods $= \frac{3}{8}$ mile.

The $\triangle ACB$ is isosceles so that $\angle CBA = \angle CAB \equiv i$. Hence, by equation (1) $\angle EBN = \angle DAO \equiv r$.

The projection of the broken line $OPCA$ on a diameter $\overline{FG}$ perpendicular to $\overline{OA}$ equals zero.

Hence,

$$\overline{IH} + \overline{HC} + \overline{CI} = 0; \qquad \overline{IH} = \overline{JP} = \overline{OP} \sin \phi = (1 - 1/\pi) \sin \phi;$$

$$\overline{HC} = - \overline{PC} \cos \phi = - \tfrac{3}{8} \cos \phi; \qquad \overline{CI} = - \overline{AC} \sin r = - 1/\pi \sin r.$$

Therefore,

$$8(\pi - 1) \sin \phi - 3\pi \cos \phi - 8 \sin r = 0. \tag{2}$$

In like manner, the projection of the broken line $NTCB$ on a diameter $\overline{KL}$ perpendicular to $\overline{NB}$ leads to

$$8 \sin (\phi + 2i - 2r) - 5\pi \cos (\phi + 2i - 2r) + 8 \sin r = 0. \tag{3}$$

Equations (1), (2), and (3) determine the acute angles ϕ, i, and r uniquely. Elimination of i and r would lead to a rational equation of degree 16 in $\tan \phi$. The coefficients would involve inconveniently high powers of π and the required root would have to be found by some method of *approximation*, such as Horner's. Consequently no mathematical self-respect is lost and an enormous amount of time is saved by employing the following process of approximation.

Assume a numerical value of ϕ and calculate r from equation (2). Next evaluate i from (1). Then substitute ϕ, i, and r in (3). Repeat until the left member of (3) vanishes. It will then be found that

$$\phi = 37° \ 33' \ 10''; \qquad i = 29° \ 40' \ 27''; \qquad r = 21° \ 47' \ 45''.$$

Let the rectangular coördinates of A and B be (x_1, y_1) and (x_2, y_2) respectively, where abscissas and ordinates are reckoned positive westward and northward in the order named.

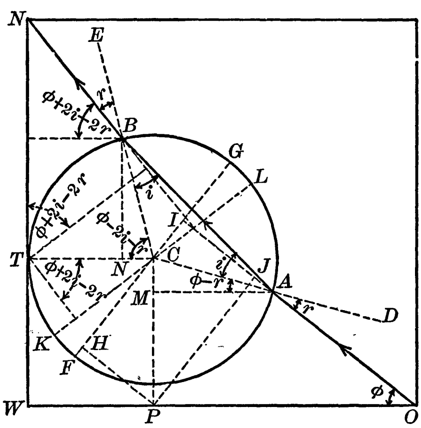


FIG. 1.

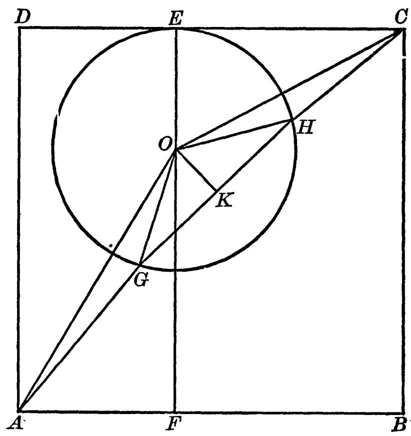


FIG. 2.

$$x_1 = \overline{OP} - \overline{AM} = \left(1 - \frac{1}{\pi}\right) - \frac{1}{\pi} \cos (\phi - r) = 0.375342 \text{ mile},$$

$$y_1 = x_1 \tan \phi = 0.288561 \text{ mile}.$$

$$l_1 \equiv \overline{OA} = x_1 \sec \phi = 0.473444 \text{ mile}, \qquad l_2 \equiv \overline{AB} = \frac{2}{\pi} \cos i = 0.553130 \text{ mile},$$

$$x_2 = \overline{OW} - (\overline{CT} - \overline{CN}) = 1 - \frac{1}{\pi} + \frac{1}{\pi} \cos (\phi + 2i - r) = 0.763511 \text{ mile}.$$

$$y_2 = \overline{PC} + \overline{NB} = \frac{3}{8} + \frac{1}{\pi} \sin (\phi + 2i - r) = 0.682615 \text{ mile},$$

$$l_3 \equiv \overline{BN} = \overline{NT} \sec (\phi + 2i - 2r) = 0.395803 \text{ mile}.$$

Hence,

$$l_1 + l_2 + l_3 = 1 \text{ mi. } 135 \text{ rd. } 0 \text{ yd. } 2 \text{ ft. } 8 \text{ in.},$$

$$t_1 = 9 \text{ min. } 28.13 \text{ sec.}, \qquad t_2 = 8 \text{ min. } 17.82 \text{ sec.}, \qquad t_3 = 7 \text{ min. } 54.96 \text{ sec.}$$

Hence,

$$t_1 + t_2 + t_3 = 25 \text{ min. } 40.9 \text{ sec.}$$

Finally, the numerical data satisfy the following check equations, obtained by projection of the path on the coördinate axes,

$$l_1 \cos \phi + l_2 \cos (\phi + i - r) + l_3 \cos (\phi + 2i - 2r) = 1,$$

$$l_1 \sin \phi + l_2 \sin (\phi + i - r) + l_3 \sin (\phi + 2i - 2r) = 1.$$

II. Solution by A. H. Holmes, Brunswick, Maine.

Let $ABCD$ (Fig. 2) be the section of land, 1 mile square, A being the southeast corner, B the northeast corner, C the northwest corner, and D the southwest corner. On DC, take $DE = 120$ rods or $\frac{3}{8}$ of a mile and draw EF perpendicular to AB at F. On EF, take $EO = 1/\pi$ of a mile to point O.

With O as a center and EO as radius describe a circle. Draw AO and CO. Let AG be the course to the circumference of track, GH the course inside the track, and HC the remainder of the course to the northwest corner C.

Draw OK perpendicular to GH at K. The ratio of GH to $AG + CH$ is the greatest when OK bisects the angle AOC. Therefore, for a minimum, the angles AOG and COH are equal.

Put $AG = x$, $CH = y$, $GH = z$, $AO = a$, $CO = b$, $1/\pi = r$, $AOG = COH = \theta$, and $AOC = 2m$.

Then

$$\frac{x+y}{3} + \frac{z}{4} = \text{minimum}.$$

In $\triangle AGO$ and CHO

$$\cos \theta = \frac{a^2 + r^2 - x^2}{2ar}, \qquad \cos \theta = \frac{b^2 + r^2 - y^2}{2br}.$$

Hence $x = \sqrt{a^2 + r^2 - 2ar \cos \theta}$, $y = \sqrt{b^2 + r^2 - 2br \cos \theta}$, and $z = 2r \sin (m - \theta)$ in which $a = .778027$, $b = .701390$, $r = .318310$, and $m = 72° \ 54' \ 9''$.

Multiplying x and y by 4 and z by 3, reducing and differentiating,

$$\frac{2a \sin \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} + \frac{2b \sin \theta}{\sqrt{b^2 + r^2 - 2br \cos \theta}} = 3 \cos (m - \theta).$$

Whence, $\theta = 12° \ 33' \ 9''$.

Knowing θ, the required course in distances and directions is easily found.

From A to G, N. 52° 45′ 48″ W. 151.17 rods; G to H, N. 44° 5′ 13″ W. 177.03 rods; H to C, N. 37° 1′ 51″ W. 126.96 rods, the length of the whole course being 455.16 rods, and made in the minimum time of 25 minutes 40.8 seconds.

MECHANICS.

303. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A pile driver weighing 500 pounds falls through 10 feet and drives a pile weighing 400 pounds 3 inches into the ground. Show that the average force of the blow is $11{,}111\frac{1}{9}$ pounds.

Solution by W. H. Williams, Oakland, California.

Let m = mass of driver in pounds, m' = mass of pile in pounds, v = velocity of driver at instant of impact in feet per second, h = height of fall of driver in feet, d = distance pile is driven into ground in feet, r = average retardation of pile and driver after impact in feet per second per second, t = time from impact until rest in seconds, and g = acceleration due to gravity in feet per second per second, F = average force of blow.

Then mv = momentum of driver at impact = momentum of driver + pile immediately after impact; $mv/(m + m')$ = common velocity of driver and pile immediately after impact.

Hence,

$$\frac{mv}{(m + m')r} = t, \qquad \text{and} \qquad d = \tfrac{1}{2}rt^2 = \tfrac{1}{2}r\frac{m^2v^2}{(m + m')^2r^2} = \tfrac{1}{2}\frac{m^2v^2}{(m + m')^2r},$$

or

$$r = \tfrac{1}{2}\frac{m^2v^2}{(m+m')^2d}.$$

But $v^2 = 2gh$, and hence,

$$r = \frac{m^2gh}{(m+m')^2d}.$$

Finally,

$$F = (m+m')r = \frac{m^2gh}{(m+m')d}\text{ poundals} = \frac{m^2h}{(m+m')d}\text{ pounds.}$$

Substituting values,

$$F = \frac{500\times 500\times 10}{(500+400)\times\frac{1}{4}} = 11{,}111\tfrac{1}{9}\text{ lbs.}$$

Also solved by A. M. HARDING and J. A. CAPARO. Erroneously solved by HERBERT N. CARLETON.

305. Proposed by B. J. BROWN, Victor, Colorado.

A particle is to be projected so as to graze the top of a wall h feet high, at a distance of a feet from the point of projection, and to strike the ground at a distance b feet from the foot of the wall. Find the velocity of projection, and the inclination of the path to the horizontal, at the ground and at the top of the wall. I. C. S. 1903.

SOLUTION BY EMMA M. GIBSON, Assistant at Drury College.

The equation of the path of the particle is

$$y = x\tan\alpha - \tfrac{1}{2}g\frac{x^2}{\nu_0^2\cos^2\alpha},$$

where ν_0 is the initial velocity and α the angle of projection.

Knowing the points (a, h), $(a+b, 0)$ to be on the curve the following satisfied relations exist:

$$h = a\tan\alpha - \tfrac{1}{2}g\frac{a^2}{\nu_0^2\cos^2\alpha} \tag{1}$$

and

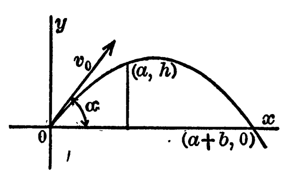
$$(a+b)\tan\alpha = \tfrac{1}{2}g\frac{(a+b)^2}{\nu_0^2\cos^2\alpha} = 0. \tag{2}$$

From equation (2),

$$\nu_0^2 = \frac{(a+b)g}{2\sin\alpha\cos\alpha}, \quad\text{or}\quad \nu_0 = \sqrt{\frac{(a+b)g}{\sin 2\alpha}}.$$

Substituting this value of ν_0 in (1) and solving for $\tan\alpha$,

$$\tan\alpha = \frac{(a+b)h}{ab}.$$



Hence,

$$\alpha = \tan^{-1}\left[\frac{(a+b)h}{ab}\right],$$

the inclination of the path at the ground.

The equation of the tangent to the curve at any point (x_1, y_1) is

$$\frac{y+y_1}{2} = \frac{x+x_1}{2}\tan\alpha - \tfrac{1}{2}g\frac{x_1x}{\nu_0^2\cos^2\alpha}.$$

At (a, h) the tangent becomes

$$y+h = (x+a)\tan\alpha - \frac{agx}{\nu_0^2\cos\alpha} = \left[\frac{\nu_0^2\sin\alpha\cos\alpha - ag}{\nu_0^2\cos^2\alpha}\right]x + a\tan\alpha.$$

The inclination of this line,

$$\tan^{-1}\left[\frac{\nu_0^2\sin\alpha\cos\alpha - ag}{\nu^2\cos^2\alpha}\right] = \tan^{-1}\left[\left(\frac{b-a}{ab}\right)h\right],$$

is the inclination of the path to the horizontal at the top of the wall.

Also solved by H. C. FEEMSTER, J. A. CAPARO, CLIFFORD N. MILLS, and HORACE OLSON. Erroneously solved by HERBERT N. CARLETON.

306. Proposed by EMMA M. GIBSON, Drury College.

A sphere is composed of a solid homogeneous hemisphere and a very thin hemispherical shell of equal mass. What is the greatest inclination of a rough plane on which the sphere can just rest in equilibrium?

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Let the radius of the sphere be a, the center at O, the center of gravity at G; then

$$OG = \frac{1}{2}\left(\frac{a}{2} - \frac{3a}{8}\right) = \frac{a}{16};$$

however the sphere is placed, G is on a circumference of radius $a/16$, centered at O. If the sphere is in equilibrium, G is vertically over the contact, C, of the sphere with the inclined plane on which it rests; the inclination of the plane is the angle COG, which has its greatest value when CG is tangent to the little circle; this value is $\sin^{-1} 1/16$. Hence, the greatest inclination of the plane is $\sin^{-1} 1/16$ or $\tan^{-1} \mu$ (μ being the coefficient of friction) whichever happens to be the less.

Also solved by J. A. CAPARO. Erroneously solved by HERBERT N. CARLETON.

NUMBER THEORY.

226. Proposed by ELBERT H. CLARKE, Purdue University.

If $0!$ is taken equal to 1, and if k is any positive integer greater than or equal to 2, show that

$$\sum_{n=0}^{\infty} \frac{n!}{(k+n)!} = \frac{1}{k-1} \cdot \frac{1}{(k-1)!}.$$

I. SOLUTION BY S. A. JOFFE, New York City.

Since

$$\frac{n!}{(k+n)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n(n+1)(n+2) \cdots (n+k)} = \frac{1}{(n+1)(n+2) \cdots (n+k)},$$

we see that

$$\sum_{n=0}^{\infty} \frac{n!}{(k+n)!} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2) \cdots (n+k)}. \tag{1}$$

Now, the series in the second member, each term of which is the reciprocal of a factorial expression, may be easily summed by an elementary theorem in finite differences or by the following equivalent simple method.

Noticing that

$$\frac{1}{(n+1)(n+2) \cdots (n+k-1)} - \frac{1}{(n+2)(n+3) \cdots (n+k)}$$
$$= \frac{(n+k)-(n+1)}{(n+1)(n+2) \cdots (n+k)} = \frac{k-1}{(n+1)(n+2) \cdots (n+k)},$$

we find that

$$\frac{1}{(n+1)(n+2) \cdots (n+k)}$$
$$= \frac{1}{k-1}\left[\frac{1}{(n+1)(n+2) \cdots (n+k-1)} - \frac{1}{(n+2)(n+3) \cdots (n+k)}\right].$$

Making in the last equation n equal successively to 0, 1, 2, $\cdots$, we get

$$\frac{1}{1\cdot 2\cdots k}=\frac{1}{k-1}\left[\frac{1}{1\cdot 2\cdots (k-1)}-\frac{1}{2\cdot 3\cdots k}\right],$$

$$\frac{1}{2\cdot 3\cdots (k+1)}=\frac{1}{k-1}\left[\frac{1}{2\cdot 3\cdots k}-\frac{1}{3\cdot 4\cdots (k+1)}\right],$$

$$\frac{1}{3\cdot 4\cdots (k+2)}=\frac{1}{k-1}\left[\frac{1}{3\cdot 4\cdots (k+1)}-\frac{1}{4\cdot 5\cdots (k+2)}\right],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

the first of which is true only if $k-1$ is at least equal to 1, that is $k \geqq 2$.

If we add these equations and observe that the second term in the brackets in each equation cancels the first term in the following equation, so that in the second member there will remain only the first term of the first equation, we shall obtain

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)\cdots(n+k)}=\frac{1}{k-1}\cdot\frac{1}{1\cdot 2\cdots (k-1)}=\frac{1}{k-1}\cdot\frac{1}{(k-1)!},$$

and in view of equation (1) this becomes

$$\sum_{n=0}^{\infty}\frac{n!}{(k+n)!}=\frac{1}{k-1}\cdot\frac{1}{(k-1)!},$$

as required.

II. Solution by L. C. Mathewson, Dartmouth College.

The statement to be proved may be put into the following form:

$$\frac{1}{k-1}\cdot\frac{1}{(k-1)!}=\frac{0!}{k!}+\frac{1!}{(k+1)!}+\frac{2!}{(k+2)!}+\cdots+\frac{n!}{(k+n)!}+\cdots. \tag{1}$$

In the first place, if k is any finite positive integer, the series in the second member is a convergent series as may be easily shown by Gauss's test;[1] for, the ratio of the $(n+1)$th term to the $(n+2)$th is

$$\frac{\alpha_{n+1}}{\alpha_{n+2}}=\frac{n+(k+1)}{n+1},$$

and since by hypothesis $k>1$, $(k+1)-1>1$, and the series is convergent, and absolutely so.

Assuming that any positive rational integer can be attained by successive additions of unity, a proof of the given formula will be made by use of mathematical induction.

If $k=2$, there results

$$\begin{aligned}1&=\frac{0!}{2!}+\frac{1!}{3!}+\frac{2!}{4!}+\cdots+\frac{n!}{(n+2)!}+\cdots\\&=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots\\&=(\tfrac{1}{1}-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\cdots\\&=1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\cdots-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}-\cdots,\end{aligned} \tag{2}$$

since the commutative law holds for an absolutely convergent series.[2] From this the sum of the first $n+1$ terms of the series is $1-1/(n+2)$, which approaches the limit 1 as n becomes infinite. Hence, the formula is true for $k=2$.[3]

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[1] Pierpont, *Theory of Functions of Real Variables*, volume 2, article 97.

[2] Pierpont, *loc. cit.*, article 110, 2.

[3] Cf. Chrystal, *Algebra*, Part II, page 120, footnote.

Now assume the given statement true if $k = r$; *i. e.*, that

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!} = \frac{0!}{r!} + \frac{1!}{(r+1)!} + \frac{2!}{(r+2)!} + \cdots + \frac{n!}{(r+n)!} + \cdots. \tag{3}$$

Transposing the first term of the second member of (3),

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!} - \frac{0!}{r!} = \frac{1!}{(r+1)!} + \frac{2!}{(r+2)!} + \cdots + \frac{r!}{(r+n)!} + \cdots, \tag{4}$$

where the second member is an absolutely convergent series. But

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!} - \frac{0!}{r!} = \frac{\frac{r}{r-1} - 1}{r!} = \frac{1}{r!(r-1)}.$$

Accordingly, (4) becomes

$$\frac{1}{r!(r-1)} = \frac{1!}{(r+1)!} + \frac{2!}{(r+2)!} + \frac{3!}{(r+3)!} + \cdots + \frac{n!}{(r+n)!} + \frac{(n+1)!}{(r+n+1)!} + \cdots. \tag{5}$$

Subtracting (5) from (3) gives, since the series are convergent,[1]

$$\begin{aligned}\frac{1}{r-1}\cdot\frac{1}{(r-1)!} - \frac{1}{r!(r-1)} &= \left(\frac{0!}{r!} - \frac{1!}{(r+1)!}\right) + \left(\frac{1!}{(r+1)!} - \frac{2!}{(r+2)!}\right) \\ &\quad + \left(\frac{2!}{(r+2)!} - \frac{3!}{(r+3)!}\right) + \cdots + \left(\frac{n!}{(r+n)!} - \frac{(n+1)!}{(r+n+1)!}\right) + \\ &= \frac{1!r}{(r+1)!} + \frac{1!(2+r)-2!}{(r+2)!} + \frac{2!(3+r)-3!}{(r+3)!} + \cdots + \frac{n!(n+1+r)-(n+1)!}{(r+n+1)!} + \cdots \\ &= r\left\{\frac{0!}{(r+1)!} + \frac{1!}{(r+2)!} + \frac{2!}{(r+3)!} + \cdots + \frac{n!}{(r+n+1)!} + \cdots\right\}. \end{aligned} \tag{6}$$

Now the first member of (6) may be simplified:

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!} - \frac{1}{r!(r-1)} = \frac{r-1}{r!(r-1)} = \frac{1}{r!}.$$

Accordingly, dividing each member of (6) by r,

$$\frac{1}{r!r} = \frac{0!}{(r+1)!} + \frac{1!}{(r+2)!} + \frac{2!}{(r+3)!} + \cdots + \frac{n!}{(r+n+1)!} + \cdots.$$

But this is exactly what (1) becomes if $r + 1$ is substituted for k. Hence, if (1) is true for a particular integral value of k ($k > 1$), then it is true for k equal to the next greater integer. But (1) was shown true if $k = 2$, accordingly, it is true for $k = 3$; if for $k = 3$, then for $k = 4$; etc.

Also solved by HORACE OLSON, FRANK IRWIN, and ELIJAH SWIFT.

[1] Pierpont, *loc. cit.*, article 111.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

I. RELATING TO THE INTERPRETATION OF THE DEGENERATE CONICS.

BY H. W. BRINKMANN, Student in the Polytechnic High School, Riverside, Calif.

In textbooks on analytic geometry the degenerate cases which appear on analyzing the general equation of the second degree are usually treated from the algebraic side alone. It is easily shown, however, that all the degenerate cases can be made to fit precisely into the definitions of the proper conics.

Consider first the case of two intersecting lines. These can be interpreted as a hyperbola as follows. If we take the bisector of one angle between them, the ratio of the distance from any point on either line to their point of intersection and the distance to the bisector is constant, viz., the secant of one half the angle between the lines. In other words, the lines can be called an hyperbola whose focus is the intersection, whose directrix is the bisector of the angle between the lines, and whose eccentricity is the secant of one half that angle. Since the secant of an angle can never be less than unity, it is clear that the lines can represent only an hyperbola.

Evidently the bisector of the second angle could be taken as directrix with the same focus. Thus we see that the pair of lines are really two coincident, conjugate hyperbolas.

The second degenerate case, namely a point, is most easily interpreted as an ellipse whose semi-axes are zero. For, if the point is also regarded as the coincident foci of the ellipse, it is clear that the sum of the distances from the point to the foci is constant, namely zero. Therefore the point is an ellipse whose semi-axes are zero.

The only degenerate case that remains is that of two parallel lines. If these are coincident, it evidently represents a parabola whose directrix is any line perpendicular to it and whose focus is their point of intersection, for then the distance from any point on this line to the directrix equals the distance from that point to the focus.

If the lines are non-coincident they are best interpreted as the limiting case of a pair of intersecting lines, if the point of intersection moves indefinitely to one side in the plane. The angle between those lines is then zero and its secant is unity. Hence the hyperbola (intersecting lines) has degenerated into a parabola, for its eccentricity is unity.

It may also be noticed that for the equation of a conic where the focus lies on the directrix we get

$$x^2(1 - e^2) + y^2 = 0$$

where e is the eccentricity. For, taking the directrix as the y-axis and the focus

as the origin, we have $\sqrt{x^2 + y^2}$ as the distance from any point (x, y) to the focus and x as the distance from that point to the directrix. Hence, if the point is on the locus,

$$\sqrt{x^2 + y^2} = ex$$

and squaring we get the required equation.

For $e > 1$, $e = 1$, $e < 1$, this reduces to the equations of two intersecting lines, two coincident lines and a point respectively.

II. TWO USEFUL RELATIONS IN TRIGONOMETRY.

By ALBERT BABBITT, University of Minnesota.

If ABC be any triangle, and if θ be the angle which a straight line CP (Fig. 1) drawn from the vertex C makes with the base of the triangle, and if we let $\angle\, PCA = \alpha$, $\angle\, BCP = \beta$, $AP = m$, $PB = n$, we then have

$$(m + n) \cot \theta = m \cot \alpha - n \cot \beta.$$

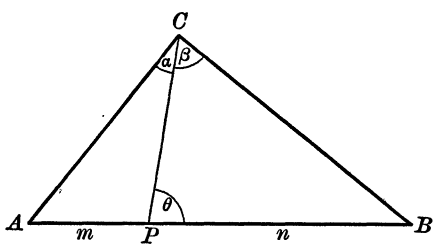

FIG. 1.

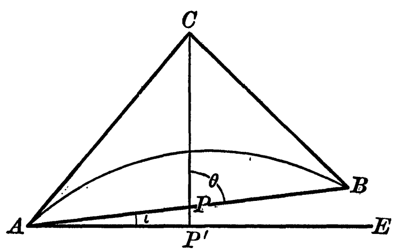

FIG. 2.

The proof of this trigonometric relation is very simple.[1] From $\triangle ACP$, we have

$$\frac{CP}{m} = \frac{\sin A}{\sin \alpha},$$

hence,

$$CP = \frac{m \sin A}{\sin \alpha} = \frac{m \sin (\theta - \alpha)}{\sin \alpha} = m (\sin \theta \cot \alpha - \cos \theta). \tag{1}$$

Similarly, from $\triangle PCB$, we have

$$CP = \frac{n \sin B}{\sin \beta} = \frac{n \sin (\theta + \beta)}{\sin \beta} = n (\sin \theta \cot \beta + \cos \theta). \tag{2}$$

From (1) and (2) it follows that

$$m(\sin \theta \cot \alpha - \cos \theta) = n(\sin \theta \cot \beta + \cos \theta).$$

Simplifying and dividing through by $\sin \theta$ (since $\sin \theta \neq 0$), we get

$$(m + n) \cot \theta = m \cot \alpha - n \cot \beta.$$

[1] Other proofs are given in, for instance, Loney's *Elements of Statics and Dynamics*, tenth edition (1896), page 92. EDITOR.

Using the same figure and notation, we can derive in a similar way the relation,

$$(m + n) \cot \theta = n \cot A - m \cot B.$$

These trigonometric relations facilitate the solution of some problems in physics.

For example, having given the direction and velocity of projection, let it be required to find the velocity with which a projectile would strike an oblique plane.

If we denote by i the inclination of the plane to the horizon (Fig. 2), and apply the second relation above derived, noting that tangents at two points A and B of a parabola meet on the diameter bisecting AB, we get

$$\cot A - \cot B = 2 \tan i.$$

Hence, we determine the angle B, since the angles A and i are known. But, if AC and BC are tangents at two points A and B on a parabolic trajectory, and V_1 and V_2 the corresponding velocities, then

$$\frac{V_1}{V_2} = \frac{\sin B}{\sin A} \quad \text{and} \quad V_2 = \frac{V_1 \sin A}{\sin B},$$

and the required velocity is determined.

THE CHARTER MEMBERSHIP OF THE ASSOCIATION.

The By-Laws of the Association provided that those who should be elected to membership before April 1, 1916, should constitute the list of Charter Members. Owing to circumstances beyond our control the issue of the Monthly containing the report of the Columbus meeting and the announcement of the conditions of membership could not be distributed to the public till well into February. Similar delays of subsequent issues likewise prevented the prompt distribution of other information, so that the great flood of applications came late in March, thus making it impossible to complete the report until the present time.

The following is a preliminary report of the charter membership giving only the name of each member and the institution represented. As soon as possible there will be published a complete directory, giving the official position and mailing address of each member. Meanwhile, we invite careful scrutiny of the present list in order to detect any omissions or errors which may have escaped the attention of the secretary in the arduous task of checking up the returns.

It was assumed that all who joined in the call for the Columbus meeting, or took part in that meeting, would consider themselves eligible to charter membership, and the executive committee of the Council so ruled. From eighteen of these five hundred persons we had not, on going to press, received any expression of their intention, but we are provisionally including their names in the list below, pending the receipt of such information.

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J. F. Messick, Alabama Polytechnic Institute.
H. A. Sayre, University of Alabama.
B. L. Shi, Alabama Polytechnic Institute.

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G. H. Cresse, University of Arizona.
C. O. Lampland, Lowell Observatory.
H. B. Leonard, University of Arizona.

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R. B. McClenon, Grinnell College.
F. M. McGaw, Cornell College.
Nina Madson, State College of Agriculture.
I. F. Neff, Drake University.
Mrs. Mary B. Norton, Cornell College.
E. A. Pattengill, State College of Agriculture.
J. F. Reilly, State University of Iowa.
Maria M. Roberts, State College of Agriculture.
J. E. Robertson, Highland Park College.
W. J. Rusk, Grinnell College.
B. F. Simonson, Upper Iowa University.
A. G. Smith, State University of Iowa.
S. G. Stein, Muscatine.
E. H. Thomas, Tabor College.
E. L. Thompson, Burlington High School.
R. N. Van Horne, Sioux City.
C. W. Wester, State University of Iowa.
John Zimmerman, Dubuque German College and Seminary.

KANSAS.

W. H. Andrews, State Agricultural College.
C. H. Ashton, University of Kansas.
Talmon Bell, Cooper College.
T. L. Bouse, Campbell College.
M. R. Bowerman, State Agricultural College.
V. B. Caris, State Manual Training Normal.
Lucy T. Dougherty, Kansas City High School.
Ottilia W. Dueker, Friends University.
Elizabeth G. Flagg, Kansas City High School.
W. H. Garrett, Baker University.
P. W. Harnly, University of Kansas.
W. A. Harshbarger, Washburn College.
A. J. Hoare, Fairmount College.
K. J. Holzinger, University of Kansas.
Emma Hyde, Kansas City High School.
Jessie M. Jacobs, University of Kansas.
H. E. Jordan, University of Kansas.
A. W. Larsen, University of Kansas.
Solomon Lefschetz, University of Kansas.
Theodore Lindquist, State Normal School.
T. E. Mergendahl, College of Emporia.
U. G. Mitchell, University of Kansas.
C. A. Nelson, University of Kansas.
Mary W. Newson, Washburn College.
H. N. Olson, Bethany College.
H. E. Porter, State Agricultural College.
B. L. Remick, State Agricultural College.
D. H. Richert, Bethel College.

J. A. G. Shirk, State Manual Training School.
E. M. Stahl, Midland College.
L. L. Steimley, University of Kansas.
E. B. Stouffer, University of Kansas.
W. T. Stratton, State Agricultural College.
J. N. Van der Vries, University of Kansas.
J. J. Wheeler, University of Kansas.
A. E. White, State Agricultural College.
Ella Woodyard, Kansas City High School.

KENTUCKY.

P. P. Boyd, University of Kentucky.
J. M. Davis, University of Kentucky.
J. E. Dotterer, University of Kentucky.
Henry Lloyd, Transylvania College.
H. R. Phalen, Berea College.
E. L. Rees, University of Kentucky.
A. L. Rhoton, Georgetown College.

LOUISIANA.

A. Bogard, Shreveport High School.
J. T. Cater, Straight College.
A. B. Dinwiddie, Tulane University.
P. T. Hedges, Louisiana State Normal.
J. W. Nicholson, University of Louisiana.
Mary C. Spencer, Newcomb College.

MAINE.

A. S. Adams, Edward Little High School, Auburn.
T. B. Ashcraft, Colby College.
B. E. Carter, Colby College.
J. A. Colson, National Bank, Searsport.
M. T. Goodrich, High School, Bingham.
J. H. Hart, University of Maine.
A. H. Holmes, Brunswick.
Alice M. Lord, Portland High School.
W. E. Milne, Bowdoin College.
W. A. Moody, Bowdoin College.
G. E. Ramsdell, Bates College.
L. J. Reed, University of Maine.
Fannie H. Robinson, Bangor High School.
H. E. Trefethen, Colby College.

MARYLAND.

Clara L. Bacon, Goucher College.
Harry Bateman, Johns Hopkins University.
Lillian O. Brown, Hood College.
Paul Capron, U. S. Naval Academy.
A. B. Coble, Johns Hopkins University.
Abraham Cohen, Johns Hopkins University.
H. A. Converse, Baltimore Polytechnic Institute.
W. C. Eells, U. S. Naval Academy.
J. B. Eppes, U. S. Naval Academy.
J. N. Galloway, Baltimore Polytechnic Institute.
Angelo Hall, U. S. Naval Academy.
S. C. Harry, Friends School, Baltimore.
A. W. Hobbs, Baltimore.
L. S. Hulburt, Johns Hopkins University.
W. W. Johnson, U. S. Naval Academy.
Florence P. Lewis, Goucher College.
Frank Morley, Johns Hopkins University.
E. B. Morrow, Gilman Country School, Baltimore.
J. R. Musselman, Johns Hopkins University.
S. F. Norris, Baltimore City College.
C. H. Rawlins, Jr., Baltimore.
H. M. Robert, Jr., Baltimore.
R. E. Root, U. S. Naval Academy.
W. F. Shenton, Johns Hopkins University.
E. R. Smith, The Park School, Baltimore.
C. P. Sousley, Mount St. Mary's College.

MASSACHUSETTS.

H. L. Agard, Williams College.
F. H. Bailey, Massachusetts Institute of Technology.
Ida Barney, Smith College.
Ralph Beatley, Milton Academy.
Susan R. Benedict, Smith College.
G. D. Birkhoff, Harvard University.
C. L. Bouton, Harvard University.
H. C. Bradley, Massachusetts Institute of Technology.
L. A. Brigham, Boston University.
R. E. Bruce, Boston University.
E. S. Bryant, Everett High School.
J. A. Bullard, Worcester Polytechnic Institute.
A. D. Butterfield, Worcester Polytechnic Institute.
Eva Chandler, Wellesley College.
J. E. Clark, Springfield.
L. L. Conant, Worcester Polytechnic Institute.
M. Imogene Cook, Walnut Hill School, Natick.
J. L. Coolidge, Harvard University.
Lennie P. Copeland, Wellesley College.
H. N. Davis, Harvard University.
C. R. Duncan, Massachusetts Agricultural College.
T. C. Esty, Amherst College.
W. C. Esty, Amherst College.
G. W. Evans, Charlestown High School, Boston.
F. C. Ferry, Williams College.
H. D. Gaylord, Browne and Nichols School, Cambridge.
G. M. Green, Harvard University.
C. E. Haigler, Wentworth Institute, Boston.
J. G. Hardy, Williams College.
Olive C. Hazlett, Cambridge.
C. A. Hobbs, Watertown.
E. V. Huntington, Harvard University.
Dunham Jackson, Harvard University.
Ervin Kenison, Massachusetts Institute of Technology.
A. E. Kennelly, Harvard University.
Carl King, Wentworth Institute, Boston.
Edward Kircher, Harvard University.
B. B. Libby, Massachusetts Institute of Technology.
Joseph Lipka, Massachusetts Institute of Technology.
U. J. Lupien, Lowell Textile School.
Emilie N. Martin, Mount Holyoke College.
Helen A. Merrill, Wellesley College.
C. L. E. Moore, Massachusetts Institute of Technology.
M. M. S. Moriarty, Holyoke High School.
R. K. Morley, Worcester Polytechnic Institute.
Florence L. Munroe, Northampton High School.
C. E. Norwood, Dartmouth College.
G. D. Olds, Amherst College.

W. F. Osgood, Harvard University.
Anna J. Pell, Mount Holyoke College.
H. B. Phillips, Massachusetts Institute of Technology.
Gertrude E. Preston, Dana Hall School, Wellesley.
W. R. Ransom, Tufts College.
C. N. Reynolds, Jr., Cambridge.
George Rutledge, Massachusetts Institute of Technology.
Clara E. Smith, Wellesley College.
Sarah E. Smith, Mount Holyoke College.
J. J. Sullivan, Jr., Roxbury.
H. W. Tyler, Massachusetts Institute of Technology.
Roxana H. Vivian, Wellesley College.
G. L. Wagar, Mount Hermon Boys' School.
A. C. Washburne, Berkshire Life Insurance Company, Pittsfield.
George Wentworth, Brookline.
A. H. Wheeler, High School of Commerce, Worcester.
F. B. Williams, Clark University.
E. B. Wilson, Massachusetts Institute of Technology.
F. S. Woods, Massachusetts Institute of Technology.
Euphemia R. Worthington, Wellesley College.
W. C. Wright, Actuary, Medford.
Mabel M. Young, Wellesley College.

MICHIGAN.

J. W. Baldwin, University of Michigan.
W. W. Beman, University of Michigan.
J. W. Bradshaw, University of Michigan.
W. H. Butts, University of Michigan.
L. C. Emmons, Michigan Agricultural College.
A. G. Erickson, Michigan State Normal College.
E. B. Escott, Peninsular Life Insurance Company, Detroit.
J. P. Everett, Western State Normal School.
Peter Field, University of Michigan.
W. B. Ford, University of Michigan.
C. H. Forsyth, University of Michigan.
W. Van N. Garretson, University of Michigan.
J. W. Glover, University of Michigan.
E. D. Grant, Michigan College of Mines.
C. L. Herron, Hillsdale College.
T. H. Hildebrandt, University of Michigan.
W. J. Hussey, University of Michigan.
L. C. Karpinski, University of Michigan.
W. W. Küstermann, University of Michigan.
E. A. Lyman, State Normal College.
J. L. Markley, University of Michigan.
G. R. Mirick, University of Michigan.
Alfred Nelson, University of Michigan.
F. N. Notestein, Alma College.
L. C. Plant, Michigan Agricultural College.
R. H. Reece, Michigan Agricultural College.
Wm. Rinck, Calvin College.
T. R. Running, University of Michigan.
E. R. Sleight, Albion College.
G. G. Speeker, Michigan Agricultural College.
C. C. Spooner, Northern Normal School.
W. J. Thome, University of Detroit.
M. O. Tripp, Olivet College.
Mary L. Welton, Union High School, Grand Rapids.
Marion B. White, Michigan State Normal College.
C. B. Williams, Kalamazoo College.
Alexander Ziwet, University of Michigan.

MINNESOTA.

Albert Babbitt, University of Minnesota.
G. N. Bauer, University of Minnesota.
W. O. Beal, University of Minnesota.
Edla G. Berger, College of St. Catherine.
W. E. Brooke, University of Minnesota.
W. H. Bussey, University of Minnesota.
H. H. Dalaker, University of Minnesota.
C. H. Gingrich, Carleton College.
G. W. Hartwell, Hamline University.
A. F. Kovarik, University of Minnesota.
L. E. Lunn, School Superintendent, Heron Lake.
B. L. Newkirk, University of Minnesota.
C. A. V. Peterson, Minneapolis.
W. D. Reeve, University of Minnesota.
R. R. Shumway, University of Minnesota.
H. L. Slobin, University of Minnesota.
J. F. Taylor, Central High School, Duluth.
A. L. Underhill, University of Minnesota.

MISSISSIPPI.

Alfred Hume, University of Mississippi.
J. M. Sharp, Mississippi College.
B. M. Walker, Agricultural and Mechanical College.

MISSOURI.

L. D. Ames, University of Missouri.
Charles Ammerman, McKinley High School, St. Louis.
A. C. Andrews, Manual Training High School, Kansas City.
C. J. Borgmeyer, St. Louis University.
M. S. Brennan, St. Louis.
Dorothy G. Calman, St. Louis.
A. D. Campbell, Washington University.
E. F. Canaday, Columbia.
Byron Cosby, State Normal School, Kirksville.
Otto Dunkel, University of Missouri.
C. A. Epperson, First District Normal School.
Zoe Ferguson, St. Joseph Junior College.
B. F. Finkel, Drury College.
R. R. Fleet, William Jewell College.
G. C. Forsman, Central High School, St. Louis.
Emma M. Gibson, Drury College.
E. R. Hedrick, University of Missouri.
C. G. Hinrichs, Consulting Chemist, St. Louis.
Louise H. Huff, McKinley High School, St. Louis.
Jewell C. Hughes, Columbia.
A. H. Huntington, Central High School, St. Louis.
Byron Ingold, Christian University.
Louis Ingold, University of Missouri.
T. W. Jackson, Fulton High School.
John James, Synodical College.

G. H. Jamison, First District Normal School.
J. R. Jenison, Tarkio College.
B. F. Johnson, State Normal School, Cape Girardeau.
Stella Johnson, Edina High School.
O. D. Kellogg, University of Missouri.
J. M. Kent, Manual Training High School, Kansas City.
Lyda Long, Cleveland High School, St. Louis.
W. A. Luby, Northeast High School, Kansas City.
A. R. Nauer, Mechanical Engineer, St. Louis.
Randolph Patton, Columbia.
J. C. Rayworth, Washington University.
W. H. Roever, Washington University.
W. G. Rowe, Smith Academy Manual Training School.
J. H. Scarborough, State Normal School, Warrensburg.
A. J. Schwartz, Grover Cleveland High School, St. Louis.
J. I. Shannon, St. Louis University.
I. C. Smith, Columbia.
H. P. Stellwagen, Yeatman High School, St. Louis.
F. C. Touton, St. Joseph Junior College.
F. W. Urban, State Normal School, Warrensburg.
C. A. Waldo, Washington University.
J. A. Waldron, Chaminade College.
Eula A. Weeks, Cleveland High School, St. Louis.
R. A. Wells, Park College.
W. D. A. Westfall, University of Missouri.
Rose B. Wood, Hardin College.
W. H. Zeigel, First District Normal School.

MONTANA.

H. F. Calderwood, Glasgow.
E. F. A. Carey, University of Montana.
N. J. Lennes, University of Montana.

NEBRASKA.

J. N. Bennett, Doane College.
Henry Blumberg, University of Nebraska.
W. C. Brenke, University of Nebraska.
G. R. Chatburn, University of Nebraska.
E. W. Davis, University of Nebraska.
R. I. Elliott, State Normal School, Kearney.
H. C. Feemster, York College.
T. J. Fitzpatrick, Bethany.
Ellen H. Frankish, Omaha High School.
J. M. Howie, State Normal School.
Mayme I. Logsdon, Hastings College.
L. E. Pratt, Tecumseh.
W. M. Reeves, Cotner University.
Oscar Schmiedel, Bellevue College.
L. C. Walker, Ceresco.

NEVADA.

Chas. Haseman, University of Nevada.
J. A. Nyswander, University of Nevada.

NEW HAMPSHIRE.

R. D. Beetle, Dartmouth College.
E. G. Bill, Dartmouth College.
C. R. Dines, Dartmouth College.
G. I. Hopkins, Manchester, High School.
L. C. Mathewson, Dartmouth College.
F. C. Moore, New Hampshire College.
F. M. Morgan, Dartmouth College.
C. C. Steck, New Hampshire State College.
H. L. Sweet, Phillips Exeter Academy.
J. W. Young, Dartmouth College.

NEW JERSEY.

E. P. Adams, Princeton University.
A. A. Bennett, Princeton University.
G. A. Bingley, Princeton.
Sister Blanche Marie, College of St. Elizabeth.
J. W. Colliton, Trenton High School.
L. S. Dederick, Princeton University.
Fletcher Durell, Lawrenceville School.
L. P. Eisenhart, Princeton University.
H. B. Fine, Princeton University.
C. O. Gunther, Stevens Institute of Technology.
William Kent, Consulting Engineer, Montclair.
I. E. Kline, Atlantic City High School.
R. W. Lord, Plainfield High School.
L. A. Martin, Jr., Stevens Institute of Technology.
Richard Morris, Rutgers College.
J. A. Nyberg, Princeton.
F. E. Seymour, State Normal School.
C. A. Stanwick, Electrical Engineer, Orange.
H. D. Thompson, Princeton University.
A. A. Titsworth, Rutgers College.
Oswald Veblen, Princeton University.
H. E. Webb, Central High School, Newark.
F. N. Willson, Princeton University.

NEW MEXICO.

W. E. Edington, University of New Mexico.
D. C. Pearson, New Mexico Military Institute.
T. G. Rodgers, New Mexico Normal University.

NEW YORK.

Joseph Allen, College of the City of New York.
Matilda Auerbach, Ethical Culture High School, New York.
D. R. Belcher, Columbia University.
Mabel R. Benway, Bay Ridge High School, Brooklyn.
C. A. Bergstresser, Boys' High School, Brooklyn.
W. J. Berry, Polytechnic Institute, Brooklyn.
Herman Betz, Cornell University.
William Betz, East High School, Rochester.
Harry Birchenough, State College for Teachers.
Joseph Bowden, Adelphi College.
Jessie W. Boyce, New York.
W. E. Breckenridge, Columbia University.
J. A. Brewster, College of the City of New York.
H. S. Brown, Hamilton College.
W. G. Bullard, Syracuse University.
R. W. Burgess, Cornell University.
J. A. C. Callan, Union University.
G. A. Campbell, Research Engineer, Am. Tel. and Telephone Company, New York.
W. M. Carruth, Hamilton College.
W. B. Carver, Cornell University.
Mary E. Caster, New York.

E. B. Chamberlain, The Franklin School, New York.
G. M. Conwell, State College for Teachers.
E. C. Cook, College of the City of New York.
Elizabeth B. Cowley, Vassar College.
E. O. Cox, New York.
Alfred Davis, Horace Mann High School, New York.
G. G. Day, Buffalo.
F. F. Decker, Syracuse University.
H. R. Dougherty, New York Military Academy.
C. H. Douglas, D. C. Heath and Company, New York.
W. P. Durfee, Hobart College.
C. P. Echols, U. S. Military Academy.
J. O. Eckersley, Consulting Engineer, New York.
T. W. Edmondson, New York University.
J. D. Eshleman, University of Rochester.
F. E. Fash, New York.
C. A. Fischer, Columbia University.
T. S. Fiske, Columbia University.
Mrs. Edward Fitch, Clinton.
W. B. Fite, Columbia University.
F. W. Frankland, Actuary, Equitable Life Assurance Society, New York.
W. S. Franklin, New York.
M. G. Gaba, Cornell University.
A. S. Gale, University of Rochester.
D. C. Gillespie, Cornell University.
Matilda Goertz, High School Teacher, New York.
Benjamin Grossbaum, New York.
C. C. Grove, Columbia University.
H. O. Hanson, East Elmhurst, Long Island.
F. M. Hartmann, Cooper Union.
H. E. Hawkes, Columbia University.
G. M. Hayes, College of the City of New York.
Robert Henderson, Actuary, Equitable Life Assurance Society, New York.
A. A. Himwich, Physician, New York.
Blanche Hirsch, Alcuin Preparatory School, New York.
F. C. Hodgdon, Ginn and Company, New York.
Anna M. Howe, Jordan.
W. A. Hurwitz, Cornell University.
S. A. Joffe, Actuary, Mutual Life Insurance Company, New York.
E. A. Johnson, College of the City of New York.
Edward Kasner, Columbia University.
D. F. Kelly, High School Teacher, New York.
E. H. Koch, Jr., High School of Commerce, New York.
F. A. Kristal, Cascadilla School, Ithaca.
Harry Langman, Metropolitan Life Insurance Company, New York.
J. A. Lanigan, Niagara Falls.
Marcia L. Latham, Hunter College.
Louis Lindsey, Syracuse University.
P. H. Linehan, College of the City of New York.
L. L. Locke, Brooklyn Training School for Teachers.
J. V. McKelvey, Cornell University.
James Maclay, Columbia University.
James McMahon, Cornell University.
E. S. Mayer, Cascadilla School, Ithaca.
Mansfield Merriman, New York.
H. B. Mitchell, Columbia University.
E. C. Molina, American Tel. and Telephone Company, New York.
A. H. Norton, Elmira College.
F. W. Owens, Cornell University.
George Paaswell, Civil Engineer, New York.
F. M. Pedersen, College of the City of New York.
H. F. Pfahl, New York.
Maximilian Philip, College of the City of New York.
President, Pi Mu Epsilon Fraternity, Syracuse University.
Arthur Ranum, Cornell University.
H. W. Reddick, Cooper Union.
Emma M. Requa, Hunter College.
F. G. Reynolds, College of the City of New York.
E. D. Roe, Jr., Syracuse University.
E. L. Sanford, St. Stephen's College.
Paul Saurel, College of the City of New York.
C. N. Schmall, Public Schools, New York.
Elmer Schuyler, Bay Ridge High School, Brooklyn.
H. M. Sheffer, College of the City of New York.
L. P. Siceloff, Columbia University.
L. L. Silverman, Cornell University.
Lao G. Simons, Hunter College.
A. W. Smith, Colgate University.
D. E. Smith, Columbia University.
R. R. Smith, with The Macmillan Company, New York.
R. H. Somers, U. S. Military Academy.
Jessie Spearing, New York.
J. J. Tanzola, Cooper Union.
J. M. Taylor, Colgate University.
W. E. Taylor, Syracuse University.
J. S. Thompson, Mutual Life Insurance Company, New York.
A. B. Turner, College of the City of New York.
C. V. Van Anda, with New York Times, New York.
Anna L. Van Benschoten, Wells College.
J. N. Vedder, Union College.
Evelyn Walker, Hunter College.
C. B. Walsh, Ethical Culture High School, New York.
Louisa M. Webster, Hunter College.
A. L. Wechsler, New York.
Mary E. Wells, Vassar College.
Bertha G. Westfall, New York.
E. E. Whitford, College of the City of New York.
G. F. Wilder, Erasmus Hall High School, Brooklyn.
Ruby Willis, Wells College.

NORTH CAROLINA.

T. C. Amick, Elon College.
Helen Barton, Salem College.
H. H. Brinton, Guilford College.
William Cain, University of North Carolina.
J. L. Douglas, Davidson.
J. W. Lasley, Jr., University of North Carolina.
Gertrude O. Mendenhall, State Normal College.
K. B. Patterson, Lenair College.

Virginia Ragsdale, State Normal College.
W. W. Rankin, Jr., University of North Carolina.
M. R. Richardson, Durham.

NORTH DAKOTA.

R. R. Hitchcock, University of North Dakota.

OHIO.

R. B. Allen, Kenyon College.
F. Anderegg, Oberlin College.
W. E. Anderson, Wittenberg College.
G. N. Armstrong, Ohio Wesleyan University.
C. L. Arnold, Ohio State University.
C. B. Austin, Ohio Wesleyan University.
Grace M. Bareis, Ohio State University.
Mrs. W. E. Beckwith, College for Women, Western Reserve University.
R. D. Bohannan, Ohio State University.
Louis Brand, Jr., University of Cincinnati.
J. B. Brandeberry, Toledo University.
Elizabeth F. Burnell, Lake Erie College.
J. M. Cain, Mechanical Engineer, Ashtabula
W. D. Cairns, Oberlin College.
A. G. Caris, Defiance College.
F. E. Carr, Oberlin College.
G. E. Carscallen, Hiram College.
E. F. Coddington, Ohio State University.
L. M. Coffin, Oberlin College.
R. M. Deming, Case School of Applied Science.
O. L. Dustheimer, Baldwin-Wallace College.
J. B. Faught, Kent State Normal College.
T. M. Focke, Case School of Applied Science.
Harriet E. Glazier, Western College for Women.
M. E. Graber, Heidelberg University.
C. B. Haldeman, Ross.
Harris Hancock, University of Cincinnati.
E. J. Hirschler, Bluffton College.
Adam Hofmann, St. Mary College.
William Hoover, Ohio University.
Miss M. C. Horn, Muskingum College.
Christian Hornung, Heidelberg University.
C. A. Hutchinson, Wittenberg College.
A. L. Jenkins, University of Cincinnati.
R. A. Johnson, Western Reserve University.
W. W. Johnson, Y. M. C. A. Night Schools, Cleveland.
J. H. Kindle, University of Cincinnati.
Emma L. Konantz, Ohio Wesleyan University.
H. W. Kuhn, Ohio State University.
Gertrude I. McCain, Oxford College for Women.
G. W. McCoard, Ohio State University.
Mrs. Eva S. Maglott, Ohio Northern University.
W. V. Metcalf, Oberlin.
F. E. Miller, Otterbein University.
C. N. Moore, University of Cincinnati.
Charlotte Morningstar, Ohio State University.
C. C. Morris, Ohio State University.
J. R. Overman, Bowling Green State Normal College.
R. E. Owen, Forest.
Anna H. Palmié, College for Women, Western Reserve University.
A. D. Pitcher, Western Reserve University.
J. B. Preston, Ohio State University.
S. E. Rasor, Ohio State University.
Hortense Rickard, Ohio State University.
Mary E. Sinclair, Oberlin College.
S. A. Singer, Capital University.
S. E. Slocum, University of Cincinnati.
E. S. Smith, University of Cincinnati.
M. J. Spinks, Bridge Engineer, Wilmington.
J. M. Stetson, Western Reserve University.
K. D. Swartzel, Ohio State University.
C. F. Thomas, Case School of Applied Science.
R. P. Thomas, College of Wooster.
S. E. Urner, Miami University.
A. W. Wallace, Franklin College.
C. J. West, Ohio State University.
Mabel G. Whiting, Antioch College.
F. B. Wiley, Denison University.
D. T. Wilson, Case School of Applied Science.
D. W. Woodard, Wilberforce University.
B. F. Yanney, College of Wooster.
E. I. Yowell, University of Cincinnati.

OKLAHOMA.

E. P. R. Duval, University of Oklahoma.
H. C. Gossard, University of Oklahoma.
Carl Gundersen, Agricultural and Mechanical College.
J. R. Livingston, School of Mines.
P. S. Morgan, Henry Kendall College.
S. W. Reaves, University of Oklahoma.
J. L. Riley, Northeastern State Normal.
W. T. Short, Oklahoma Baptist University.

OREGON.

E. E. De Cou, University of Oregon.
F. L. Griffin, Reed College.
H. H. Ludlow, U. S. Army, Fort Stevens.
A. A. Merriss, Portland.
Emily G. Palmer, Salem High School.
Gilbert Thayer, Engineer, Portland.
E. D. West, Pacific University.

PENNSYLVANIA.

O. P. Akers, Allegheny College.
A. T. G. Apple, Franklin and Marshall College.
C. S. Atchison, Washington and Jefferson College.
A. C. Baird, Fifth Avenue High School, Pittsburgh.
J. A. Bauman, Muhlenberg College.
O. F. H. Bert, Washington and Jefferson College.
F. L. Bishop, University of Pittsburgh.
J. C. Bland, Bridge Engineer, Pittsburgh.
J. F. Burley, Philadelphia.
G. G. Chambers, University of Pennsylvania.
R. L. Charles, Lehigh University.
J. J. Clark, International Correspondence School.
J. A. Clarke, West Philadelphia High School for Boys.
J. W. Clawson, Ursinus College.
E. S. Crawley, University of Pennsylvania.
J. E. Davis, Pennsylvania State College.
C. G. Dill, Drexel Institute.
G. G. Durham, Philadelphia.
H. B. Evans, University of Pennsylvania.
G. E. Fisher, University of Pennsylvania.
F. A. Foraker, University of Pittsburgh.

O. E. Glenn, University of Pennsylvania.
W. A. Granville, Pennsylvania College.
T. E. Gravatt, Pennsylvania State College.
J. H. Griffith, Bureau of Standards, Pittsburgh.
H. V. Gummere, Drexel Institute.
J. W. Haines, Central High School, Philadelphia.
W. S. Hall, Lafayette College.
N. B. Heller, Temple University.
F. J. Holder, University of Pittsburgh.
H. A. Kiess, Albright College.
P. A. Lambert, Lehigh University.
W. W. Landis, Dickinson College.
M. A. Linton, Provident Life and Trust Company, Philadelphia.
Barry MacNutt, Lehigh University.
Emory McClintock, Mutual Life Insurance Company, Philadelphia.
R. W. Marriott, Swarthmore College.
J. A. Miller, Swarthmore College.
R. L. Moore, University of Pennsylvania.
M. T. Nolan, Dunmore High School.
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Central College, Fayette, Mo.

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Baylor University, Waco, Texas.
Middlebury College, Middlebury, Vt.
University of Virginia, University, Va.
University of Washington, Seattle, Wash.

Summary of Charter Membership.

Individual members whose acceptances have been received	1,028
Individuals elected as signers of the "Call" or participants in the Columbus meeting, from whom no word has been received	18
Institutional members	52
Total	1,098

In addition to the above figures, there are on the MONTHLY subscription list the names of 141 persons who have not as yet become members of the ASSOCIATION, and of 176 libraries not included among the institutional members.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University.

Dr. DUNHAM JACKSON has been promoted to an assistant professorship of mathematics at Harvard University.

Mr. C. GARLOUGH has been appointed instructor in mathematics at Wheaton College, Illinois.

The death is announced of Professor J. W. RICHARD DEDEKIND, of the technical school of Brunswick, Germany.

In *Science* of March 3, Professor E. V. HUNTINGTON has a short and interesting discussion on "The fundamental equation of mechanics."

Miss S. F. RICHARDSON, assistant professor of mathematics in Vassar College, died on Feb. 2, 1916. Miss Richardson was a graduate of Vassar, and for thirty years has been a member of the teaching staff.

Professor ARNOLD EMCH, of the University of Illinois, has an interesting non-technical article in *The Scientific Monthly*, March, 1916, on "The representation of large numbers and infinite processes."

At Smith College, Miss HARRIET R. COBB has been promoted to a professorship of mathematics, Miss PAULINE SPERRY has been appointed assistant professor of mathematics, and Dr. MARY B. HOPKINS has been promoted to an associate professorship of astronomy.

In the *Journal of Educational Psychology* for February, Professor H. L. RIETZ has a statistical paper on "The correlation of marks of students in mathematics and in law."

A paper prepared by President Emeritus CHARLES W. ELIOT for the General Education Board on "Changes needed in American secondary education" appeared in *School and Society,* March 16.

A Plane Geometry by C. G. PALMER, associate professor of mathematics in the Armour Institute, and D. P. TAYLOR of the Oak Park High School, Chicago, has appeared from the press of Scott, Forsman & Co.

The following Cambridge tracts in mathematics are announced to appear soon: "The definite integral," by E. W. HOBSON; "An introduction to the theory of attraction," by T. J. BROMWICH; "Pascal's hexagon," by H. W. RICHMOND; "Lemniscate functions," by G. B. MATHEWS; "Chapters on algebraic geometry," and "Integrals of algebraic functions," by T. H. BAKER.

The publishing house of Lyons and Carnahan, Chicago, have just issued a text on Solid Geometry for secondary schools by J. H. WILLIAMS, teacher of mathematics in the Urbana, Ohio, High School, and K. P. WILLIAMS, assistant professor of mathematics at Indiana University. This book is the sequel to the authors' Plane Geometry, which appeared some months ago.

A second edition of E. T. WHITTAKER'S "A course of modern analysis" has appeared from the Cambridge University Press. The same publishers also announce "Euclid's book on division of figures" with a restoration based on Woepcke's text, by Professor R. C. ARCHIBALD, of Brown University. This book of 188 pages is a restoration of Euclid's ninth book, which has not heretofore been attempted in English.

The January number of *Proceedings of the London Mathematical Society* contains the following papers: "On Dirichlet's divisor problem," by G. H. HARDY; "On periodic irrotational waves at the surface of deep water," by W. BURNSIDE; "Proof of the complementary theorem," by J. C. FIELDS; "On integrals and derivatives with respect to a function," by W. H. YOUNG; "The effect on the tides of variation in the depth of the sea," by G. R. GOLDSBROUGH; "The second theorem of consistency for summable series," by G. H. HARDY.

The American Journal of Mathematics for January contains the following papers: "The oscillations of an orthogonal set," by O. D. KELLOGG; "On some properties of the medians of closed continuous curves formed by analytic arcs," by ARNOLD EMCH; "Theorems on groups of isomorphisms of certain groups," by L. C. MATHEWSON; "Self-projective rational sextics," by R. M. WINGER; "On linear difference and differential equations," by C. E. LOVE; "The uniform motion of a sphere through a viscous liquid," by R. W. BURGESS; "Note on the theory of optical images," by G. STEIN.

A reprint from the Napier Tercentenary memorial volume, published by the Royal Society of Edinburgh, contains a short article by the veteran mathematician, ARTEMAS MARTIN, LL.D., of Washington, on "A method of finding without the use of tables the number corresponding to a given natural logarithm." The memorial volume also contains an address by Professor D. E. SMITH on "The law of exponents in the works of the sixteenth century," delivered at the Napier Tercentenary.

The Mathematics Teacher, September, 1915, contains an instructive address by Professor A. D. PITCHER, of Dartmouth College, delivered before the Association of Mathematics Teachers of New England on the "Reorganization of the mathematical curriculum in the secondary schools." Professor Pitcher's conclusions indicate that: (1) No serious change should be made in the curriculum of the secondary school until after careful consideration based upon skilled experimentation; (2) much of the criticism against the curriculum will disappear with the increasingly careful and thorough preparation of teachers; (3) we should let our conservatism operate only to prevent the adoption of reform measures without due experimentation and experimental evaluation; (4) the curriculum at present is good; however, the following minor changes are suggested: (*a*) omission of difficult technique of algebra; (*b*) intuitional approach to geometry; (*c*) introduction of the function concept early in algebra; (*d*) introduction of the elements of trigonometry.

The Mathematics Club of Mt. Holyoke College is an organization of junior and senior students in mathematics, and the members of the mathematical teaching staff, one of whom is on the executive committee. The purpose of the club as stated in its constitution is "to present to its members a broader view of mathematics." Attendance and membership are voluntary. Regular meetings are held once a month, a number during the year being social gatherings; the remaining meetings consist of regular programs for presentation of papers and reports by student and faculty members, upon various historical and non-technical features of mathematics.

Summer courses in graduate and undergraduate mathematics are offered by a large number of colleges and universities. Below are enumerated those courses already announced, so far as they have come to the notice of the editor.

DARTMOUTH COLLEGE: Summer session, July 16 to August 16. By Professor J. W. YOUNG: The reorganization of secondary school mathematics.—By Professor E. G. BILL: Plane analytic geometry; Projective geometry.

INDIANA UNIVERSITY: Summer session, June 15 to August 11. By Professor S. C. DAVISSON: Differential geometry, 5 hrs.; Projective geometry, 3 hrs.—By Professor D. A. ROTHROCK: Fourier series, 3 hrs.; Differential equations, 5 hrs. —By Professor U. S. HANNA: Solid analytic geometry, 5 hrs.; Elementary calculus, 5 hrs.—By Professor K. P. WILLIAMS: Advanced calculus, 5 hrs. Courses in college algebra, trigonometry, and analytic geometry will also be given.

UNIVERSITY OF MICHIGAN: Summer session, July 30 to August 25. By Professor W. W. BEMAN: Differential equations; Teacher's course in algebra and geometry.—By Professor J. L. MARKLEY: Functions of a complex variable; Advanced analytics; Advanced algebra.—By Professor W. B. FORD: Advanced calculus; Theory of potential; Infinite series and products.—By Professor L. C. KARPINSKI: History of mathematics.—By Dr. C. H. FORSYTH: Mathematical theory of finance, insurance and statistics.—By Dr. V. C. POOR: Calculus.—By Mr. COE: Calculus; Mechanics. Courses in elementary algebra, trigonometry and analytics are also announced.

UNIVERSITY OF WISCONSIN: Summer session, June 26 to August 14. By Professor E. B. SKINNER: Differential geometry; Linear substitutions; Complex numbers.—By Professor H. W. MARCH: Theoretical mechanics; Infinite series. —By Professor H. C. WOLFF: Probabilities and statistics; Differential calculus. —By Professor W. W. HART: Analytic geometry.—By Mr. TAYLOR: Elementary analysis for engineering students.—By Mr. PAINE: Integral calculus.—By Mr. SIMPSON: Mathematical theory of investment for students of commerce. The usual elementary courses in geometry, algebra and trigonometry are also provided.

THE UNIVERSITY OF CHICAGO: Summer session, June 17 to September 1. By Professor E. H. MOORE: Theory of limits (first term); Integral equations in general analysis (first term).—By Professor G. A. BLISS: Integral calculus; Theory of functions of a real variable.—By Professor L. E. DICKSON: Substitution groups and algebraic equations; Solution of numerical equations (first term); Determinants and symmetric functions (second term).—By Professor H. E. SLAUGHT: Differential calculus; Elliptic integrals.—By Professor F. R. MOULTON: Theory of infinite series (second term); Theory of functions of infinitely many variables (second term).—By Professor J. W. A. YOUNG: Selected topics in mathematics.—By Professor W. D. MACMILLAN: Descriptive astronomy; Introduction to celestial mechanics.—By Professor ARTHUR RANUM: Metric differential geometry.—By Dr. C. R. DINES: Differential equations. Courses in trigonometry, college algebra, and plane analytic geometry are announced; and also courses in reading and research for advanced students. The mathematical club meets each week for the reports of research and the consideration of questions of teaching, alternating these topics.

IMPORTANT NOTICE TO FORMER MONTHLY SUBSCRIBERS.

Henceforth, the subscription price of the MONTHLY will be *three dollars net to all non-members of the* ASSOCIATION.

The following adjustments are proposed for those subscribers who failed to apply for membership in the ASSOCIATION before April first.

(1) Any person who had already paid \$2.00 on the old basis for the year 1916 is asked to send *one dollar additional* to complete the subscription. If such a person desires also to become a member of the ASSOCIATION, an initation fee of \$2.00 must now be paid.

(2) Those whose subscriptions for 1916 are entirely unpaid will now pay \$3.00 for the subscription or \$5.00 for membership in the ASSOCIATION, including the subscription.

In the case of subscriptions under (1) or (2) which expire *before* the end of 1916, please add *thirty cents extra* for each copy needed to complete the year. *Hereafter all subscriptions will date from January of each year.*

(3) An institution in which the Calculus is taught may become an *institutional member* of the ASSOCIATION by the payment of *five dollars* annually, which will entitle the library to receive *two copies of the* MONTHLY and the institution to send a voting delegate to all meetings of the ASSOCIATION. Institutions in which the Calculus is taught, whose libraries have already renewed their subscriptions for 1916, are asked to send *three dollars additional* and thus become institutional members of the ASSOCIATION.

Other institutions, and those not wishing to become institutional members, whose library subscriptions have already been renewed for 1916, are asked to send *one dollar additional* to complete the advanced price of the MONTHLY. No further subscriptions will be received at the old rate of two dollars, *and no discount from the advanced rate of three dollars will be allowed on subscriptions under any circumstances.*

(4) The obligations of the MONTHLY for 1916 will, of course, be fulfilled on the former basis in the case of any individual or institution whose subscription has already been paid, and who may decline to make the adjustment on the new basis, but it is hoped that all will comply, since even the advanced rate is only the actual cost of producing the journal.

(5) Please note that all subscriptions to the MONTHLY and dues in the ASSOCIATION are to be paid to the SECRETARY-TREASURER, Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

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A CURIOUS CONVERGENT SERIES.

By FRANK IRWIN, University of California.

1. Under the above title Dr. Kempner has shown, in the issue of the MONTHLY for February, 1914, that the series derived from the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, by striking out those terms whose denominators contain the digit 9 is a convergent series. It has seemed to me that it would be not without interest to attempt to extend this result, to see if one could not considerably cut down the class of numbers omitted from the harmonic series, and still get a convergent series. The result reached is as follows:

Proposition. If we strike out from the harmonic series those terms whose denominators contain the digit 9 at least a times, and, *at the same time*, the digit 8 at least b times, the digit 7 at least c times, and so on, to the digit 0 at least j times ($a, b, c, \cdots j$ being any given integers), the series so obtained will converge.

It should be explicitly pointed out that each condition added, or, to put the same thing in another way, each increase in any of the numbers $a, b, c, \cdots$, makes the class of numbers stricken out *smaller*. For instance, with $a = 4$, $b = 2$, $c = 1$, we should leave out the number 1/899979897, but not if $a = 4$, $b = 3$, $c = 1$.

Special Case. We first prove that the series derived from the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, by striking out those terms whose denominators contain the digit 9 at least a times converges, and that this is true for any other digit, including 0, equally well with 9. That this holds for $a = 1$ is what Dr. Kempner has proved (the case of the digit 0 will be treated specially at the end). We have, then, merely to prove that, if it holds for any given a, it will hold for $a + 1$.

Let us compare the two series:

Series I. The series obtained from the harmonic series by striking out terms containing at least a 9's in the denominators.

Series II. The series obtained by striking out terms containing at least $(a + 1)$ 9's in the denominators.

What we wish to prove is that, if series I converges, then will also series II. The principle we shall employ is that, if the series formed of the terms that do not occur in I, but have been introduced into II, converges, then series II, as the sum of two convergent series, will itself converge. Now, these terms are evidently the terms containing *exactly* a 9's.

How many such terms are there? That is, how many are there among those whose denominators consist of n digits in all? This may be readily treated as a problem in permutations. Two cases occur. First, if the *first* digit be a 9, we can pick out among the remaining $n - 1$ digits $a - 1$ places in which to put the remaining $a - 1$ 9's in $\binom{n-1}{a-1}$ ways; then for each one of these choices we can fill each of the remaining $n - a$ places with any one of the digits $0, 1, 2, \cdots 8$, that is, we can fill them in 9 ways each, or 9^{n-a} ways altogether; there are, then, $\binom{n-1}{a-1} 9^{n-a}$ numbers of n digits beginning with a 9 and containing exactly $a - 1$ other 9's. In the same way, secondly, we may determine that there are $\binom{n-1}{a} \cdot 8 \cdot 9^{n-a-1}$ numbers of n digits not beginning with a 9 but containing exactly a 9's (the factor 8 arises from the fact that we can put one of the digits $1, 2, \cdots 8$ only in the first place, *not* the digit 0).

Each of the terms of the harmonic series having one of these numbers for its denominator is less than $1/10^{n-1}$; together, then, their sum is less than

$$\binom{n-1}{a-1} \cdot 9^{n-a}/10^{n-1} + \binom{n-1}{a} \cdot 8 \cdot 9^{n-a-1}/10^{n-1};$$

so that the series formed of these terms with exactly a 9's will certainly converge if each of the series,

$$\sum_n \binom{n-1}{a-1} \cdot 9^{n-a}/10^{n-1} \quad \text{and} \quad \sum_n \binom{n-1}{a} \cdot 8 \cdot 9^{n-a-1}/10^{n-1},$$

converges. But each of these series may be shown to converge by applying the Cauchy ratio test.

Our general proposition may now be proved from this special case by mathematical induction. For, what this special case tells us is that the proposition is true when $b = c = \cdots = j = 0$. From this we can prove the proposition for the case $c = d = \cdots = j = 0$, $d = e = \cdots = j = 0$, and so on. As an example, then, take the following.

Suppose we are given that the series obtained from the harmonic series by striking out the terms in which the denominators contain the digit 9 at least a times and, at the same time, the digit 8 at least b times converges; this is the case $c = d = \cdots = j = 0$; call this series III. We shall prove that then the same

will be true of Series IV, obtained by striking out the terms containing at least a 9's and b 8's and c 7's; this is the case $d = e = \cdots = j = 0$.

It will be sufficient, as before, to show that the terms not occurring in series III that have been introduced into series IV form, of themselves, a convergent series. To this end it is merely necessary to notice that none of these terms contains more than $(c - 1)$ 7's; for since, by our special case, the series consisting of *all* terms that do not contain more than $c - 1$ 7's is convergent, *à fortiori* will any series made up of a selection of such terms be convergent, since all these terms are positive.

This completes the proof except for the special treatment demanded by the digit 0. It will, perhaps, suffice to indicate how Dr. Kempner's proposition would be extended to cover this case, leaving to the reader the proof of our "special case" for the digit 0. We are to show, then, that the series obtained from the harmonic by striking out terms containing the digit 0 converges. There are 9^n numbers of n digits not containing 0; the sum of their reciprocals is less than $9^n/10^{n-1}$; and our series is less than $\Sigma 9^n/10^{n-1}$, *i. e.*, 90.

2. Let us return to Dr. Kempner's series, namely that obtained from the harmonic series by striking out terms whose denominators contain the digit 9. This series converges; what is its sum? Dr. Kempner merely shows that it is less than 90. Its value actually lies between 22.4 and 23.3. This result may be obtained with no great amount of labor, and a considerably closer approximation might be reached, if desired, by the methods here employed.

Consider those terms of our series that have two digits in the denominator:

$$a_2 = \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{18} + \frac{1}{20} + \frac{1}{21} + \cdots + \frac{1}{28} + \cdots + \frac{1}{80} + \frac{1}{81} + \cdots + \frac{1}{88}.$$

Compare with these the terms with three digits in the denominator. The first nine of them, $1/100 + 1/101 + \cdots + 1/108$, are each less than or equal to $1/100$, and their sum is less than $9/100$, that is, less than $9/10$ of $1/10$, the first number of a_2. In the same way the next set of nine, $1/110 + 1/111 + \cdots + 1/118$, is less than $9/10$ of $1/11$, the second number of a_2 and so on: the sum of all the terms with three digits is less than $9/10$ of a_2. Similarly, the sum of all terms with four digits is less than $(9/10)^2$ of a_2; and in general, the sum of all terms with n digits is less than $(9/10)^{n-2}$ of a_2. Therefore the sum of our series is less than

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \left[1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \cdots\right] a_2 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} + 10a_2.$$

The value of a_2 may be quickly computed with the help of a table of reciprocals. We have:

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} < 2.72$$
$$a_2 < 2.058; \quad 10a_2 < 20.58$$

The sum of the series is less than 23.3

To obtain an inferior limit for the sum of the series, we show in the same way

that the sum of the terms with n digits in the denominator is greater than $(9/10)^{n-2} \cdot a_2'$, where

$$a_2' = \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{21} + \frac{1}{22} + \cdots + \frac{1}{29} + \cdots + \frac{1}{81} + \frac{1}{82} + \cdots + \frac{1}{89};$$

and, therefore, the series is greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + [9/10 + (9/10)^2 + \cdots]a_2',$$

that is, greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + 9a_2',$$

which turns out to be greater than 22.4.

A still closer approximation may be found by starting with a_3 and a_3', that is, with the terms having three digits in the denominator and with

$$a_2' = \frac{1}{101} + \cdots + \frac{1}{109} + \frac{1}{111} + \cdots + \frac{1}{119} + \cdots + \frac{1}{881} + \cdots + \frac{1}{889}.$$

ON THE MATRIX EQUATION $BX = C$.[1]

By H. T. BURGESS, University of Wisconsin.

Section 1. To Find the Matrix X. The problem is to calculate the elements of the matrix X to satisfy the matrix equation $BX = C$:

$$\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Vmatrix} \begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{Vmatrix} = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{Vmatrix}.$$

If we compute the matrix product BX, we get

$$BX = \begin{Vmatrix} \Sigma b_{1\epsilon} x_{\epsilon 1} & \Sigma b_{1\epsilon} x_{\epsilon 2} & \cdots & \Sigma b_{1\epsilon} x_{\epsilon n} \\ \Sigma b_{2\epsilon} x_{\epsilon 1} & \Sigma b_{2\epsilon} x_{\epsilon 2} & \cdots & \Sigma b_{2\epsilon} x_{\epsilon n} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma b_{n\epsilon} x_{\epsilon 1} & \Sigma b_{n\epsilon} x_{\epsilon 2} & \cdots & \Sigma b_{n\epsilon} x_{\epsilon n} \end{Vmatrix},$$

where the summation runs for $\epsilon = 1, 2, \cdots, n$.

The conditions to be fulfilled are obtained by setting the elements of the product BX equal to the corresponding elements of C. Taking these by columns we get the following n-sets of simultaneous linear equations:

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[1] For the elementary properties of matrices the reader may conveniently consult Bôcher's *Introduction to Higher Algebra*, using the index to find the appropriate sections.

$$\text{I.}\qquad \begin{matrix} b_{11}x_{1\kappa} + b_{12}x_{2\kappa} + \cdots b_{1n}x_{n\kappa} = c_{1\kappa}, \\ b_{21}x_{1\kappa} + b_{22}x_{2\kappa} + \cdots b_{2n}x_{n\kappa} = c_{2\kappa}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ b_{n1}x_{1\kappa} + b_{n2}x_{2\kappa} + \cdots b_{nn}x_{n\kappa} = c_{n\kappa}, \end{matrix} \qquad \kappa = 1, 2, \cdots, n.$$

The matrix B on the n-sets of unknowns is the same for each of the n-sets of equations, but the column of c's is different for each set. These n-sets may all be solved at once by the following simple device: Write out the two matrices B and C in juxtaposition as one matrix in the form

$$\begin{Vmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} & c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} & c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} & c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{Vmatrix}.$$

The following operations may be performed on this matrix which give an equivalent matrix in the sense that the n-sets of equations written down from the resulting matrix will have the same solutions as the systems I.

(1) Any two rows may be interchanged,

(2) Any row may be multiplied by a constant not zero,

(3) Any row may be multipled by a constant and added to any other row.
By (1) the element b_{11} can be made different from zero, by (2) it can then be made unity. Next by (3) with $-b_{21}, -b_{31}, \cdots, -b_{n1}$, as multipliers of the first row, all the remaining elements of the first column can be replaced by zeros.

If the matrix B is non-singular, this process can be continued with the successive columns of the matrix until the matrix B is reduced to the unit matrix I and the matrix C is simultaneously reduced to X. For when B is reduced to I, the n-sets of equations have the form

$$\text{II.}\qquad \begin{matrix} x_{1\kappa} & & & & = \bar{c}_{1\kappa}, \\ & x_{2\kappa} & & & = \bar{c}_{2\kappa}, \\ & & x_{3\kappa} & & = \bar{c}_{3\kappa}, \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & x_{n\kappa} & = \bar{c}_{n\kappa}, \end{matrix} \qquad \kappa = 1, 2, \cdots, n.$$

Illustration: To determine the matrix X to satisfy the equation

$$\begin{Vmatrix} 1 & 5 & 1 \\ 3 & 4 & 2 \\ 2 & 1 & 3 \end{Vmatrix} \begin{Vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{Vmatrix} = \begin{Vmatrix} 18 & 11 & 3 \\ 20 & 11 & 7 \\ 10 & 4 & 8 \end{Vmatrix}$$

we write

$$\begin{Vmatrix} 1 & 5 & 1 & 18 & 11 & 3 \\ 3 & 4 & 2 & 20 & 11 & 7 \\ 2 & 1 & 3 & 10 & 4 & 8 \end{Vmatrix}.$$

Reducing the first column by (3), we get

$$\begin{Vmatrix} 1 & 5 & 1 & 18 & 11 & 3 \\ 0 & -11 & -1 & -34 & -22 & -2 \\ 0 & -9 & 1 & -26 & -18 & 2 \end{Vmatrix}.$$

Reducing the third column by (3), we get

$$\begin{Vmatrix} 1 & 14 & 0 & 44 & 29 & 1 \\ 0 & -20 & 0 & -60 & -40 & 0 \\ 0 & -9 & 1 & -26 & -18 & 2 \end{Vmatrix}.$$

Dividing the second row by -20 by use of (2), and reducing the second column by (3), we get

$$\begin{Vmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{Vmatrix}. \quad \text{Hence } X = \begin{Vmatrix} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{Vmatrix}.$$

The method applies to $XB = C$ by use of the conjugates, for $B'X' = C'$.

Section 2. To Find the Inverse of a Matrix. A very useful application occurs when C is replaced by the unit matrix I, for in this case X becomes the inverse of B. The amount of work required to calculate the inverse of a matrix by this method is practically the same as that required to compute one of its elements by the ordinary method.

Illustration: To compute the inverse of the matrix

$$B = \begin{Vmatrix} 1 & -1 & -5 \\ -1 & 3 & -10 \\ -1 & 0 & 12 \end{Vmatrix} \text{ we write } \begin{Vmatrix} 1 & -1 & -5 & 1 & 0 & 0 \\ -1 & 3 & -10 & 0 & 1 & 0 \\ -1 & 0 & 12 & 0 & 0 & 1 \end{Vmatrix}.$$

Reducing column one by (3):

$$\begin{Vmatrix} 1 & -1 & -5 & 1 & 0 & 0 \\ 0 & 2 & -15 & 1 & 1 & 0 \\ 0 & -1 & 7 & 1 & 0 & 1 \end{Vmatrix}.$$

Interchanging rows two and three by (1), changing signs in row two by (2), and reducing column two by (3):

$$\left\|\begin{array}{cccccc} 1 & 0 & -12 & 0 & 0 & -1 \\ 0 & 1 & -7 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & 2 \end{array}\right\|.$$

Changing the signs in the last row by (2) and reducing column three by (3):

$$\left\|\begin{array}{cccccc} 1 & 0 & 0 & -36 & -12 & -25 \\ 0 & 1 & 0 & -22 & -7 & -15 \\ 0 & 0 & 1 & -3 & -1 & -2 \end{array}\right\|.$$

Hence,

$$B^{-1} = \left\|\begin{array}{ccc} -36 & -12 & -25 \\ -22 & -7 & -15 \\ -3 & -1 & -2 \end{array}\right\|.$$

CENTERS OF SIMILITUDE AND THEIR N-DIMENSIONAL ANALOGIES.

By BANCROFT HUNTINGTON BROWN, Brown University.

1. In the January, 1915, issue of the MONTHLY[1] "Centers of similitude of circles and certain theorems attributed to Monge" were discussed. The theorems there given are the following:

(A) *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*[2]

(B) *The vertices of the six common tangent cones of three spheres, taken in pairs, lie by threes on four straight lines.*

(C) *Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If the six other tangent cones be drawn, then their vertices lie by threes in planes*[3] *with threes of the first group.*

It was shown:

(1) That theorem (A) was, in all probability, known to the Greeks of two thousand years ago;

(2) That Fuss found, with regard to the *external* centers of similitude of coplanar circles, that:

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[1] R. C. Archibald, THE AMERICAN MATHEMATICAL MONTHLY, Vol. XXII, pp. 6–12.

[2] Symbolically, if $E_{m,\,n}$ denote the external, and $I_{m,\,n}$ the internal centers of similitude of the circles C_m and C_n ($m, n = 1, 2, 3.\ \ m \gtrless n$), the following groups of points are collinear:

$$E_{1,\,2},\ E_{1,\,3},\ E_{2,\,3};\ E_{1,\,2},\ I_{1,\,3},\ I_{2,\,3};\ I_{1,\,2},\ E_{1,\,3},\ I_{2,\,3};\ I_{1,\,2},\ I_{1,\,3},\ E_{2,\,3}.$$

[3] These planes have been called "planes of similitude."

(*D*) *For n coplanar circles, the* $n(n-1)/2!$ *external centers of similitude are situated, in general, by threes on* $n(n-1)(n-2)/3!$ *different straight lines;*[1]

(3) That the centers of similitude of the spheres of theorem (*C*) lie by sixes on eight different planes. Further generalization of these results does not seem to have been published. It is the object of this paper to give indications of such generalizations.

2. Consider first the number of centers of similitude of n coplanar circles; any circle combined with each of the $(n-1)$ remaining circles determines 2 centers of similitude, and since in this way each pair of centers of similitude is enumerated twice we arrive at the result:

n coplanar circles have, in general, $n(n-1)$ *centers of similitude.*[2]

Three circles may be selected from n circles in $n(n-1)(n-2)/3!$ different ways. With any such set of three circles we may by theorem (*A*) associate a group of four lines on which the centers of similitude lie by threes. Hence we have:

(*E*) *The* $n(n-1)$ *centers of similitude of n coplanar circles are situated, in general, by threes on* $4n(n-1)(n-2)/3!$ *different straight lines.*

Some particular cases may be noted: (*a*) If three (or more) circles have two common external tangents, then three (or more) of the centers of similitude coincide and *any* line through this point would contain at least three centers of similitude; moreover all of the centers of similitude lie on a single line. (*b*) Again, if two or more pairs of circles are in perspective[3] from the same point, then at least four axes of similitude coincide. For if $E_{1,2}$ and $E_{3,4}$ are coincident, this point is collinear with $E_{1,4}$ and $E_{2,4}$, with $E_{2,4}$ and $E_{2,3}$, and with $E_{1,3}$ and $E_{1,4}$; hence, these three lines coincide with the fourth line containing $E_{2,3}$, $E_{2,4}$, and $E_{3,4}$. Similar reasoning may be employed if $E_{1,2}$ and $I_{3,4}$ coincide. (*c*) If, of two pairs of equal circles, the lines of centers are parallel, the circles may be said to be in perspective from a point at infinity, in the direction of their lines of centers, and their external centers of similitude coincide at infinity. This is then a particular case of the preceding. (*d*) If three or more circles are tangent at one point, since this point of tangency is a center of similitude (external or internal) of any two of the circles, *any* line through this point contains at least three (coincident) centers of similitude; and, as in the case of every set of coaxial circles, all the centers of similitude are collinear. (*e*) If, of three circles, at least two are concentric, the six (or five) centers of similitude are collinear.

3. Monge was the first to give theorem (*C*) and he indicated five such planes:

$$E_{1,2},\ E_{1,3},\ E_{1,4},\ E_{2,3},\ E_{2,4},\ E_{3,4};\quad E_{1,2},\ E_{1,3},\ I_{1,4},\ E_{2,3},\ I_{2,4},\ I_{3,4};$$
$$E_{1,2},\ I_{1,3},\ E_{1,4},\ I_{2,3},\ E_{2,4},\ I_{3,4};\quad I_{1,2},\ E_{1,3},\ E_{1,4},\ I_{2,3},\ I_{2,4},\ E_{3,4};$$
$$I_{1,2},\ I_{1,3},\ I_{1,4},\ E_{2,3},\ E_{2,4},\ E_{3,4}.$$

[1] These straight lines have been called "axes of similitude."

[2] If the circles (spheres) are equal and non-concentric, one center of similitude is at infinity, and the other bisects the line joining their centers. If two circles (spheres) are concentric we may say that the two centers of similitude coincide with the common center of the circles (spheres) or that one center of similitude coincides with the common center, the other being indeterminate.

[3] Any two circles are in perspective from either center of similitude.

There are three others, which he overlooked, making eight in all:

$I_{1,2}, I_{1,3}, E_{1,4}, E_{2,3}, I_{2,4}, I_{3,4}$; $I_{1,2}, E_{1,3}, I_{1,4}, I_{2,3}, E_{2,4}, I_{3,4}$; $E_{1,2}, I_{1,3}, I_{1,4}, I_{2,3}, I_{2,4}, E_{3,4}$.

That, in general, not more than six such centers are coplanar, and that not more than eight such planes exist, may be shown as follows: The six points in a plane are L's (I's or E's) with the six *different* pairs of suffixes. $E_{1,2}$ and $I_{1,2}$ cannot lie in one of these planes, for, if they did, the centers of spheres 1 and 2, and then of all the spheres would lie in this plane. Hence, we can determine all the planes by taking three L's with the same suffixes, *e. g.*, 1, 2, 1, 3, 1, 4; that is, in $2 \times 2 \times 2$, or eight, and only eight, different ways.

If now we consider n spheres, having $n(n-1)$ centers of similitude, we can choose 4 spheres in $n(n-1)(n-2)(n-3)/4!$ ways. And since with each of these sets of four spheres we can associate 8 planes containing 6 centers of similitude, we have

(*F*) *The* $n(n-1)$ *centers of similitude of* n *spheres lie, in general, by sixes, in* $8n(n-1)(n-2)(n-3)/4!$ *planes.*

4. Consider now five four-dimensional hyper-spheres. Four points $L_{i,j}$ determine a hyper-plane in which six other centers of similitude lie. For example, if we take $E_{1,2}$, $I_{1,3}$, $E_{1,4}$, $E_{1,5}$; applying theorem (*A*) to spheres and hyper-spheres, by considering the circles of plane sections, we know $E_{1,2}$, $I_{1,3}$, $I_{2,3}$ are collinear; also $E_{1,2}$, $E_{1,4}$, $E_{2,4}$; $E_{1,2}$, $E_{1,5}$, $E_{2,5}$; $I_{1,3}$, $E_{1,4}$, $I_{3,4}$; $I_{1,3}$, $E_{1,5}$, $I_{3,5}$; and $E_{1,4}$, $E_{1,5}$, $E_{4,5}$. Hence $E_{1,2}$, $I_{1,3}$, $E_{1,4}$, $E_{1,5}$, $I_{2,3}$, $E_{2,4}$, $E_{2,5}$, $I_{3,4}$, $I_{3,5}$, $E_{4,5}$ lie in one hyper-plane. By a proof similar to the one used in the case of spheres, (1) not more than 10 centers of similitude can, in general, lie in a hyper-plane, and (2) there can, in general, be 16, and only 16, such hyper-planes, since L_{ij} can be chosen in only $2 \times 2 \times 2 \times 2$ or 16 ways. We have then

(*G*) *The* 20 *external and internal centers of similitude of* 5 *four-dimensional hyper-spheres lie, in general, by tens in* 16 *hyper-planes.*

From among n hyper-spheres we may choose 5 in $n(n-1)(n-2)(n-3)(n-4)/5!$ different ways. And since each of these sets of hyper-spheres has the property of theorem (*G*), in general, there follows

(*H*) *The* $n(n-1)$ *centers of similitude of* n *four-dimensional hyper-spheres lie, in general, by tens in* $16n(n-1)(n-2)(n-3)(n-4)/5!$ *hyper-planes.*

5. We are now in a position to extend the investigation to $(p+1)$ p-dimensional hyper-spheres, lying in a p-dimensional space. Generalizing from the preceding theorems we see that $(p+1)p/2$ of the $(p+1)p$ centers of similitude should lie in a $(p-1)$-dimensional space. Further, there will be 2^p such spaces. Hence,

(*K*) *The* $(p+1)p$ *centers of similitude of* $(p+1)$ p*-dimensional hyper-spheres lie, in general, in* 2^p $(p-1)$*-dimensional spaces containing* $(p+1)p/2!$ *centers of similitude.*

Next, consider n p-dimensional hyper-spheres, where $n \geqq p+1$. We have

(*L*) *The* $n(n-1)$ *centers of similitude of* n p*-dimensional hyper-spheres lie,*

in general, in $2^p\ n(n-1)\cdots(n-p)/(p+1)!$ $(p-1)$*-dimensional hyper-planes each containing* $(p+1)p/2!$ *centers of similitude.*

Finally, we may slightly generalize this result and arrive at

(M) *The* $n(n-1)$ *centers of similitude of* n m*-dimensional hyper-spheres lie, in general, in* $2^p\ n(n-1)\cdots(n-p)/(p+1)!$ $(p-1)$*-dimensional hyper-planes containing* $(p+1)p/2!$ *centers of similitude,* where $n > m \geqq p$.

6. For $(p+1)$ p-dimensional hyper-spheres there exists an interesting relation with regard to the number of exterior and interior centers in each $(p-1)$-dimensional hyper-plane.

We saw that for $p = 2$, there was one line containing 3 external, and 3 lines containing 1 external and 2 internal centers.

Monge showed for $p = 3$, that 1 plane contained 6 external centers, and 4 planes 3 external and 3 internal centers. The three which he omitted contain 2 external and 4 internal centers.

The results for $p = 4, 5, \cdots$ up to 10, may be readily obtained, and are sufficient to indicate a general relation. A display in tabular form will simplify the presentation. In the first table is shown the number of hyper-planes (planes, lines); and in the second is indicated, in a corresponding place, the number of external, and of internal centers of similitude contained in each corresponding hyper-plane (plane, line).

TABLE I.

$p = 2$	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
3	4	5	6	7	8	9	10	11
	3	10	15	21	28	36	45	55
			10	35	56	84	120	165
					35	126	210	330
							126	462

TABLE II.

$p = 2$	3	4	5	6	7	8	9	10
3, 0	6, 0	10, 0	15, 0	21, 0	28, 0	36, 0	45, 0	55, 0
1, 2	3, 3	6, 4	10, 5	15, 6	21, 7	28, 8	36, 9	45, 10
	2, 4	4, 6	7, 8	11, 10	16, 12	22, 14	29, 16	37, 18
			6, 9	9, 12	13, 15	18, 18	24, 21	31, 24
					12, 16	16, 20	21, 24	27, 28
							20, 25	25, 30

It will be observed that in the first table the numbers, with the exception of the last in the columns where p is odd, are the regular coefficients in the expansion of $(a+b)^{p+1}$; the last numbers, when p is odd, are half the middle coefficients of the expansion. The composition of the second table is not especially complicated, but perhaps it will be worth while to append a series of formulæ for the general case.

When *p is odd:*

Number of $(p-1)$-dimensional hyperplanes	External Centers of Similitude	Internal Centers of Similitude
1	$\frac{p^2+p}{2}$	0
$p+1$	$\frac{p^2-p}{2}$	p
$\frac{(p+1)p}{2!}$	$\frac{p^2-3p+4}{2}$	$(p-1)2$
$\frac{(p+1)p(p-1)}{3!}$	$\frac{p^2-5p+12}{2}$	$(p-2)3$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots(p-k+2)}{k!}$	$\frac{p^2-(2k-1)p+(k-1)2k}{2}$	$(p-(k-1))k$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+5}{2}\right)}{\left(\frac{p-1}{2}\right)!}$	$\frac{p^2-4p+3}{4}$	$\left(\frac{p+3}{2}\right)\left(\frac{p-1}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+3}{2}\right)}{2\cdot\left(\frac{p+1}{2}\right)!}$	$\frac{p^2-1}{4}$	$\left(\frac{p+1}{2}\right)^2$

When *p is even,* the last two lines become:

$\frac{(p+1)p(p-1)\cdots\left(\frac{p+6}{2}\right)}{\left(\frac{p-2}{2}\right)!}$	$\frac{p^2+8}{4}$	$\left(\frac{p+4}{2}\right)\left(\frac{p-2}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+4}{2}\right)}{\left(\frac{p}{2}\right)!}$	$\frac{p^2}{4}$	$\frac{p}{2}\left(\frac{p+2}{2}\right)$

HISTORICAL NOTE ON CENTERS OF SIMILITUDE OF CIRCLES.

By RAYMOND CLARE ARCHIBALD, Brown University.

In the paper on "Centers of Similitude of Circles and certain Theorems attributed to Monge. Were they known to the Greeks?"[1], I endeavored to show, especially through consideration of a problem in the book *On Tangencies*

[1] AMERICAN MATHEMATICAL MONTHLY, January, 1915, vol. 22, pp. 6–12. *Addenda:* In note 2, page 10, line 5, *for* cones *read* conics; on page 12, line 2, *for* side *read* sides. The two following historical notes may also be given:

(*a*) The theorem that "*The six centers of similitude of three coplanar circles lie by threes on four straight lines*" has been attributed to Monge who published it in 1798. Proof by analytical geometry that the three *external* centers of similitude are collinear was given by L. PUISSANT in his *Recueil de diverses propositions de géométrie,* Paris, 1801, pp. 50–56; it was probably here that the particular case of two of the circles being equal was first considered in recent times. No reference is made to Monge. There was a German edition by Hahn (Berlin, 1806). In the second French edition (Paris, 1809) mention is made (pp. 131–137) of Monge and the *general* theorem.

by Apollonius of Perga (about 225 B. C.), that several of the now well-known properties of centers of similitude of circles were also familiar to the Greeks. When writing the paper it did not occur to me to reinforce my argument by reference to another work by Apollonius, namely that *On Plane Loci*.[1] Pappus's account[2] of this work has been the basis of restorations and discussions by Fermat,[3] Schooten,[4] Simson,[5] Camerer,[6] Lhuilier,[7] Bonnycastle,[8] Breton (de Champ)[9] and Zeuthen.[10]

The *Plane Loci* consisted of two books which contained 147 propositions and figures, and 8 lemmas. Pappus states a general proposition, the detailed discussion of which he seems to indicate as the chief original contribution of Apollonius to the first book of his work.[11] This general proposition is as follows:

(*b*) To Monge (1798) is due the theorem (emended form of that given on p. 7 of the above mentioned paper): "*Given any four spheres in space, fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane.*" *If the six other tangent cones be drawn then the twelve conical vertices lie by sixes in five planes.* (This theorem is referred to by Mr. Brown in the preceding paper.) Monge overlooked the fact that there were three other planes with a similar property. This seems to have been first remarked by J. B. Durrande in *Annales de mathématiques* (Gergonne), Juillet, 1820, tome 11, pp. 18–20. He there calls the 8 planes "plans de similitude" and also gives synthetic proof of Monge's theorems for any positions of circles and spheres. The proofs of Monge are only applicable for circles and spheres external to one another. Durrande also used the term "axes de similitude" in connection with collinear centers of similitude of circles and spheres (pp. 10 and 17).

[1] The most recent historical sketch is by G. Loria in *Le Scienze esatte nell' antica Grecia.* Seconda edizione totalmente reviduta. Milano, 1914. Pp. 393–396 and 440–443. A reference may also be given to G. S. Klügel's *Mathematisches Wörterbuch*, erste Abtheilung, dritter Theil, Leipzig, 1808. Pp. 697–698.

[2] Pappi Alexandrini, *Collectionis*, edidit F. Hultsch, vol. 2, Berolini, 1877, pp. 660–671, 852–865. In C. I. Gerhardt's Greek-German edition, Halle, 1871, pp. 20–29; 166–175.

[3] *Varia opera mathematica*, D. Petri de Fermat. Tolosae, 1679, pp. 12–43; facsimile edition, Berolini, 1861—E. Brassine's *Précis des Oeuvres mathématiques de P. Fermat*, Toulouse, 1853, pp. 39–41—*Oeuvres de Fermat*, Paris, tome 1, 1891, pp. 3–51; tome 3, 1896, pp. 3–48. See also tome 2, pp. 5, 30, 56, 74, and 100.

[4] *Exercitationum mathematicarum. Liber III. Continens Apollonii Pergaei Loco Plana restituta.* Lugd. Batav. 1656, pp. 191–292—*Derde Bouck der mathematische Oeffeningen begrijpende, Apollonii Pergaei herstelde Vlacke Plaetsen* . . . door Franciscus Van Schooten. Amsterdam, 1660, pp. 185–272.

[5] *Apollonii Pergaei Locorum Planorum Libri II. Restituti* a Roberto Simson. Glasguæ, 1749. 252 pp.

[6] *Apollonius von Pergen ebene Oerter.* Wiederhergestellt von Robert Simson . . . Aus dem Lateinischen übersezt, mit . . . Aufgaben begleitet von J. W. Camerer. Leipzig, 1796. 455 pp. + 17 plates.

[7] *Éléments d'analyse géométrique et d'analyse algébrique appliquées a la recherche des lieux géométriques.* Par S. Lhuilier. Paris, 1809. "Lieux traités par Apollonius, suivant Simson" and "additions diverses aux lieux plans d'Apollonius," pp. 35–111.

[8] J. Bonnycastle, *Elements of Geometry.* Sixth edition, London, 1818. "De Locis Planis," pp. 371–375.

[9] *Recherches nouvelles sur les porismes d'Euclide.* Paris, 1855. Pp. 91–95. Also in *Journal de mathématiques pures et appliquées*, tome 20, 1855. Pp. 299–303.

[10] H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum.* Deutsche Ausgabe von R. v. Fischer-Benzon. Kopenhagen, 1886. Pp. 207–212.

[11] The early propositions of the *second* book include the following familiar results: (1) The locus of points, the difference of the squares of whose distances from two fixed points is constant, is a straight line perpendicular to the straight line joining the points; (2) The locus of the points,

"Two straight lines are drawn either from the same fixed point or from two fixed points, in the same direction or in such a way as to form a fixed angle; the lengths of these lines are in a constant ratio to one another or their rectangle is constant. If the extremity of one of them describes a plane locus given in position, the extremity of the second will also describe a plane locus, given in position, which is either of the same or of different species from the first."

A particular proposition included in this general one may be formulated as follows:

Through a point O draw lines OP_1, OP_2, OP_3, $\cdots$ to the various points P_1, P_2, P_3, $\cdots$ on a circle.[1] *Divide OP_1, OP_2, OP_3, $\cdots$ internally at Q_1, Q_2, Q_3, $\cdots$ respectively, and such that $OP_1 : OQ_1 = OP_2 : OQ_2 = \cdots =$ a const., and externally at R_1, R_2, R_3, $\cdots$ respectively, such that $OP_1 : OR_1 = OP_2 : OR_2 = \cdots =$ a const. Then the locus of the Q's is a circle,*[1] *and the locus of the R's is a circle.*[1]

Here O is the external center of similitude of the circles (P) and (Q), and the internal center of similitude of the circles (P) and (R). It is exactly such a proposition which Simson and others (*l. c.*) consider in their restorations in various cases when the circles (1) are exterior to one another; (2) intersect; (3) are such that one is inside of the other. The property of parallel radii joining corresponding points of the pairs of circles, arises in the course of the proofs.

My earlier argument that Apollonius was familiar with the centers of similitude of circles and some of their chief properties has thus been reinforced through consideration of another of his works.

A CIRCLE THEOREM.

By ROGER A. JOHNSON, Adelbert College, Western Reserve University.

THEOREM. *If three equal circles are drawn through a point, the circle through their other three intersections is equal to each of them.*

Proof. Denote the centers of the circles (see figure 1 of the next paper) by C_1, C_2, C_3, the intersections of C_2 and C_3 by O and P_1, those of C_3 and C_1 by O and P_2, those of C_1 and C_2 by O and P_3. Then $OC_2P_1C_3$ is a rhombus, and so is $OC_3P_2C_1$. Hence, C_2P_1 and C_1P_2 are equal and parallel, $C_1C_2P_1P_2$ is a parallelogram, and P_1P_2 is equal to C_1C_2. Thus the triangles $C_1C_2C_3$ and $P_1P_2P_3$ are congruent, and have equal circumcircles. But the circumcircle of the former has its center at O, and is equal to each of the given circles. Hence, the circle through P_1, P_2, P_3 is equal to each of the given circles.

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the ratio of whose distances from two fixed points is constant, is either a straight line or a circle —the Circle of Apollonius. Eutocius gives the proof of Apollonius for this latter locus (Apollonius, ed. Heiberg, Vol. 2, pp. 180–185). The name "Circle of Apollonius" is, however, a misnomer, since the construction of this locus connected with his name appears in exactly the same form at a much earlier date in Aristotle's *Meteorologica*, III, 5, 376 f.

[1] "Circle" is here considered as a curved line. The cases of this proposition when we substitute "straight line" for "circle" were discussed by Euclid in Propositions 35–36 of his *Data* (Simson's edition, Prop. 39—for example: *Elements of Elucid . . . also Euclid's Data*, 9th ed., Edinburgh, 1793, pp. 393–394).

COROLLARY. *Each of the four points O, P_1, P_2, P_3, is the orthocenter of the triangle of the other three, and the set of points has all the well-known properties of an orthocentric system.*

Singularly enough, this remarkable theorem appears to be new. A rather cursory search in several of the treatises on modern elementary geometry fails to disclose it, and the author has not yet found any person to whom it was known. On the other hand, the figure is so simple (especially as it can be drawn and the theorem verified with a coin or other circular object) that it seems almost out of the question that the fact can have escaped detection. Even if geometers have overlooked it, someone must have noticed it in casually drawing circles. But if this were the case, it seems like a theorem of sufficient interest to receive some prominence in the literature, and therefore ought to be well known. It is hoped that if any reader recognizes the theorem, or knows where it has already been given, he will report the same. Of course, the converse theorem that the four circumcircles of an orthocentric system are equal, is well known.

REMARKS ON THE FOREGOING CIRCLE THEOREM.

By ARNOLD EMCH, University of Illinois.

1. The foregoing theorem proved by Mr. Johnson gains additional interest in connection with the theory of circular inversion.

Before this fact is pointed out, another proof of the theorem, equally notable on account of its extreme simplicity, will be given. Using the same notation as Mr. Johnson, and denoting the given circles through O by $\alpha_1, \alpha_2, \alpha_3$, as shown[1] in Fig. 1, we have $\measuredangle OP_3P_2 = \measuredangle OP_1P_2$, because the circles α_1 and α_3 are equal and have the common chord OP_2 subtending those angles. Likewise, $\measuredangle OP_2P_3 = \measuredangle OP_1P_3$. Consequently $\measuredangle OP_3P_2 + \measuredangle OP_2P_3 = \measuredangle OP_1P_2 + \measuredangle OP_1P_3 = \measuredangle P_2P_1P_3$, so that $\measuredangle P_2OP_3$ and $\measuredangle P_2P_1P_3$ are supplementary. But also $\measuredangle P_2AP_3$ and $\measuredangle P_2OP_3$ are supplementary (A is any point on α_1); hence $\measuredangle P_2P_1P_3 = \measuredangle P_2AP_3$. From this follows immediately that the circle α_4 through $P_1P_2P_3$ is equal to the circle α_1, and consequently to α_2 and α_3. As the sum of the six angles in the equality

$$\measuredangle OP_2P_1 + \measuredangle OP_2P_3 + \measuredangle OP_3P_2 = \measuredangle OP_3P_1 + \measuredangle OP_1P_3 + \measuredangle OP_1P_2$$

is equal to the sum of the angles in the triangle $P_1P_2P_3$, the left and right hand sides are equal to a right angle, *i. e.*, P_2P_3Q is a right triangle, and consequently P_3Q is perpendicular to P_1P_2. Likewise P_2O and P_1O prolonged are perpendiculars to P_1P_3 and P_3P_2, respectively; and O is the orthocenter of the triangle $P_1P_2P_3$. In a similar manner it can be shown, that P_1, P_2, P_3, are orthocenters of the corresponding triangles OP_2P_3, OP_3P_1, OP_1P_2.

[1] For figures in which O is without the triangle $P_1P_2P_3$, the proposition can be proved in a similar manner by angular relations.

2. Consider in Fig. 2 any triangle $P_1'P_2'P_3'$ with its circumscribed circle α_4' and the inscribed circle I with center O. Denote the sides of the triangle respectively by α_1', α_2', α_3', and invert the whole figure with respect to I as the circle of inversion. Let Q_1, Q_2, Q_3 be the points of tangency of the inscribed circle with the sides of the triangle; then α_1', α_2', α_3', supposed to be indefinitely extended,

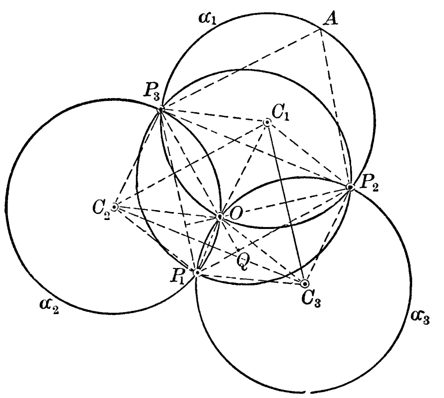


Fig.1

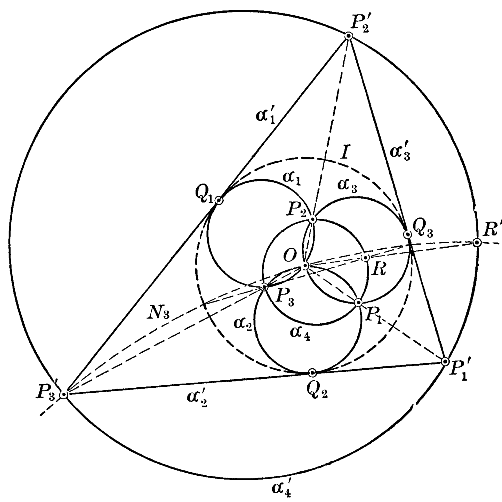


Fig.2

are inverted into three circles α_1, α_2, α_3, which pass through O and touch I at Q_1, Q_2, Q_3, respectively. They intersect in three points P_1, P_2, P_3 which in the same order are the inverse of P_1', P_2', P_3', and are evidently equal circles. The circle α_4 through P_1, P_2, P_3 is the inverse of the circumscribed circle α_4' and, according to the circle theorem, is equal to α_1, α_2, α_3. We have therefore the

THEOREM: *The indefinitely extended sides of a triangle and its circumscribed circle are inverted into four equal circles, with the inscribed* (*or escribed*) *circle of the triangle as the circle of inversion.*

Conversely, the figure of the circle theorem (Fig. 1) may always be inverted into a triangle with its in- (or escribed) and circumscribed circles.

3. Among the number of other propositions which may be derived from this theorem I shall prove only one.

Through any of the vertices, say P_3', of the triangle (Fig. 2) and the point O pass a circle N_3 cutting α_4' orthogonally, and let R' be the other point of intersection of N_3 with α_4'. The question is, what relation does the point R' have with respect to the triangle? Invert N_3 with I as the circle of inversion. The inverse is a straight line P_3R which intersects α_4 orthogonally. P_3R is therefore a diameter of α_4, and R is therefore the point of tangency of the circle having P_3 as a center and touching each of the three equal circles α_4, α_1, α_2. Inverting back, we find, that R' is the point of tangency of a circle which is tangent to the

circumscribed circle of the triangle and the two sides of the triangle that meet at P_3'. A similar result is obtained by passing circles through P_1', P_2' and O, orthogonal to α_4'. The result may be stated as the

THEOREM: *A circle which passes through a given vertex of a triangle and the center of the inscribed circle, and is orthogonal to the circumscribed circle, cuts the latter in another point which is the point of tangency of a circle that is tangent to the circumscribed circle and to the two sides of the triangle meeting at the given vertex.*

KANSAS SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The first meeting of the Kansas Section of the Mathematical Association of America was held at Lawrence, Kansas, on Saturday, March 18, 1916, in room 105, Administration Building, University of Kansas.

Following are the names of those present, together with the institutions represented:

University of Kansas: J. N. Van der Vries, C. H. Ashton, U. G. Mitchell, H. E. Jordan, J. J. Wheeler, S. Lefschetz, A. W. Larsen, K. L. Holzinger, L. L. Steimley, C. A. Nelson, J. M. Jacobs, P. W. Harnley.

Kansas State Agricultural College: B. L. Remick, A. E. White, W. T. Stratton, H. E. Porter.

Kansas State Normal: Theodore Lindquist.

Fairmount College: A. J. Hoare.

College of Emporia: T. E. Mergendahl.

Washburn College: W. A. Harshbarger, Mary Newson.

Baker University: W. H. Garrett.

McPherson College: A. B. Frizell.

Bethel College: D. H. Richert.

Friends University: O. W. Dueker.

Campbell College: T. L. Bouse.

Kansas City, Kan., High School: Elizabeth G. Flagg, Emma Hyde, and Lucy Dougherty.

This was the second meeting of an association formed in 1915 for the improvement of the teaching of collegiate mathematics in the state of Kansas.

The Kansas Association has now become a section of the national organization, The Mathematical Association of America, founded at Columbus, Ohio, December 30, 1915.

The meeting was called to order by Professor A. J. Hoare, of Fairmount College.

Professor U. G. Mitchell, of the University of Kansas, gave a report on the organization meeting of the National Association at Columbus, Ohio. Professor Mitchell was the delegate of the Kansas Section, which was the first body to make application for admission as a section of the Association.

Members of the Kansas Section of the Mathematical Association of America, in Session at Lawrence, Kansas, March 18, 1916.

Professor J. N. Van der Vries, of the University of Kansas, gave a paper on "Geometry for College Juniors and Seniors." The discussion of this paper was led by Professor Mary Newson, of Washburn College. The paper will be published in a later issue of the MONTHLY.

The committee in charge consisted of Professor A. J. Hoare, *Chairman*, Dr. S. Lefschetz, Professor B. L. Remick, Professor W. A. Harshbarger, and Professor T. E. Mergendahl, *Secretary*.

The next meeting will be held at the time of the State teachers' association in November, 1916. This meeting will be devoted to a consideration of a rearrangement of the freshman college algebra work, due to a reduction by legislative enactment of one-half unit in the requirement for admission.

The meeting next spring will be devoted to a consideration of the algebra course for college juniors and seniors, to be arranged in accordance with the freshman work decided upon at the fall meeting.

T. E. MERGENDAHL,
Secretary of the Kansas Section.

A TRIBUTE TO ANDREW WHEELER PHILLIPS.

In the death of Professor Andrew Wheeler Phillips, January 20, 1915, Yale University lost one of her truly great men, and without doubt the most beloved teacher ever connected with that institution.

Born in 1844 amid the rocks and hills of Connecticut, he passed his early life as a typical New England farmer boy. His home training was under a father and mother thrifty, intelligent, and devoutly religious. The district school and three summer vacations spent in a select school at Jewett City constituted his elementary schooling. His liking for mathematics came early, for in a letter he wrote: "When I got bigger and was trusted to go off alone and harrow a field, how many times have I let the oxen rest under the shade of a big tree as I smoothed off the rough ground and drew geometrical diagrams and solved knotty problems, while the oxen looked at me with tender eyes and chewed their cuds with contentment and happiness!"

He taught his first school when a lad of sixteen, and at the age of twenty he became a teacher in the Cheshire School. Though he had never been a student in an academy or high school, he soon ranked with the best teachers in the academy at Cheshire. During the forty-seven years, from his appointment at Cheshire until his retirement from the Yale Graduate School in 1911, he did every year full work as a teacher and administrative officer. This shows that the education which fitted him so well for his life work was largely the product of his own private study and keen observation. Professor Newton admitted him to his class in mathematics for graduates as a special student in the fall of 1870, and placed the recitations on Saturday so that Phillips might attend. After three years he received the degree of Bachelor of Philosophy, and in the winter

of 1876 he left the academy at Cheshire to become a tutor in mathematics at Yale. He continued his studies in mathematics and in other departments, and his advancement to the Doctor's degree and to higher appointments was steady and natural. Anyone who never knew him might ask how a man who never was an undergraduate student for a day in a college, or even a pupil in a high school, should become the head of the Yale Graduate School.

Professor Phillips had much technical mathematical ability and wrote several books and numerous articles for the mathematical journals. It is a significant fact that each one of his books was written in collaboration with one of his colleagues. Dean Henry P. Wright said of him: "He lived to do good and make others happy, and could do nothing to lessen the happiness even of those who had injured him. He was wholly unselfish. I cannot think of him as forming any plan solely for his own advancement. He seemed to ignore his own interests in his devotion to the interests of others. He was always busily occupied, but it was for other persons or for other objects than his own—for the Cheshire school, or the Hopkins grammar school, or the Hotchkiss school; for the college; for the graduate department; for the Bicentennial Fund; for St. Thomas's church in New Haven; for his students, and those who had been his students; for his colleagues; for his friends, or his family; for the assistants in his office, or for the servants in his house. He was always ready to help when an opportunity offered."

As a teacher of young men, he was firm and kind. Every student who came in contact with Andy, as we used to call him, immediately felt his wonderful personality. His love and enthusiasm for work were so great that even the laziest student somehow managed to work some for him. One of them once said: "Work is catching in Andy's class." His interest in his students was so great that every student who took any course with him felt that he was an intimate and personal friend. His memory is now dear to all his students, not for the mathematics he taught so well, but for the high ideals he inculcated by his beautiful character and example. His students and friends will always think of him as they knew him, full of life, sunshine, and human sympathy.

H. T. Burgess.

University of Wisconsin.

BOOK REVIEWS.

Send all communications to W. H. Bussey, University of Minnesota.

Diophantine Analysis. By Robert D. Carmichael. John Wiley and Sons, Inc., New York, 1915. vi+118 pages. $1.25.

Greek theory of numbers, like Greek geometry, has come down to us in a remarkably disconnected and unsystematized form. The method used in solving one problem gives little hint of the method to be used in solving a second which may be apparently closely related to the first. The theorems in Euclid are

arranged in masterly sequence, but the proofs stand by themselves, a different scheme for each. In Diophantine analysis, not only do the solutions stand apart from each other, but the theorems themselves seem unrelated and patchy. This characteristic has made it the favorite hunting ground for the amateur, who finds there little necessity for relating his work to other theories with which he may not be on speaking terms. This same characteristic has also attracted to the subject some of the finest minds in the world of science: men who have labored to pick out of this vast, confused heap of unrelated materials some connecting thread with which to bind the whole theory together. No one thing indicates more clearly the miraculous insight of Fermat than his instinctive appreciation of the importance of the so-called Pellian equation. Whether he had the key to his famous last theorem or not, he certainly had some clear notion of the place it should occupy in the Theory of Numbers. Fermat tried to do for Diophantine Analysis what his great contemporary, Desargues, did for pure geometry. Since Fermat's discovery of the method of "infinite descent" very few general methods have been found applicable to this kind of analysis. Since all solutions of the Pellian equation seem to be only disguises of the solution by continued fractions, it is not improbable that Fermat was familiar with this powerful algorithm.

Professor Carmichael in his book calls attention to two other methods which he calls the "method of multiplicative domains," and the "method of functional equations." As an illustration of what he means by the "method of multiplicative domains" he derives from the identity:

$$(m^2 + n^2)^3 = (m^3 - 3mn^2)^2 + (3m^2n - n^3)^2$$

the following two double-parameter solutions of the equation,

$$x^2 + y^2 = z^3:$$

$$x = m^3 + mn^2, \qquad y = m^2n + n^3, \qquad z = m^2 + n^2;$$

$$x = m^3 - 3mn^2, \qquad y = 3m^2n - n^3, \qquad z = m^2 + n^2.$$

The theory of the "multiplicative domain" is developed at some length in Chapter II. The above illustration indicates what may be deduced from the fact that the sum of two squares multiplied by the sum of two squares is again the sum of two squares. This general multiplicative property does not hold for the sum of two cubes, and Professor Carmichael shows how by increasing the number of variables a new set of numbers may be obtained which includes the given set and enjoys the multiplicative property. Thus, instead of the numbers represented by $x^3 + y^3$, which do not form a multiplicative domain, he uses the numbers $x^3 + y^3 + z^3 - 3xyz$, which do form such a domain. The underlying theory is contained in the theory of composition of forms. Thus the theorem that "No form can be transformed into the product of two forms of the same sort (irreducible forms being understood to have at least one invariant different from zero), unless the number of its indeterminates is a multiple of its order," indicates why the numbers $x^3 + y^3$ do not form a multiplicative domain.

In the "method of functional equations" Professor Carmichael makes use of rational solutions of certain types of Diophantine problems of importance in the work of Diophantus itself. A general systematic development of this method is not attempted.

Chapters III and IV, on the equations of the third and fourth degrees, are very valuable in giving in elegant form the principal results so far obtained in this field. Professor Carmichael has done good service in collecting and arranging not only his own researches but the results of the labor of others as well. A short account is given in Chapter V of certain higher equations, ending with a very complete statement of our knowledge regarding Fermat's Last Theorem.

Professor Carmichael has omitted entirely the theory of binary quadratic forms and has made no use of the theory of continued fractions. This has resulted in some rather unsightly patches in the book. The discussion of the equation $x^2 - Dy^2 = z^2$ makes the author almost as much trouble, and takes up almost as much space as a straightforward discussion of the continued fraction representing a quadratic surd would require.

The book will serve a valuable end in stimulating interest in this delightful branch of mathematics. There are many interesting and suggestive problems at the end of each chapter, some of which are excellent material for further research.

D. N. LEHMER.

UNIVERSITY OF CALIFORNIA.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

457. Proposed by FRANK IRWIN, University of California.

If a be any number prime to m and m/a be developed as a continued fraction,

$$\frac{m}{a} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k},$$

with $a_1 \neq 0$, then there will exist a number b such that $m/b = a_k + 1/a_{k-1} + \cdots + 1/a_2 + 1/a_1$. Show that $ab \equiv \pm 1 \pmod{m}$ and determine the sign.

458. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that n terms of the series $1 + 3 + 4 + 6 + 7 + 9 + 10 + \cdots$ is $\frac{1}{4}(n+1)(3n-1)$ when n is odd, and $n/2[(3n/2) + 1]$ when n is even.

459. Proposed by C. N. SCHMALL, New York City.

By d'Alembert's test, or otherwise, show that in the infinite series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots + \frac{n^nx^n}{n!} + \cdots$$

the upper limit of the interval of convergence is $1/e$ where e is the Naperian base, *i. e.*, $x < 1/e$ when the series is convergent. (Bromwich's *Infinite Series*, pp. 28, 33.)

GEOMETRY.

489. Proposed by NATHAN ALTSHILLER, The University of Colorado.

The parallels to the asymptotes a, b of a given hyperbola, drawn from a variable point of the curve, meet a and b in P, Q respectively. The line PQ envelops an hyperbola whose asymptotes are a and b.

490. Proposed by ELMER E. MOOTS, University of Arizona.

In any quadrilateral $ABCD$, let AC and BD be the diagonals intersecting in K. On AC, lay off CR equal to AK. Join B and R. Connect the middle point G of BR with D. On GD, lay off GM equal to $\frac{1}{3}GD$. Show that M is the center of gravity of the quadrilateral.

491. Proposed by N. P. PANDYA, Sojitra, India.

In a triangle $mx = b$ and $nx = c$, determine a relation between m, n, x, A and s and solve it for x.

CALCULUS.

407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

408. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The ellipse $(x^2/81) + (y^2/16) = 1$ is revolved around the y-axis. Find the area of the surface generated.

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

MECHANICS.

326. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A uniform beam, of length $2l$, rests in equilibrium against a smooth vertical wall and upon a peg at a distance a from the wall; show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{a}{l}\right)^{\frac{1}{3}}.$$

327. Proposed by C. N. SCHMALL, New York City.

An inclined plane makes an angle ϕ with the horizontal plane, and from its foot a body is projected upward at an angle ψ to the plane, and with velocity v. Show that it will strike the plane *perpendicularly* if $\tan \psi = \frac{1}{2} \cot \phi$ and that its range up the plane in that case will be

$$\frac{2v^2 \sin \phi}{g(1 + 3\sin^2 \phi)}.$$

NUMBER THEORY.

244. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine the rational value of x that will render $x^2 + px + q$ a perfect square. What value of x will render $x^2 - 7x + 2$ a perfect square?

245. Proposed by NORMAN ANNING, Chilliwack, B. C.

When all the letters denote positive integers and when the a's are primes of the form $4k + 1$, the equation

$$x^2 + y^2 = (a_1a_2a_3 \cdots a_m)^n,$$

has, in Legendre's notation, $E((n+1)^m/2)$ solutions. Show that in 2^{m-1} of these solutions x and y are relatively prime.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

445. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a}.$$

I. Solution by the Proposer.

This is a generalization of the series forming the first member of equation (2) in the Proposer's solution of problem No. 424 (*June* 1915 issue, p. 205), obtained by multiplying the upper indices $n, n-1, n-2, \cdots$ by the constant factor c. Following the method employed in that solution, we find that the given series equals

$$\Delta^a\binom{cn-ca}{a},$$

the finite differences being taken with respect to n.

Now

$$\Delta_x\binom{cx}{a} = \binom{c\,\overline{x+1}}{a} - \binom{cx}{a},$$

the second member of which may be written in the following form:

$$\binom{cx+c}{a} - \binom{cx+c-1}{a} + \binom{cx+c-1}{a} - \binom{cx+c-2}{a} + \binom{cx+c-2}{a} - \cdots$$
$$+ \binom{cx+1}{a} - \binom{cx}{a},$$

since all these terms, except the first and last, alternately cancel each other. Combining the terms in pairs and noticing that

$$\binom{cx+c}{a} - \binom{cx+c-1}{a} = \binom{cx+c-1}{a-1},\quad \binom{cx+c-1}{a} - \binom{cx+c-2}{a} = \binom{cx+c-2}{a-1},$$

etc., we have

$$\Delta_x\binom{cx}{a} = \binom{cx+c-1}{a-1} + \binom{cx+c-2}{a-1} + \cdots + \binom{cx+1}{a-1} + \binom{cx}{a-1},$$

which means that the first difference, taken with respect to x, of the binomial coefficient $\binom{cx}{a}$ having for its lower index a, equals the sum of c binomial coefficients, each having for its lower index $a-1$.

In the same manner, the second difference $\Delta_x{}^2\binom{cx}{a}$ may be expressed as the sum of $c \cdot c = c^2$ binomial coefficients, each having $a-2$ for its lower index; and continuing this process, we find that the ath difference $\Delta_x{}^a\binom{cx}{a}$ may be expressed as the sum of c^a binomial coefficients, each having for its lower index $a - a = 0$ and hence each equal to 1. In other words,

$$\Delta_x{}^a\binom{cx}{a} = c^a,$$

and, similarly,

$$\Delta_n{}^a\binom{c\,\overline{n-a}}{a} = c^a.$$

We thus arrive at the result:

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a} = c^a.$$

II. Solution by Norman Anning, Chilliwack, B. C.

Define operators E and Δ as follows:

$$E^h f(n) = f(n+h), \qquad \Delta f(n) = f(n+1) - f(n) = (E-1)f(n).$$

Then the given expression,

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a}$$

$$= (1-E^{-1})^a\binom{cn}{a} = (\Delta E^{-1})^a\binom{cn}{a} = \Delta^a E^{-a}\binom{cn}{a} = \Delta^a\binom{cn-ca}{a}.$$

Now

$$\binom{cn-ca}{a} = \frac{1}{a!}\{c^a n^a + \text{terms in } n \text{ of degree lower than } a\}.$$

Hence,

$$\Delta^a\binom{cn-ca}{a} = \frac{1}{a!}\{c^a\cdot a! + 0\} = c^a.$$

446. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Solve the equations

$$x^2(y-z) = l^2(m-n), \qquad y^2(z-x) = m^2(n-l), \qquad z^2(x-y) = n^2(l-m).$$

Solution by H. S. Uhler, Yale University.

An equation involving only one variable x may be obtained in the following manner. Equating the expressions for z derived from the first two given equations we find

$$y - l^2(m-n)/x^2 = x + m^2(n-l)/y^2$$

or

$$x^3y^2 - x^2y^3 + m^2(n-l)x^2 + l^2(m-n)y^2 = 0. \tag{1}$$

Substituting the value of z from the first in the third given equation and reducing, we obtain

$$x^5y^2 - x^4y^3 - n^2(l-m)x^4 - 2l^2(m-n)x^3y + 2l^2(m-n)x^2y^2 + l^4(m-n)^2x - l^4(m-n)^2y = 0. \tag{2}$$

Multiplying (1) by x^2 and subtracting (2) from the product gives, after removing the factor $m-n$,

$$(lm - mn + nl)x^4 - 2l^2x^3y + l^2x^2y^2 + l^4(m-n)x - l^4(m-n)y = 0. \tag{3}$$

Multiplying (1) by l^2, (3) by y, adding, and dividing by x,

$$(lm - mn + nl)x^3y - l^2x^2y^2 + l^2m^2(n-l)x + l^4(m-n)y = 0. \tag{4}$$

Adding (3) and (4) and dividing by x,

$$(lm - mn + nl)x^3 + (lm - mn + nl - 2l^2)x^2y + l^2[l^2(m-n) + m^2(n-l)] = 0. \tag{5}$$

Substituting the expression for y from (5) in (4) and removing the factors $(l-m)(l-n)$ we derive the following quadratic in x^3, namely

$$(lm - mn + nl)^2x^6 - 2l^4[m^2(l-n) + n^2(l-m)]x^3 - l^6(mn - nl + lm)(nl - lm + mn) = 0.$$

Consequently

$$x^3 = l^3$$

and

$$x^3 = \frac{l^3[l^2(m-n)^2 - m^2n^2]}{(lm - mn + nl)^2}.$$

Finally, the six values of x are

$$l,\ l\omega,\ l\omega^2,\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}}{(lm-mn+nl)^{\frac{2}{3}}},\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}\omega}{(lm-mn+nl)^{\frac{2}{3}}},\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}\omega^2}{(lm-mn+nl)^{\frac{2}{3}}},$$

where

$$\omega \equiv (-1+i\sqrt{3})/2.$$

The corresponding values of y and z may be obtained at once by cyclical permutation of the letters l, m, and n.

Remarks: The above analysis was verified by direct substitution of the six sets of values in the given equations. The elimination of y between equations (1) and (2) can be performed in a more elegant manner by Sylvester's dialytic method, but the simplification of the determinant of the eighth order requires too much space for publication.

Also solved by A. H. HOLMES and NORMAN ANNING.

GEOMETRY.

471. Proposed by C. N. SCHMALL, New York City.

In the ellipse $x^2/a^2 + y^2/b^2 = 1$, an equilateral hexagon is inscribed with two sides parallel to the major axis. In the major auxiliary circle the same thing is done. If H_1 and H_2 be the sides of the hexagons, and e the eccentricity of the ellipse, show that $H_1 : H_2 :: 4 - 2e^2 : 4 - e^2$.

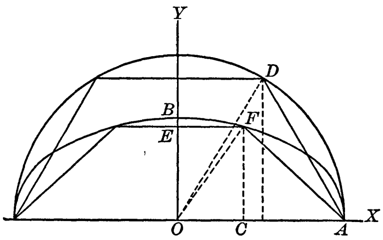
SOLUTION BY J. A. CAPARO, University of Notre Dame.

Since AD is the side of a regular hexagon inscribed in the major circle of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we have

$$DA = OA = a = H_2$$

and since

$$e^2 = \frac{a^2-b^2}{a^2}, \qquad b^2 = H_2^2(1-e^2), \qquad \therefore\ \frac{x^2}{H_2^2} + \frac{y^2}{H_2^2(1-e^2)} = 1. \tag{1}$$



Now $EF = H_1/2$; therefore, from (1) we have

$$FC^2 = y^2 = \frac{(4H_2^2 - H_1^2)(1-e^2)}{4}.$$

Also $CA = OA - OC$. Hence, $CA = H_2 - (H_1/2)$; and since $FA^2 = FC^2 + CA^2$, we have

$$4H_1^2 = (4H_2^2 - H_1^2)(1-e^2) + (2H_2 - H_1)^2.$$

Write $H_1/H_2 = x$. Then, substituting and reducing, we have

$$x^2(4-e^2) + 4x + 4(e^2-2) = 0.$$

Solving for x, we have

$$x = \frac{-2 \pm 2(e^2-3)}{4-e^2}.$$

Using the negative sign, we have $x = (4-2e^2)/(4-e^2)$; and since $x = H_1/H_2$,

$$H_1 : H_2 :: 4 - 2e^2 : 4 - e^2.$$

Also solved by C. E. HORNE, FRANK IRWIN, HAROLD T. DAVIS, NORMAN ANNING, ELIJAH SWIFT, and C. N. SCHMALL.

473. Proposed by FRANK R. MORRIS, Gendale, Calif.

What is the length of the longest rectangle an inch wide that can be placed inside another rectangle 12 inches long and 8 inches wide? Obtain the result correct to the third decimal.

SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

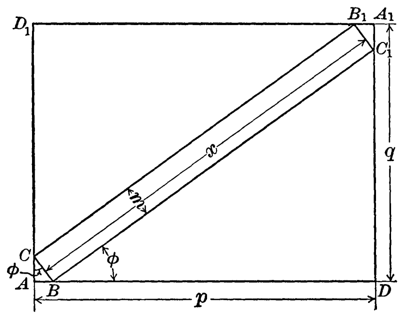
Using the notation of the figure, we have $AB = m \sin \phi$, $BD = x \cos \phi$ and $AB + BD = p$. Then,

$$m \sin \phi + x \cos \phi = p. \tag{1}$$

$AC = m \cos \phi$, $CD_1 = DC_1 = x \sin \phi$ and $AC + CD_1 = q$. Then,

$$x \sin \phi + m \cos \phi = q. \tag{2}$$

Solving (1) and (2) for $\sin \phi$ and $\cos \phi$, we get



$$\sin \phi = \frac{pm - qx}{m^2 - x^2}, \quad \text{and} \quad \cos \phi = \frac{qm - px}{m^2 - x^2}.$$

Squaring, adding, and putting $\sin^2 \phi + \cos^2 \phi = 1$, we get

$$\left[\frac{pm - qx}{m^2 - x^2}\right]^2 + \left[\frac{qm - px}{m^2 - x^2}\right]^2 = 1.$$

Expanding, clearing of fractions and reducing we obtain

$$x^4 - (p^2 + q^2 + 2m^2)x^2 + 4pmqx - (p^2 + q^2 - m^2)m^2 = 0.$$[1]

Putting, $p = 12$, $q = 8$, and $m = 1$, we get

$$x^4 - 210x^2 + 84x - 207 = 0.$$

Solving this equation by Horner's method, we find $x = 13.5176$. Therefore, the length of the inscribed rectangle is 13.5176 inches.

Also solved by H. C. FEEMSTER and J. A. CAPARO.

CALCULUS.

390. Proposed by WILSON L. MISER, University of Arkansas.

Show that the triangle whose area is a constant and whose perimeter is a minimum is equilateral.

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let ABC be the given triangle of constant area equal to k. Denote the base AB of the triangle by x, the perpendicular CD upon AB by y, the perimeter by P and the angle CAB by α.

[1] See Merriman's *The Solution of Equations*, p. 20, 4th ed.

Then,

$$k = \frac{xy}{2}. \tag{1}$$

Now, $P = AC + CB + AB$, and for a minimum $dP/dy = 0$ and $\delta P/\delta\alpha = 0$. But, $AC = y/\sin\alpha = y\csc\alpha$; and since $\overline{CB}^2 = \overline{CD}^2 + \overline{DB}^2$ and $DB = AB - AD = x - y\cot\alpha$,

$$CB = [y^2 + (x - y\cot\alpha)^2]^{\frac{1}{2}} = \left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{\frac{1}{2}}.$$

Then

$$P = y\csc\alpha + \left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{\frac{1}{2}} + \frac{2k}{y}.$$

$$\therefore\quad \frac{dP}{dy} = \csc\alpha + \frac{1}{2}\left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{-\frac{1}{2}}\left(2y\csc^2\alpha - \frac{8k^2}{y^3}\right) - \frac{2k}{y^2} = 0,$$

which can be written

$$(y^2\csc\alpha - 2k)[(y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha)]^{\frac{1}{2}} - (2k + y^2\csc\alpha) = 0.$$

Hence,

$$y^2\csc\alpha - 2k = 0 \tag{2}$$

and

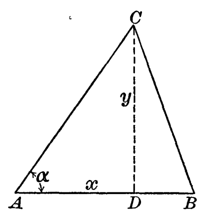
$$y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha = (2k + y^2\csc\alpha)^2,$$

which reduces to

$$-4ky^2\cot\alpha = 4ky^2\csc\alpha.$$

Hence,

$$-\cot\alpha = \csc\alpha \qquad \text{or} \qquad \alpha = 180°.$$



Substituting (1) in (2) we have $y^2\csc\alpha = xy$ or $\sin\alpha = y/x$; and since $\sin\alpha = y/AC$, $AC = x$, that is, $AC = AB$. Also

$$dP/d\alpha = -y\csc\alpha\cot\alpha + \tfrac{1}{2}(y^2\csc\alpha + 4k^2/y^2 - 4k\cot\alpha)^{-\frac{1}{2}}(-2y^2\csc^2\alpha\cot\alpha + 4k\csc^2\alpha) = 0;$$

or reducing,

$$\cos\alpha\cdot(y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha)^{\frac{1}{2}} = 2k - y^2\cot\alpha.$$

Squaring both sides and reducing, we get

$$(y^2\cot\alpha - k)(1 - \cos^2\alpha) = 0.$$

Hence,

$$1 - \cos^2\alpha = 0 \qquad \text{and} \qquad y^2\cot\alpha - k = 0. \tag{3}$$

Solving (2) and (3) for α, we get $\tan\alpha = 2\sin\alpha$. Hence $\alpha = 60°$; and since we have shown above that $AB = AC$ and $\alpha = 60°$ is the included angle between these two lines, it follows that $CB = AB = AC$.

Also solved by FRANK IRWIN.

391. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $0 < \lambda < 1$ and $0 < x < \pi$ show that the function $(\sin\lambda x)/(\sin x)$ increases as x increases.

I. SOLUTION BY H. S. UHLER, Yale University.

Applying the well-known formula

$$\sin\theta = \theta\left[1 - \left(\frac{\theta}{\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{2\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{3\pi}\right)^2\right]\cdots, \qquad [\theta^2 < \infty],$$

to the problem under consideration we get

$$\frac{\sin \lambda x}{\sin x} = \frac{\lambda\left[1-\left(\frac{\lambda x}{\pi}\right)^2\right]\left[1-\left(\frac{\lambda x}{2\pi}\right)^2\right]\left[1-\left(\frac{\lambda x}{3\pi}\right)^2\right]\cdots}{\left[1-\left(\frac{x}{\pi}\right)^2\right]\left[1-\left(\frac{x}{2\pi}\right)^2\right]\left[1-\left(\frac{x}{3\pi}\right)^2\right]\cdots}.$$

Consider now the ratio of the nth binomial factor of the numerator to the nth or corresponding factor of the denominator:

$$\frac{1-\left(\frac{\lambda x}{n\pi}\right)^2}{1-\left(\frac{x}{n\pi}\right)^2} = \frac{n^2\pi^2 - \lambda^2 x^2}{n^2\pi^2 - x^2} = 1 + \frac{(1-\lambda^2)x^2}{n^2\pi^2 - x^2}.$$

For any chosen value of n (1, 2, 3, $\cdots$) the denominator of the last fraction decreases while the numerator increases as x grows larger. The fraction is always finite and positive because of the hypotheses $0 < \lambda < 1$ and $0 < x < \pi$. Consequently, the ratio of any binomial in the numerator of the expression for $(\sin \lambda x)/(\sin x)$ to the corresponding binomial in the denominator increases with x. It is accordingly manifest that the product of an infinite number of such converging ratios increases as x increases, and so the problem is solved.

II. Solution by Frank Irwin, University of California.

We shall show that, for the values of x in question, the derivative of the function is positive. This derivative is

$$\frac{\lambda\cdot\cos\lambda x\cdot\sin x - \sin\lambda x\cdot\cos x}{\sin^2 x}$$

or

$$\frac{1}{x\cdot\sin^3 x\cdot\sin\lambda x}(\lambda x\cdot\cot\lambda x - x\cdot\cot x).$$

This will be positive if $\lambda x\cdot\cot\lambda x > x\cdot\cot x$, that is, if $y\cdot\cot y$, let us say, continually decreases as y increases from 0 to π. This is so, since its derivative, $\cot y - y\csc^2 y$, or $(\sin 2y - 2y)/2\sin^2 y$, is always negative.

A solution similar to the second was received from Elijah Swift.

392. Proposed by HORACE OLSON, Student at The University of Chicago.

Two right cylinders of radii a and b, respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

Solution by Elijah Swift, University of Vermont.

Assume $a > b$. Let the axis of the smaller cylinder be the z-axis, that of the larger, the y-axis. We see that the volume is

$$V = 8\int_0^b\int_0^{\sqrt{b-x^2}}\sqrt{a^2-x^2}\cdot dy\cdot dx = 8\int_0^b \sqrt{(a^2-x^2)(b^2-x^2)}\cdot dx.$$

This latter integral is elliptic, and may be expressed in terms of complete elliptic integrals of the first and second kinds. Letting $x = b\,\mathrm{sn}\,(y, b/a)$, the integral becomes

$$8ab^2\int_0^K\left\{1 - \frac{a^2+b^2}{a^2}\mathrm{sn}^2\, y + \frac{b^2}{a^2}\mathrm{sn}^4\, y\right\}dy.$$

This, in turn, may be integrated by reduction formulas and gives finally for the volume,

$$V = \tfrac{8}{3}a\left[(b^2-a^2)K\left(\frac{b}{a}\right) + (a^2+b^2)E\left(\frac{\pi}{2}, \frac{b}{a}\right)\right],$$

where K denotes the complete elliptic integral of the first kind, E the elliptic integral of the second kind.

Note.—For this same problem, see Byerly's *Integral Calculus*, Chap. XVI, Elliptic Integrals. In the 1902 edition, however, an incorrect answer is given.

Also solved by H. S. UHLER.

MECHANICS.

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

SOLUTION BY H. S. UHLER, Yale University.

For brevity let

$$l \equiv \cos WOX = \cos XOW, \quad p \equiv \cos XOY = \cos YOX,$$
$$m \equiv \cos WOY = \cos YOW, \quad q \equiv \cos XOZ = \cos ZOX,$$
$$n \equiv \cos WOZ = \cos ZOW, \quad r \equiv \cos YOZ = \cos ZOY.$$

Since the forces are in equilibrium the (algebraic) sum of their projections on any straight line must vanish. Hence, by projecting the forces successively upon the lines of action of W, X, Y, and Z, respectively, we obtain the following redundant set of homogeneous equations:

$$W + lX + mY + nZ = 0,$$
$$lW + X + pY + qZ = 0,$$
$$mW + pX + Y + rZ = 0,$$
$$nW + qX + rY + Z = 0.$$

By applying a well-known theorem of determinants to the cofactors of the first row we find

$$W : X :: \begin{vmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: -\Delta_1^2 : \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}. \tag{1}$$

Similarly, for the second line or row

$$W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} 1 & m & n \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : -\Delta_2^2. \tag{2}$$

The determinants constituting the fourth and third terms of proportions (1) and (2), respectively, are equal because the rows of one are the same as the corresponding columns of the other, hence the product of (1) and (2) is

$$W^2 : X^2 :: \Delta_1^2 : \Delta_2^2,$$

or, since we are only dealing with the arithmetical magnitudes of the forces,

$$W : X :: \Delta_1 : \Delta_2.$$

Obviously cyclical permutation of the symbols leads to the required proportions

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4.$$

Remark: From the principles of statics we know that the four original scalar equations must be compatible or simultaneous and, from the theory of determinants, that the necessary and sufficient condition for this compatibility is the vanishing of the determinant Δ of the coefficients of W, X, Y, and Z. Hence, the six cosines l, m, n, p, q, r pertaining to a closed space quadrilateral are not all independent but are mutually connected by the relation $\Delta = 0$, that is,

$$l^2 + m^2 + n^2 + p^2 + q^2 + r^2 - l^2r^2 - m^2q^2 - n^2p^2 - 2lmp$$
$$- 2lnq - 2mnr - 2pqr + 2lmqr + 2lnpr + 2mnpq = 1.$$

310. Proposed by EMMA M. GIBSON, Drury College.

"A particle moveable on a smooth spherical surface of radius a is projected along the horizontal great circle with a velocity v which is great compared with $\sqrt{(2ga)}$. Prove that its path lies between this great circle and a parallel circle whose plane is at a depth $2ga^2/v^2$ below the centre, approximately."

From Lamb's *Dynamics*, p. 334, Ex. 3.

Solution by H. S. Uhler, Yale University.

Since the particle has only two degrees of freedom let us choose as coördinates the angles θ and ϕ which the radius terminating in the particle (of mass m) makes with any fixed vertical plane which contains the center of the sphere and with the horizontal plane passing through this center, respectively. Since none of the forces acting on the particle has a moment about the vertical diameter of the sphere it follows that the moment of momentum about this axis must remain invariable. At the instant of projection the moment of momentum equals $a \cdot mv$. At some later time the moment of momentum will be $r \cdot mr\dot{\theta}$ or $ma^2(\cos^2\phi)\dot{\theta}$, because $r = a\cos\phi$. Consequently

$$a(\cos^2\phi)\dot{\theta} = v. \tag{1}$$

Another relation between θ and ϕ may be derived from the principle of the conservation of energy. At the instant of projection the total energy equals $\frac{1}{2}mv^2 + V$, where V symbolizes the potential energy at the level of the center of the sphere. In general, at any later time the components of the kinetic energy in the horizontal and vertical planes will be respectively $\frac{1}{2}ma^2(\cos^2\phi)\dot{\theta}^2$ and $\frac{1}{2}ma^2\dot{\phi}^2$, while the potential energy will be $V - mga\sin\phi$. Therefore,

$$a^2(\cos^2\phi)\dot{\theta}^2 + a^2\dot{\phi}^2 - 2ag\sin\phi = v^2. \tag{2}$$

Substituting the value of $\dot{\theta}$ from (1) in (2) we find

$$a^2\dot{\phi}^2 = 2ag\sin\phi - v^2\tan^2\phi$$

or

$$a\dot{\phi} = (2ag\sin\phi - v^2\tan^2\phi)^{\frac{1}{2}}.$$

The last equation shows that ϕ cannot be negative with $\dot{\phi}$ real, in other words, the particle will not rise above the plane of projection. Writing this equation in the form

$$2a\sqrt{2ag}(\cos\phi)\dot{\phi} = [(\sin\phi)(\sqrt{v^4 + 16a^2g^2} + v^2 + 4ag\sin\phi)(\sqrt{v^4 + 16a^2g^2} - v^2 - 4ag\sin\phi)]^{\frac{1}{2}},$$

it becomes self-evident that the greatest value which ϕ can attain without making $\dot{\phi}$ complex is given by the formula

$$4ag\sin\phi_0 = \sqrt{v^4 + 16a^2g^2} - v^2.$$

The corresponding vertical distance below the horizontal diametral plane is $a\sin\phi_0 = d$. Then, rigorously,

$$d = \frac{1}{4g}(\sqrt{v^4 + 16a^2g^2} - v^2).$$

When $4ag/v^2$ is less than unity we may expand the radical by the binomial theorem to obtain an approximate rational expression for d. Then

$$d = \frac{v^2}{4g}\left[\left(1 + \frac{8a^2g^2}{v^4} - \frac{32a^4g^4}{v^8} + \cdots\right) - 1\right]$$

or, as a first approximation,

$$d = 2ga^2/v^2.$$

Remark: If $v = 0$, (1) gives $\dot{\theta} = 0$ or $\theta = constant$, showing that the motion is now uniplanar. Under these conditions (2) reduces to $a\dot{\phi}^2 = 2g \sin \phi$, since $\dot{\phi} \neq 0$. Then $a\ddot{\phi} = g \cos \phi$ or putting $\phi = (\pi/2) + \xi$, $a\ddot{\xi} = - g \sin \xi$ which is the familiar form of the equation of motion of a simple pendulum.

311. Proposed by B. J. BROWN, Student in Drury College.

A particle oscillates in a straight line about a center of force varying as the distance, and is subject to a retardation $k \times (\text{vel.})^2$. If a, b be two successive elongations, on opposite sides, prove that

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

What form does the result take if a is infinite?

From Lamb's *Dynamics*, p. 299, ex. 14.

I. Solution by H. S. Uhler, Yale University.

Let the displacement of the particle at any time t from the center of attraction be denoted by x. To fix the ideas, consider the forces acting on the particle of mass m at an instant when its displacement and velocity are both positive. Then

$$m\frac{d^2x}{dt^2} = - cmx - km\left(\frac{dx}{dt}\right)^2,$$

where c is a positive constant, or

$$\frac{d^2x}{dt^2} + k\left(\frac{dx}{dt}\right)^2 + cx = 0. \tag{1}$$

Equation (1) is a standard form of which the first integral can be obtained at once, for the solution of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 f(y) + F(y) = 0$$

is

$$\frac{dy}{dx}e^{\int f(y)dy} = \left\{ A - 2\int dyF(y)e^{2\int f(y)dy} \right\}^{\frac{1}{2}}.$$

See Forsyth's *Differential Equations*, third ed., Art. 67.
Hence

$$\frac{dx}{dt}e^{kx} = \left\{ A + \frac{ce^{2kx}}{2k^2}(1 - 2kx) \right\}^{\frac{1}{2}}, \tag{2}$$

The constant of integration A in equation (2) may be found from the datum that $dx/dt = 0$ when $x = - a$, whence

$$\frac{dx}{dt}e^{kx} = \frac{1}{k}\left\{ \frac{c}{2}\left[(1 - 2kx)e^{2kx} - (1 + 2ka)e^{-2ka} \right] \right\}^{\frac{1}{2}}. \tag{3}$$

Again, by hypothesis, $dx/dt = 0$ when $x = b$ so that equation (3) gives the required relation

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

When a becomes infinite we have the so-called indeterminate form

$$\frac{1 + 2ka}{e^{2ka}} = 0 + \frac{z}{e^z} = \frac{\infty}{\infty}.$$

In general,

$$\frac{z}{e^z} = \left[z^{-1} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n + 1)!} + \cdots \right]^{-1};$$

consequently as z increases indefinitely ze^{-z} approaches zero as a limit. The same result follows from the well-known fact that "any infinite number is an infinity of higher order than any power of its logarithm."

Finally, since b is not negative, $e^{2kb} \neq 0$ and

$$b = \frac{1}{2k}.$$

Remark: If k were zero the last result would become $b = \infty$ which is consistent with the fact that equation (1) now represents rectilinear simple harmonic motion with infinite amplitude $a = b = \infty$.

II. Solution by Horace Olson, Chicago, Illinois.

From the conditions of the problem, $dv/dt = -cx \mp kv^2$, if the center of force is at $x = 0$. The upper or lower sign of kv^2 is taken according as the particle is moving in a positive or a negative direction.

From the identity $dv/dt = dv/dx \cdot dx/dt = v dv/dx$, we have $v dv/dx = -cx \mp kv^2$. Set $w = v^2$. Then $dw/dx = -2cx \mp 2kw$, a linear differential equation. Whence,

$$we^{\pm 2kx} = \mp \frac{cxe^{\pm 2kx}}{k} + \frac{ce^{\pm 2kx}}{2k^2} + C.$$

v, and, therefore, $w = 0$ when $x = a$ or b. Hence,

$$0 = \left(\frac{ac}{k} + \frac{c}{2k^2}\right) e^{-2ka} + C = \left(-\frac{bc}{k} + \frac{c}{2k^2}\right) e^{2kb} + C;$$

whence

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

When $a = +\infty$, the first member of this equation becomes 0 and $(1 - 2kb)e^{2kb} = 0$. Hence, if b is not $-\infty$, $2kb = 1$.

When $a = -\infty$, we have $-\infty = (1 - 2kb)e^{2kb}$; whence $b = +\infty$.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence, Kansas.

NEW QUESTIONS.

32. In a discussion of the Peaucellier[1] Cell by analytic methods the following equations are obtained:

$$(1)\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\ (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\ (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\ (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\ x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\ x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\ x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating $x_1, y_1, x_2, y_2, x_3, y_3$ gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimina tion.

I. What element of the situation is left unused by the following procedure in the elimination?

(*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$f_1(x_1, y_1) = 0. \tag{8}$$

(*b*) From equations (2), (4), (6) eliminate x_3 and y_3, and obtain an equation

$$f_2(x_1, y_1) = 0. \tag{9}$$

(*c*) From equations (7), (8), (9) eliminate x_1 and y_1, and obtain the desired equation.

II. How should this procedure be supplemented to secure the result?

[1] If reference is made to the article on "Linkages" in the December (1915) Monthly by Mr. Leavens, the following coördinates may be applied to his figure: $O(o, o)$; $C(c, o)$; $P_1(x_1, y_1)$; $M(x_2, y_2)$; $M_1(x_3, y_3)$; $P_2(XY)$.

DISCUSSIONS.

RELATING TO INDETERMINATE FORMS.

I. A QUESTION WITH RESPECT TO 0/0.

By J. W. Nicholson, Louisiana State University.

Has 0/0, (*a*) in particular cases a definite value, or (*b*) is it always indeterminate?

A generation ago mathematicians, in general, answered (*a*) affirmatively and (*b*) negatively; but in recent years, as a rule, they do just the reverse. This change is due to the ruling out of division by zero, because, as is asserted, it is impossible in cases like $a/0$ and indeterminate in cases like 0/0.

Without advocating the one contention above the other, I invite a careful consideration of the following argument, hoping that some one will point out the fallacy, if fallacy there be.

The locus of

$$y = \frac{x^2 - a^2}{x - a} \tag{1}$$

consists of the two straight lines[1]

$$y - x - a = 0 \tag{2}$$

and

$$x - a = 0, \tag{3}$$

as shown by clearing of fractions, transposing and factoring.

Now when $x = a$ in (1), *y has a definite and also an indeterminate value.*

For, according to the definitions of loci and their equations, the value of y when $x = a$ is the ordinate of the point of intersection of the loci of $x - a = 0$ and (1); that is, of the line $x - a = 0 \cdots$ (4) and the two lines (2) and (3).

But the point of intersection of (4) and (2) is $(a, 2a)$, and that of (4) and (3) is $(a, 0/0)$.

Therefore, when $x = a$ in (1) we have

$$y = 2a \quad \text{and} \quad y = 0/0.$$

It will be seen that the above process of determining the definite value $2a$ is the simple operation of finding the intersection of two given lines, and is independent of the idea or doctrine of limits.

By the same method the following more general proposition may be proved: *When $x = x'$ the definite value of*

$$\frac{f(x) - f(x')}{x - x'} \quad \text{is} \quad f_1(x),$$

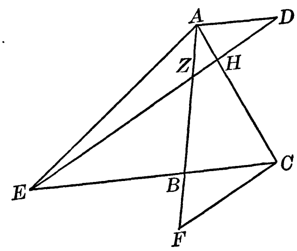
in which $f_1(x)$ is the derivative of $f(x)$.

[1] See, for instance, J. W. Young's *Fundamental Concepts of Algebra and Geometry*, p. 214; Granville's *Differential and Integral Calculus*, p. 171; Townsend and Goodenough's *Essentials of Calculus*, p. 21; McMahon and Snyder's *Differential Calculus*, p. 116; and many others.

II. A GEOMETRICAL ILLUSTRATION OF THE FORM ∞/∞.

By James H. Weaver, High School, West Chester, Pa.

In the American Mathematical Monthly, Vol. XXIII (1916), pp. 41–44, Professor Lovitt has given some interesting illustrations of indeterminate forms. The following is an example of the form ∞/∞ and involves the invariance of cross-ratio. The example and the proof have been taken from Pappus.[1] Some



slight modifications in the statement and proof of the theorem have been made to suit the present needs.

Theorem. *Let there be four lines, AE, AB, AC and AD concurrent in A, and let these be cut by a line $EZHD$ and let EBC be drawn parallel to AD. Then $ED \cdot ZH : EZ \cdot DH = CB : BE$.*

Proof. Through C draw CF parallel to ED, cutting AB produced in F. Since CF is parallel to HZ, $CA : AH = CF : ZH$, and since EC is parallel to AD, $CA : AH = ED : DH$. Therefore $ED : DH = CF : ZH$ and $ED \cdot ZH = CF \cdot DH$. Hence

$$\begin{aligned} ED \cdot ZH : EZ \cdot DH &= CF \cdot DH : EZ \cdot DH, \\ &= CF : EZ, \\ &= CB : BE. \end{aligned}$$

Now AD and BC will intersect at infinity and because of the invariance of the cross-ratio under a projective transformation, we have

$$ED \cdot ZH : EZ \cdot DH = CB \cdot \infty : BE \cdot \infty.$$

In this connection it is interesting to note that the above theorem is the foundation for the celebrated Configuration of Pappus.[2]

RELATING TO THE MECHANICAL TRISECTION OF AN ANGLE.

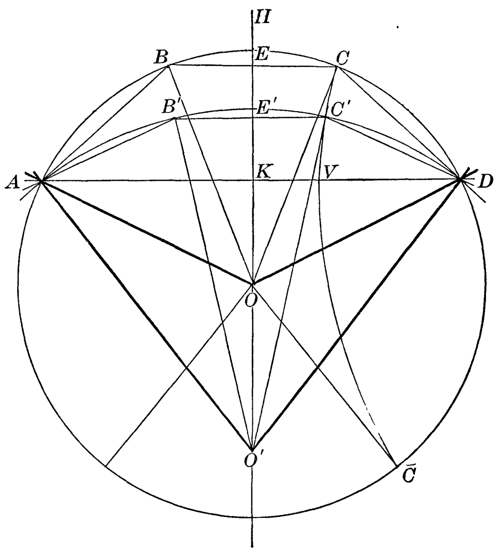
By Bernhart Ingimundson, Student in the University of Manitoba.

Note.—Interest in the trisection problem is perennial. It was old in the time of Plato and is new today. It is probably worth while to republish solutions occasionally in order to remind readers that the problem is readily solvable by means other than ruler and compass. Trisection

[1] Pappus Alexandrinus, *Collectio*, ed. Hultsch, p. 882.
[2] Cf. Veblen and Young, *Projective Geometry*, Vol. I, p. 99.

by means of the hyperbola dates back to the time of Pappus (about 300 A.D.) and the use of the hyperbola of eccentricity 2 is by no means new;[1] but we believe many of our readers will enjoy the solution given below, sent in by a first year engineering student.—EDITOR.

Let AOD be any angle, acute, obtuse or reflex. With radius OA describe a circle intersecting the sides of the angle at A and D. Suppose the arc AD trisected at points B and C and join AB, BC and CD. Bisect the angle AOD by the line OH, cutting BC at E. Then OH is the perpendicular bisector of BC and AD. Hence $CE = CD/2$.



Similarly, if O' be any other point on OH and the angle $AO'D$ be trisected by $O'B'$ and $O'C'$, $C'E' = C'D/2$. Hence, as the point O moves along the line OH the point C describes a curve such that the ratio of CE to CD is constant and equal to $\frac{1}{2}$. Therefore the locus of C is an hyperbola with focus at D, directrix OH and eccentricity 2. Similarly, it may be shown that the reflex angle AOD is trisected by $O\bar{C}$.

We have, therefore, a mechanical means of trisecting any given angle AOD as follows:

Cut off $OA = OD$ and draw OH perpendicular to and bisecting AD at K.

With D as focus and OKH as directrix construct an hyperbola of eccentricity 2 cutting the circle whose center is O and radius OD at C.

CO then trisects the given angle AOD.

The required hyperbola could be constructed by the ordinary pin, thread and ruler method, or, if one uses the same length of chord AD in each case, then the arc of an hyperbola which is such that the distance from the focus to the vertex is $\frac{1}{3}AD$ may be used to obtain the required division.

[1] Cf. articles by Candy in *Kansas University Quarterly*, Vol. II (1893), pp. 35–45; Crawford, *Edinburgh Math. Soc. Proc.*, Vol. XVI, pp. 42–45; Genese, *Messenger of Math.*, Vol. I (1872), pp. 103, 181; and various others, which may be readily found.

SUMMER MEETING OF THE ASSOCIATION.

In response to the cordial invitation of the Massachusetts Institute of Technology, the Council has decided to announce a summer meeting of the Association in Cambridge, Massachusetts. The meeting will be held Friday and Saturday, September 1 and 2, in the new buildings of the Massachusetts Institute of Technology. These dates immediately precede the dates for the summer meeting of the American Mathematical Society. It is hoped that the attendance at both meetings may thus be large. Specially low round-trip rates are already announced from Chicago to Boston, good till October first, and doubtless equally favorable rates will be in effect from other western points.

Plans are in preparation for social meetings on Friday and Saturday evenings, and perhaps for a nearby, week-end excursion between the meetings. Arrangements will be made for luncheon on Friday and Saturday, and information in regard to hotels and boarding places in Boston will be furnished on application to Professor H. W. TYLER, Chairman of the committee on arrangements, Massachusetts Institute of Technology.

The program committee consists of Professor H. B. FINE of Princeton University, Professor C. S. SLICHTER of the University of Wisconsin, and Professor DUNHAM JACKSON of Harvard University. A preliminary announcement of the program will be made at an early date, and a fuller announcement may be ready in time for printing in the June issue.

THE LIBRARY OF THE ASSOCIATION.

As announced earlier, it is the purpose of the Association to organize a library as rapidly as possible. Plans are already under way for placing the facilities of such a library within the reach of any member of the Association who may desire to take advantage of it. Further announcement of particulars along this line will be made at an early date. Meanwhile, any authors of books or publishers who are willing to contribute volumes to the Association Library may send them directly to Professor W. D. CAIRNS, Oberlin College, Oberlin, Ohio. The following books have already been presented for this purpose: Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, by LOUIS C. KARPINSKI, University of Michigan, published by the Macmillan Company; the Napier Tercentenary Memorial Volume, edited by CARGILL G. KNOTT, published for the Royal Society of Edinburgh by Longmans, Green and Company; Fundamental Conceptions of Mathematics, by ROBERT P. RICHARDSON and EDWARD H. LANDIS, published by the Open Court Publishing Company; and Fundamental Sources of Efficiency, by FLETCHER DURELL, published by J. B. Lippincott Company.

As previously stated, a complete set of the MONTHLY has been made available through the courtesy of Professor E. H. MOORE, who has been a subscriber since the establishment of the journal in 1894. These volumes will be bound and placed in the Association Library at an early date.

Unfortunately, complete sets of Volume XX (the first year after the reorganization of the MONTHLY) are nearly exhausted, so that it will be impossible to supply any more libraries or individuals with these volumes, though the calls for them are numerous. But it is proposed to bind the few sets of this volume which still remain and place them in the library in order that they may be loaned to members who may desire to see them. Sets of Volumes XXI and XXII are also running low, but several more may be had at two dollars and fifty cents per volume by those who apply early for them. It has been our purpose to make the issues of Volume XXIII sufficiently large to cover all contingencies; but even here, the unexpected demands have drawn heavily on our supply.

No complete sets of the MONTHLY for the years preceding the reorganization in 1913 are available, but numerous separate volumes and single copies can be had at two dollars per volume or twenty-five cents for single copies; and the Managing Editor will be pleased to assist members of the Association, as far as possible, in completing broken sets or volumes.

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. R. W. BURGESS, of Cornell University, has been appointed instructor in mathematics at Brown University.

In *Science* for April 21 is a four-page paper on "A new method for the graphic solution of algebraic equations," by Professor H. G. DEMING, of the University of the Philippines.

At the spring meeting of the New England Association of Mathematical Teachers held at the Massachusetts Institute of Technology on May 6, Professor R. C. Archibald read a paper on "Euclid and his works."

The one hundred eighty-fourth regular meeting of the American Mathematical Society was held in New York City on Saturday, April 29, 1916. Twenty-six papers were presented at the two sessions. There were present fifty-one members.

Doctor HENRY M. SHEFFER is instructor in philosophy at the College of the City of New York. Doctor Sheffer is a member of the American Mathematical Society and of the Mathematical Association of America and has many interests on the mathematical side of philosophy.

The Benjamin Peirce instructorships at Harvard University have been filled by the appointments of Dr. E. KIRCHER and Dr. W. L. HART. The latter has just received the doctorate at the University of Chicago.

At Sheffield Scientific School of Yale University, Dr. E. J. MILES has been promoted to an assistant professorship of mathematics. Dr. Miles graduated at Indiana University in 1906, was granted the A.M. degree at Swarthmore College in 1907, and received the doctorate at the University of Chicago in 1910.

On the evening of March 4, 1916, Dean GEORGE D. OLDS, head of the department of mathematics in Amherst College, spoke at Mount Holyoke College to the students in the mathematics classes. His subject was "Reminiscences of my German teachers in the early eighties."

The sixth regular meeting of the American Mathematical Society at Chicago, being the thirty-seventh regular meeting of the Chicago Section, was held at the University of Chicago, Friday and Saturday, April 20 and 21, 1916. There were present forty-six members. Thirty papers were presented.

At the March meeting of the Brown University Mathematical Club, sixty-eight members were present, and Professor N. F. DAVIS presided. The following papers were presented by the students: "History and mathematics of the sundial," by Miss RUTH H. HALL; "Linkages with numerous models," by B. H. BROWN.

"The Mathematical Association of America," is the title of an article by President E. R. HEDRICK in *School and Society* for March 11, 1916. In this paper President Hedrick sets forth the history culminating in the organization of the Association, especially in its relation to the American Mathematical Society.

Miss OLIVE C. HAZLETT has published her Doctor's dissertation in the *American Journal of Mathematics*. The title is: "On the classification and invariantive characterization of nilpotent algebras." Doctor Hazlett is to begin work as a member of the faculty of Bryn Mawr College in September, 1916.

An indication of the great decrease in the number of new publications in the natural and exact sciences, occasioned by the European war, is the size of Vol. 37 (1915) of "Naturæ Novitates." It contains only 340 pages, as compared to more than 500 pages in 1914, and upwards of 620 pages each for the years of 1912 and 1913.

Upon the occasion of the celebration of the seventieth birthday of the distinguished Swedish mathematician, Professor M. G. MITTAG-LEFFLER, he and his wife set aside their entire fortune to be used in founding an international institute for pure mathematics. The *Acta Mathematica* has been edited by Professor Mittag-Leffler since its founding in 1882.

The first meeting of the Ohio Section of the ASSOCIATION was held at Columbus, on April 21, 22, 1916. This section was formed at Columbus at the same time that the national association was organized and application for admission was made then. A full report of this meeting will be printed in the June issue.

"Studies on Divergent Series and Summability," is the title of a monograph by Professor W. B. FORD, which constitutes Volume II of the Scientific Series of Publications by the University of Michigan. These studies are published by the Macmillan Company. The first volume, which was recently reviewed in the MONTHLY was "Robert of Chester's Latin translation of the algebra of Al-Khowarizmi," by Professor KARPINSKI.

In the publications of the General Education Board, ABRAHAM FLEXNER has discussed the topic, "A Modern School," in which his reference to mathematics is much along the same line as that of some other writers, such as the commissioner of education of Massachusetts and the state superintendent of New Hampshire. This document is quite fully outlined in *School and Society* for April 22, 1916.

A number of reprints have been received from the *Proceedings of the Edinburgh Mathematical Society,* entitled: "Research papers of the mathematical department of the University of Edinburgh." The authors are: E. T. WHITTAKER, E. LINDSAY RICE, L. R. FORD, A. G. BURGESS, EDWARD BLADES, and ARCHIBALD MILNE. This mathematical laboratory is under the direction of Professor E. T. WHITTAKER, and the publications relate for the most part to the field of applied mathematics.

The initial number of a mathematical periodical, *Revista de Matematicas,* edited by MANUEL GUITARTE, Buenos Aires, Argentina, appeared recently, and contains articles as follows: "Mathematics, its nature and importance," by C. C. DASSEN; "On algebraic division," by E. REBUELTO; "Questions of elementary mathematics related to the theory of groups and the differential calculus," by J. DUCLOUT; "The potential function in hyperspace," and "Area of rectilinear triangles, some particular cases," by M. GUITARTE. The *Revista* is published in the Spanish language and is to appear bi-monthly.

At the Kansas State Agricultural College, a mathematical club has been organized and in operation for several years. The club is composed of students and members of the departmental faculty; it meets bi-weekly and presents interesting programs upon pedagogical, historical and applied subjects. The printed program for the current year shows a very interesting selection of topics. The membership as indicated upon the program consists of one hundred and twenty-four persons.

At the twenty-eighth Educational Conference of the academies and high schools in relations with the University of Chicago, on Friday and Saturday, April 14 and 15, 1916, the mathematical section, which was in charge of Professor J. W. A. YOUNG, had for its general subject, "The consideration of the relative importance of topics usually treated in algebra and plane and solid geometry with reference especially to the credit value of the unit." There were also

discussions of current committee reports on the "definition of the units in mathematics" and on "mathematics in the secondary school of tomorrow."

The Mathematical Club of Smith College was organized in 1899. Its membership is limited to students of the senior and junior classes who are carrying major courses in mathematics. Professor Cushing is president and other members of the faculty and the graduates living in the city are admitted to honorary membership. The club meets once in three weeks for the consideration of papers presented by the students or the faculty, and two social meetings are held annually. During the last year a study was made of the history of mathematics, considering first the general development of the subject, and later some of the great names in mathematical literature.

Mr. PHILIP E. B. JOURDAIN, of Fleet, near London, contributes to *Mind* (Vol. 25, pp. 42–55) an article on "Zeno's Flying Arrow: An Anachronism," in which Zeno is represented in hades as discussing with Socrates the views regarding motion held in ancient and modern times. Zeno comments even on Bertrand Russell's Lowell Lectures of 1914. The article is extremely entertaining. Jourdain holds that Zeno desired to combat the Pythagorean belief that lines are made up of points. The version of the Arrow by Sextus Empiricus, "a man never dies, for if a man die, it must be either at a time when he is alive or at a time when he is not alive," is compared with the claim of the Chinese philosopher, Hui Tzŭ, that "a motherless colt never had a mother; for when it had a mother it was not motherless, and at every other moment of its life it had no mother."

At the meeting of the Iowa Academy of Science in Des Moines on April 27, the members of the Mathematical Association of America who were present held a conference and decided to petition the Council for the establishment of an Iowa section. An organization was effected with Professor A. G. SMITH, of the State University, as president, Professor G. A. CHANEY, of the State College of Agriculture and Mechanic Arts, as vice-president, and Professor I. F. NEFF, of Drake University, as secretary-treasurer. Professor R. B. McClenon of Grinnell College, and Professor J. F. Riley of the State University were appointed a committee on membership. The officers were instructed to prepare and present a petition to the Council for the establishment of the Iowa Section, the petition to be sent for signatures to all the thirty-two members of the ASSOCIATION resident in the State of Iowa.

"Constructive Geometry" is the title of a pamphlet of 75 pages by E. R. HEDRICK, recently published by the Macmillan Company. It is in the form of a notebook with blank pages interspersed. The constructions and suggestions are exceedingly interesting and well organized. The following quotation from the preface adequately states the character and purpose of the work: "The saying is trite that students who enter formal courses in Euclidean Geometry have to learn both the strange methods of formal logic and the equally strange geometric forms. A course to acquaint students with the elementary forms and

constructions is valuable particularly to those who never go on to a more formal course and it furnishes a basis for a truer comprehension by those who do go on. Such courses are deservedly popular in Europe, but no good American geometric notebook exists. This is modeled after those long used successfully in England, some of which have been extensively used in America."

The Twenty-First Summer Meeting and Eighth Colloquium of the American Mathematical Society will be held at Harvard University during the week beginning Monday, September 4, 1916. The first two days will be devoted to the regular sessions for the presentation of papers. The Colloquium will open on Wednesday morning and close on Saturday morning. Two courses of five lectures each will be given by Professors G. C. EVANS, of Rice Institute, and OSWALD VEBLEN, of Princeton University. A brief outline of the courses is given in the May issue of the *Bulletin*. The year 1916 marks the twenty-fifth anniversary of the broadening out of the Society toward a national organization and of the founding of the *Bulletin*. It is proposed to arrange an appropriate celebration of this event at the summer meeting. Some seventy-five of the members of the Society in the year 1891 have retained their membership through these twenty-five years. Many of these will, doubtless, attend the celebration, which is itself a notable milestone in the Society's progress. The younger generation, on whom rests the Society's future, should also be there in large numbers, to take over the responsibility of guiding the destinies of the Society for the next quarter century.

The annual meeting at Ann Arbor of the Mathematics Section of the Michigan Schoolmasters' Club was opened by a luncheon Thursday noon, March 30, Newberry Hall, at which some sixty were present. Several five-minute talks contributed to the pleasure of the occasion. After the luncheon the meeting adjourned to the auditorium of the New Science Hall, where Professor Karpinski presented an address, illustrated by stereopticon, on The Story of Algebra. A second meeting was held on Friday at which several short papers were presented on the relation of higher mathematics to elementary mathematics. Professors Glover, Bradshaw, Running, Field, and Dr. Hopkins participated in the discussion, with reference to actuarial mathematics, projective geometry, calculus, mechanics and analytical geometry, respectively. The remainder of the program was devoted to various phases of the teaching of high-school mathematics. Of particular interest was an account of the continuation-school work at the Cass Technical School, with pupils drawn largely from the workers in the Detroit automobile plants. Mr. Lewis Hayes, an experienced and graduate engineer, who is in charge of the work in mathematics, gave an illuminating account of the preparation in arithmetic, algebra, geometry, and trigonometry which is actually required by the skilled workers in manufacturing plants. At the meeting next year it is proposed to discuss the question of forming a Michigan Section of the Mathematical Association of America. Mr. John Craig, of Muskegon, and Mr. W. V. Garretson, of Ann Arbor, were elected chairman and secretary, respectively, of the Section for the coming year.

VOLUME XXIII JUNE, 1916 NUMBER 6

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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FIRST ANNUAL MEETING OF THE OHIO SECTION.

The Ohio Section of the Mathematical Association of America was organized at Columbus, December 30–31, 1915, in connection with the meetings for the founding of the national Association. Twenty-five Ohio teachers of collegiate mathematics were registered in the organization meetings. A constitution was adopted and the following officers elected: Professor R. B. ALLEN, Kenyon College, Chairman; Professor G. N. ARMSTRONG, Ohio Wesleyan University, Secretary-Treasurer; and Professor C. C. MORRIS, Ohio State University, third member of the Executive Committee. The Section applied to the national Association for a charter, which was granted March 1, 1916.

The first annual meeting of the Section was held at the Ohio State University, Columbus, on April 21–22, 1916, in connection with the meetings of The Ohio College Association, The Ohio Academy of Science, The Ohio Society of College Teachers of Education, and The Association of Ohio Teachers of Mathematics and Science.

The following forty persons were in attendance, of whom all but the last six had already joined the Association:

R. B. Allen, Kenyon College; F. Anderegg, Oberlin College; W. E. Anderson, Wittenberg College; G. N. Armstrong, Ohio Wesleyan University; C. L. Arnold, Ohio State University; C. B. Austin, Ohio Wesleyan University; Miss Grace M. Bareis, Ohio State University; J. B. Brandeberry, Toledo University; Miss Elizabeth Burnell, Lake Erie College; A. G. Caris, Defiance College; George E. Carscallen, Hiram College; Oscar Dustheimer, Baldwin-Wallace College; J. B. Faught, Kent State Normal College; T. M. Focke, Case School of Applied Science; Miss Harriet E. Glazier, The Western College; M. E. Graber, Heidelberg University; Harris Hancock, University of Cincinnati; E. J. Hirschler, Bluffton College; William Hoover, Ohio University; Christian Hornung, Heidelberg University; Charles A. Hutchinson, Wittenberg College; H. W. Kuhn, Ohio State University; Miss Gertrude McCain, Oxford College for Women; G. W. McCoard,

Ohio State University; Charles N. Moore, University of Cincinnati; C. C. Morris, Ohio State University; A. D. Pitcher, Western Reserve University; S. E. Rasor, Ohio State University; Miss Hortense Rickard, Ohio State University; Karl D. Swartzel, Ohio State University; L. E. Urner, Miami University; C. J. West, Ohio State University; Forbes B. Wiley, Denison University; B. F. Yanney, College of Wooster; Miss Lily Batterham, Ohio State University; Miss Lucile Brown, Ohio State University; C. W. Keyser, Zanesville; Miss Hazel E. Schoonmaker, Denison University; T. Elmer Trott, Mount Union College; A. E. Young, Miami University.

The members of the Section dined together at the Ohio Union on Friday at six o'clock, and later in the evening attended an address by Professor Charles H. Judd, of the University of Chicago, on "The more complete articulation of higher institutions with the high school."

On Saturday morning a symposium was conducted by all the associations and the Section was represented by Professor A. D. Pitcher, of Western Reserve University, who spoke on "Mathematics and the college curriculum." Following this a joint session was held with the Association of Ohio Teachers of Mathematics and Science.

At the business sessions the following motions were passed: (1) That one meeting should be held each year in the spring in conjunction with the Ohio College Association; (2) that a collection of twenty-five cents each be taken to meet the expenses of this meeting for printing and postage; (3) that the national Association be asked to appropriate five per cent. of the annual dues and at least one half of the initiation fees of new members in each section for financing the sections and stimulating them to retain and to gain members for the Association. The officers elected for the following year are: Professor T. M. Focke, Case School of Applied Science, President; Professor G. N. Armstrong, Ohio Wesleyan University, Secretary-Treasurer; Professor C. C. Morris, Ohio State University, member of Executive Committee.

The following formal papers were presented:

(1) "What elective courses following the calculus should the average college offer?" Professor A. E. Young, Miami University. (Introduced by the Chairman of the Section.)

The discussion was led by Professors C. C. Morris, S. E. Rasor, and M. E. Graber.

(2) The Chairman's address on "Hypercomplex number systems." Professor R. B. Allen, Kenyon College.

(3) "Divergent series and their applications." Professor C. N. Moore, University of Cincinnati.

(4) "The bearing of recent legislation on the teaching of mathematics." Professor B. F. Yanney, College of Wooster.

(5) "What courses should be offered for prospective teachers of secondary mathematics?" Professor Harriet Glazier, Western College for Women.

(6) "Minimum requirements for unqualified endorsement for a teacher of high-school mathematics." Professor J. B. Faught, Kent State Normal College.

The following brief notes were presented:

(7) "A device for the trisection of an angle." Professor Christian Hornung, Heidelberg University.

(8) "The degree of convergence of a Fourier series of a limited number of terms." Professor T. M. Focke, Case School of Applied Science.

(9) "The evaluation of a certain volume integral." Professor F. Anderegg, Oberlin College.

(10) "Some cases of the cubic equation." Professor William Hoover, Ohio University.

(11) "Descriptive geometry analyses for common geometry constructions." Professor G. N. Armstrong, Ohio Wesleyan University.

(12) "An interesting combination of figures." Professor F. B. Wiley, Denison University.

(Note: Owing to lack of time (11) and (12) were omitted.)

Following are abstracts of the papers presented.

(1) In considering the question as to what elective courses should follow the calculus in the ordinary college, Dr. A. E. Young suggested that all such courses could be classified as algebraic, geometric, functional, or applied mathematics. He favored courses taken from the first two classifications for the average student, a functional course for the exceptionally brilliant, and courses in applied mathematics for the prospective engineer.

In discussing this paper, Professor Morris called attention to the fact that the ordinary freshman course covers college algebra, trigonometry, and analytical geometry. These courses are so fundamental and are done in such a superficial manner that he advocates following the calculus with a thorough course in plane and solid analytical geometry in the junior year, followed in the senior year with a course in the theory of equations and the elements of the theory of numbers. A student with this preparation is prepared to proceed in any one of many directions.

In making up a twelve-hour program beyond the calculus, Professor Rasor would include a course in algebra, one in geometry, say plane and solid analytics; and then without question a course in differential equations, or advanced calculus inclusive of differential equations—something to give contact with the practical and usable.

In his discussion, Professor Graber noted that trigonometry and analytical geometry are generally presented by the text book and problem method, but in the calculus freer and more general methods of instruction may be introduced, leading up to the lecture plan. Most of our colleges offer at least one advanced course in each of the four divisions, analysis, mechanics, algebra, and geometry, without special attempt to correlate these subjects with one another or with science and engineering. There should be a well-defined connectivity in the post-calculus courses both with the different fields of mathematics and the applications of science. A spiral extension of elementary mathematics into the outlying

fields of number theory, modern geometry, substitution groups, mechanics, and the fundamental notions of modern mathematics in general, affords abundant material for lectures in the courses following the calculus.

(2) Professor Allen gave a few of the fundamental theorems on hypercomplex number systems, with enough of the proofs to indicate their elementary character; and by use of these showed the only real number systems in which division is unambiguous to be the real system, the ordinary complex system, and the real quaternion system.

(3) In Professor Moore's paper, a brief account of the history of divergent series was given. Some of the principal methods of summation were described, and a few of their more important applications were indicated.

(4) Professor Yanney showed that it is possible, as the laws in Ohio are administered, for a college graduate to qualify for a provisional state certificate to teach in the high schools without any special scholastic training in any subject. This is at variance with what experience has shown to be the best preparation for efficient teaching in secondary schools. However, in spite of this deficiency in the laws, there is good evidence that the better colleges, pretending to prepare secondary teachers, demand fully two years in college mathematics including calculus, followed by special courses in the history and teaching of mathematics. The present situation offers the new Association an opportunity to use its influence in the direction of a better public sentiment in the matter of scholastic equipment of the secondary teacher of mathematics.

(5) Professor Glazier insisted that the college must consider the needs of the secondary teachers in mathematics. What they teach, and how they teach it, will determine largely the place of mathematics in the high school curriculum. In addition to the regular undergraduate courses in mathematics the college should also give: (1) A survey course dealing with the fundamental concepts of algebra and geometry, and relating the college mathematics to the secondary. (2) A pedagogical course for consideration of questions definitely concerned with the teaching of mathematics. (3) A history course which should give the historical background of the subject and familiarity with the mathematical literature and mathematical activities.

(6) According to Professor Faught, the minimum academic requirements for unqualified endorsement for a teacher of high-school mathematics should be (1) a four-year high school course with three years of mathematics and one year of physics. (2) A college course with two years of mathematics, including college algebra, trigonometry, analytical geometry, and calculus, and one year of physics. The professional requirements should be (1) one year of educational psychology, principles of teaching and history of education. (2) One year of special methods in algebra and geometry, with the history of mathematics. (3) One half year of observation and practice teaching of algebra and geometry.

(7) Professor Hornung described a piece of mechanism invented by one of his pupils, about eighteen years of age, whereby any angle may be trisected. It consists of a pencil, a string, and a right triangular prism of small height, so

connected that when the base of the prism is moved along the bisector of the angle, as directrix, the pencil will describe a curve, which, by a proper selection of starting point and focus, will pass through a trisection point of any circular arc subtending the angle whose center is at the vertex of the angle. The curve so drawn is an hyperbola.

(8) It can be shown, theoretically, that the r terms of a series of the usual Fourier form that will best represent all the points on a given curve between 0 and 2π, will be the first r terms of the infinite Fourier series. Professor Focke showed, by means of machine drawn curves, that it is possible to find curves whose equations are of the same form and number of terms, which will give a much better approximation to the given curve over a very considerable part of the interval.

(9) On encountering the problem of finding the volume of the solid bounded by the surface $(x^2/a^2 + y^2/b^2 + z^2/c^2)^2 = x^2/a^2 + y^2/b^2 - z^2/c^2$, it occurred to Professor Anderegg that the solution could be simplified by using an auxiliary solid whose surface is obtained from the given surface by putting $x/a = x'$, $y/b = y'$, $z/c = z'$, so that

$$V = \int\int\int dxdydz = abc \int\int\int dx'dy'dz' = abcV'.$$

The auxiliary surface $(x'^2 + y'^2 + z'^2)^2 = x'^2 + y'^2 - z'^2$ is obtained by revolving the lemniscate $(x^2 + z^2)^2 = x^2 - z^2$, or $\rho^2 = \cos 2\theta$ about the Z-axis, and the volume bounded by it is easily found. The solution of many problems in quadrature and cubature can be simplified by this device.

(10) The instances of cubic equations presented by Professor Hoover consisted mainly of a number of interesting cases of the cubic resulting from discussions in the applications of pure mathematics over a considerable range of subjects, giving concreteness to the instances adduced.

In his address before the general session on Saturday morning, Professor Pitcher called attention to the responsibility of a given college department to those students who can take but little work in the department but who would acquire as broad a knowledge of the subject as possible. The problem of discharging this responsibility is a very difficult one for mathematics on account of its technical character but very important on account of its many values, especially on account of its central position in scientific learning. Probably an improvement of current conditions may be secured by offering a modified first course to be taken by all alike. Such a course could contain a somewhat broader selection of material than at present and omit certain technique without interfering with the values usually attached to a first course. A further help would be an historical synoptic course open to juniors and seniors who by their first two years of college work have shown themselves capable of mature thought and serious work.

G. N. Armstrong,
Secretary.

PASCAL LINE EQUATIONS AND SOME CONSEQUENCES.

By R. D. BOHANNAN, Ohio State University.

I know of no attempt to give the equations of the Pascal lines of the inscribed hexagon of a conic except that made by Salmon in Articles 267–8 of his *Conic Sections*, and this does not reach any definite form.

Salmon takes *abcdef* as the hexagon order; $ab = 0$, as the equation of *ab* (*specifying no particular form of equation*).

The conic circumscribing *abcd*, *defa* is either

$$ab \cdot cd = ad \cdot bc \tag{1}$$

or,

$$de \cdot fa = ad \cdot ef; \tag{2}$$

$$\therefore \quad ab \cdot cd - de \cdot fa = ad(bc - ef). \tag{3}$$

The left hand side of (3) (a conic about $ab - de - cd - af$) thus splits into two factors; one is the diagonal *ad*; therefore the other, $bc - ef = 0$, is the other diagonal, or the Pascal line, since (3) shows that the points *ab*, *de*; *cd*, *fa*; *bc*, *ef*, are on this line.

All that this shows is that the equation of the Pascal may be expressed in terms of those of a pair of opposite sides, and that for any particular hexagon and conic it could be gotten by an appropriate process of subtraction and factoring.

Clearly, Salmon's result (the difference of the equations of a pair of opposite sides) would not, in general, hold if the equations were in "normal form."

Moreover if Salmon's process in Art. 268, to prove concurrency of Pascal lines in a Steiner's g-point, is applied to a Kirkman's h-point, *we may have exactly the same sets of equations to prove concurrency for two entirely different sets of lines* (one line in common).

Consider the g-point and h-point:[1]

[1] This notation is different from Salmon's and better. Salmon indicates the Pascal lines by pairs of horizontal lines, $\left(\frac{ab}{ed}\right)$ indicating the point of intersection of lines *ab*, *ed*, but his horizontal lines do not give the hexagons involved. In both (g) and (h) below, the horizontal lines give the hexagons involved, and, taken in pairs, the Pascal lines of these hexagons. In (g), line (2) is formed from line (1) by writing under each segment of (1) its opposite segment in (1), reversing, in each segment, the hexagon order of letters in (1); (3) from (2) as (2) from (1); lines (1), (2) indicate the Pascal of (1); (2), (3) that of (2); (3), (1) that of (3). In (h) the same procedure is followed, except that the hexagon order is retained in one column (in this particular h-point, the third) and reversed in the other two; (1), (2) give the Pascal of (1); (2), (3) that of (2); (3), (1′) that of (3); (1′) is (1).

In this notation also the quadrilaterals to which to refer the conic to prove concurrency for (g), (h) are indicated. For concurrency of (g) the Pascals may be expressed in terms of the opposite sides in any column, and the reference quadrilaterals are the lines of the two remaining columns. For concurrency of (h), we must express the Pascals in terms of the sides of the 3d column (always that column in which the hexagon order is held) and the reference quadrilaterals are the lines of the first and second columns. Note now that the third columns of (g) and (h) indi-

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & af, & cb \\ cf, & eb, & ad \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \quad (g); \qquad \begin{bmatrix} ab, & cd, & ef \\ ed, & af, & bc \\ bf, & ec, & da \\ dc, & ba, & fe \end{bmatrix} \begin{matrix} (1) \\ (5) \\ (6) \\ (1') \end{matrix} \quad (h).$$

Selecting as reference quadrilaterals for (g)

$$abcd, \quad edaf, \quad cfeb,$$

the conic is

$$ab \cdot cd = ad \cdot bc \quad (1),$$

or,

$$ed \cdot af = ad \cdot ef \quad (2),$$

or,

$$cf \cdot eb = ef \cdot bc \quad (3). \qquad (A)$$

Subtracting in pairs

$$\left.\begin{aligned} ab \cdot cd - ed \cdot af &= ad\,(bc - ef), \\ ed \cdot af - cf \cdot eb &= ef\,(ad - bc), \\ cf \cdot eb - ab \cdot cd &= bc\,(ef - ad). \end{aligned}\right\} \qquad (B)$$

And the Pascals are,

$$bc - ef = 0, \qquad (4)$$

$$ad - bc = 0, \qquad (5)$$

$$ef - ad = 0. \qquad (6)$$

Identifying the conic with quadrilaterals in the first and second columns of (h), the conic is

$$ab \cdot cd = ad \cdot bc \qquad (7)$$

or

$$ed \cdot af = ad \cdot ef \qquad (8)$$

or

$$bf \cdot ec = ef \cdot bc. \qquad (9)$$

On subtracting in pairs,

$$\left.\begin{aligned} ab \cdot cd - ed \cdot af &= ad\,(bc - ef), \\ ed \cdot af - bf \cdot ec &= ef\,(ad - bc), \\ bf \cdot ec - ab \cdot cd &= bc\,(ef - ad). \end{aligned}\right\} \qquad (D)$$

And again the Pascals are,

$$bc - ef = 0, \qquad (10)$$

$$ad - bc = 0, \qquad (11)$$

$$ef - ad = 0, \qquad (12)$$

cate *the same set of opposite sides.* Thus, Steiner's theorem, as expressed in a g-point, and Kirkman's theorem, as expressed in an h-point, are capable of a single expression, namely, *if three hexagons have for pairs of opposite sides the sides of a triangle, taken in pairs, the Pascal lines of these hexagons are concurrent.* Concerning duality of properties of the complete Pascal hexagon see account of investigation by G. Veronese on pages x–xii of Vol. II, 6th ed. of Salmon-Fiedler's *Kegelschnitte.* Concerning advantages of this notation, see "Hexagon Notation" by R. D. Bohannan, in *Ohio Journal of Science* for February, 1916.

concurrent lines, with the same equations as (4), (5), (6), while (11), (12) are entirely different lines from (5), (6). (B) shows that (5) is the line ed, cf; af, eb; bc, ad; while (D) shows that (11), with the same equation as (5), is the line ed, bf; af, ec; bc, ad. And (B) shows that (6) is the line cf, ab; eb, cd; ef, ad, while (D) shows that (12), with the same equation as (6), is the line bf, dc; ec, ab; ef, ad.

The trouble is that if $ab = 0$ is the equation of ab, so is $k \cdot ab = 0$.

If equations in normal form (or any other specified form) are used, different multipliers are necessary to render the equations (A) identical; likewise for those in (C). If l_1, l_2, l_3 are such multipliers for (A), the Pascals are

$$l_1 \cdot bc - l_2 \cdot ef = 0,$$

$$l_2 \cdot ad - l_3 \cdot bc = 0,$$

$$l_3 \cdot ef - l_1 \cdot ad = 0,$$

concurrent lines.

Now a different set of multipliers would be used in (C) and thus different equations would result therefrom.

If $ab = 0$ is the equation in *normal form* of ab (and all other sides of the hexagon are assumed to be expressed in *normal form*), the equation of the Pascal line of

$$fabcde \tag{1}$$

may be written

$$f\binom{ad}{ab} \cdot ab = c\binom{ad}{de} \cdot de, \tag{p}$$

where $f\binom{ad}{ab}$ denotes the ratio of the perpendiculars from f on ad and ab.

For (p) is evidently a line passing through the intersection of ab, de, a pair of opposite sides of (1). And (p) may also be written:

$$f\binom{ad}{ab} \cdot ab - ad = c\binom{ad}{de} \cdot de - ad,$$

which represents a line passing through the intersection of af, cd, another pair of opposite sides of (1).

(The manner of writing (p) from (1) is evident at a glance.)

If (1) is reversed and written

$$c\ b\ a\ f\ e\ d$$

its Pascal (in terms of ba, ed) is by (p),

$$c\binom{be}{ba} \cdot ba = f\binom{be}{ed} \cdot ed \tag{p'}$$

But (p') is identically the same as (p). For if the conic is

$$ab \cdot ed = m \cdot be \cdot da,$$

then

$$f\binom{ab}{ad} = mf\binom{be}{ab}$$

and

$$c\binom{de}{ad} = mc\binom{be}{ba}.$$

Thus the ratios of the coefficients in (p), (p') are the same.

Thus the equation of the Pascal of any hexagon may be expressed definitely in terms of any pair of opposite sides by (p). (Three equations of the same form.)

Taking the hexagons of the g-point

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & af, & cb \\ cf, & eb, & ad \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \qquad (g)$$

in the forms

$$bcdefa, \qquad (1)$$

$$dafcbe, \qquad (2)$$

$$febadc, \qquad (3)$$

and applying (p) to each form, we have, as the equations of the Pascals of (1), (2), (3),

$$b\binom{cf}{cd}\cdot cd = e\binom{cf}{fa}\cdot fa, \qquad \text{(I)}$$

$$d\binom{ab}{af}\cdot af = c\binom{ab}{be}\cdot be, \qquad \text{(II)}$$

$$f\binom{ed}{eb}\cdot eb = a\binom{ed}{dc}\cdot dc. \qquad \text{(III)}$$

But (I), (II), (III) are concurrent.

$$\therefore \quad \frac{a(ed)\cdot b(dc)\cdot e(cf)\cdot d(fa)\cdot c(ab)\cdot f(be)}{a(cd)\cdot f(de)\cdot c(eb)\cdot d(ba)\cdot e(af)\cdot b(fc)} = \text{unity}, \qquad (E)$$

where $a(ed)$ denotes the perpendicular from a on ed.

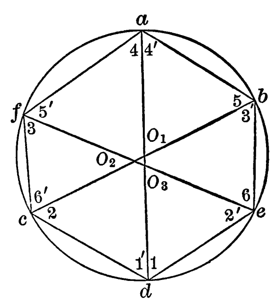
The hexagon, *abedcf*, appearing in the numerator and denominator here, is one of the three hexagons of which the selected point is the g-point. If we had expressed the Pascals in terms of the equations of pairs of opposite sides shown in any column of (g) other than the second (as above), another hexagon of which (g) is the g-point would have appeared. In the notation used in (g) the hexagons of which (g) is the g-point are shown in the columns, namely, *ab*, *ed*, *cf*; *cd*, *af*, *eb*; *ef*, *cb*, *ad*.

By (E) the g-point gives this theorem for any hexagon inscribed in a conic:

The product of the perpendiculars from each vertex on the second side ahead (that through the opposite vertex) in going around the hexagon in one direction, is equal to the product of the perpendiculars on the second side ahead going in the opposite direction.

And there are the usual variations for the pentagon and quadrilateral by letting two vertices of the hexagon be replaced by a tangent.

In the case of a triangle the perpendiculars going one way are the same as those going the other way.



This theorem is self-evident for a circle since it is equivalent to

$$\frac{\sin\angle 1\cdot\sin\angle 2\cdot\sin\angle 3\cdot\sin\angle 4\cdot\sin\angle 5\cdot\sin\angle 6}{\sin\angle 5'\cdot\sin\angle 4'\cdot\sin\angle 3'\cdot\sin\angle 2'\cdot\sin\angle 1'\cdot\sin\angle 6'} = \text{unity} \qquad (E') \quad \text{(see the Fig.)},$$

in which equal angles are the one under the other. Thus (E), (E') are projective relations, (E') remaining true in any conic, where the angles are not equal in pairs. The reason is that this theorem can be proved directly as a consequence of the well-known theorem, "*if from each of two points pairs of parallel secants of a conic are drawn, the ratio of the rectangle of secants from the first point is the same as that from the second*" (F). And this is a projective relation.

Draw the hexagon, *abedcf*, in this order, on a conic. Let the lines joining opposite vertices meet: *ad*, *bc* in O_1; *bc*, *ef* in O_2; *ef*, *ad* in O_3.

$$\begin{aligned} a(de) &= ad\cdot\sin\angle 1, \\ d(fa) &= ad\cdot\sin\angle 4, \\ b(dc) &= bc\cdot\sin\angle 2, \\ c(ab) &= bc\cdot\sin\angle 5, \\ e(cf) &= ef\cdot\sin\angle 3, \\ f(be) &= ef\cdot\sin\angle 6. \end{aligned}$$

Treat the denominator in the same way and (E) will finally reduce to

$$\frac{O_1a\cdot O_1d}{O_3a\cdot O_3d}\cdot\frac{O_3e\cdot O_3f}{O_2e\cdot O_2f}\cdot\frac{O_2b\cdot O_2c}{O_1b\cdot O_1c}. \tag{G}$$

Draw a line through O_1 parallel to $O_2O_3(ef)$, cutting the conic in A, B.

$$\therefore\quad \frac{O_1a\cdot O_1d}{O_1A\cdot O_1B}=\frac{O_3a\cdot O_3d}{O_3e\cdot O_3f}$$

and

$$\frac{O_1A\cdot O_1B}{O_1b\cdot O_1c}=\frac{O_2e\cdot O_2f}{O_2c\cdot O_2b}.$$

Eliminate $O_1A\cdot O_1B$. The result is

$$(E) = \text{unity}.$$

(G) may be written thus:

$$\frac{O_1a\cdot O_1d\cdot O_2b\cdot O_2c\cdot O_3e\cdot O_3b}{O_1b\cdot O_1c\cdot O_2e\cdot O_2f\cdot O_3a\cdot O_3d}=1 \tag{G'}$$

and formulated into a proposition: "*if the opposite vertices of a hexagon are joined and the lines meet in three points the ratio of the product of rectangles as given in* (G') *is unity;* and corresponding propositions for the pentagon and quadrilateral.

Thus Steiner's theorem in the (g) point is a consequence of the theorem (F).

If we take the h-point

$$\begin{bmatrix} ab, & cd, & ef \\ ed, & fa, & cb \\ ca, & be, & fd \\ fe, & dc, & ba \end{bmatrix}\begin{matrix}(1)\\(2)\\(3)\\(1')\end{matrix} \tag{H}$$

and write the hexagons in the order

$$bcdefa \tag{1}$$

$$dfacbe \tag{2}$$

$$abefdc \tag{3}$$

and apply (p) to these, the Pascals are

$$b\begin{pmatrix}cf\\cd\end{pmatrix}\cdot cd = e\begin{pmatrix}cf\\fa\end{pmatrix}\cdot fa,$$

$$d\begin{pmatrix}fb\\fa\end{pmatrix}\cdot fa = c\begin{pmatrix}fb\\be\end{pmatrix}\cdot be,$$

$$a\begin{pmatrix}bd\\be\end{pmatrix}\cdot be = f\begin{pmatrix}bd\\dc\end{pmatrix}\cdot dc,$$

and these lines are concurrent.

$$\therefore\quad \left\{\frac{a(be)\cdot d(fa)\cdot b(cd)}{e(fa)\cdot c(be)\cdot f(dc)}\right\}\cdot\left\{\frac{f(bd)\cdot c(fb)\cdot e(cf)}{b(cb)\cdot d(fb)\cdot a(bd)}\right\}=\text{unity}. \tag{I}$$

The hexagon appearing here (outside the parentheses) is *adbfce*, the hexagon to which (H) is unique (none of its sides appear in (H)); *be, fa, cd* appearing in parentheses in the first fraction are diagonals joining opposite vertices. And the proposition in (I) is this:

If we go half way around a hexagon dropping perpendiculars from the vertices on the second diagonal ahead and the rest of the way dropping perpendiculars from the vertices on the side next behind; and then reverse, going in the opposite direction, the products of the perpendiculars in the two circuits are equal.

If we write the hexagons of (H)

$$edcbaf \tag{1}$$

$$cafdeb \tag{2}$$

$$febacd \tag{3}$$

and apply (p), the Pascals are

$$e\begin{pmatrix} da \\ dc \end{pmatrix} \cdot dc = b\begin{pmatrix} da \\ af \end{pmatrix} \cdot af,$$

$$c\begin{pmatrix} ae \\ af \end{pmatrix} \cdot af = d\begin{pmatrix} ae \\ eb \end{pmatrix} \cdot eb,$$

$$f\begin{pmatrix} ec \\ eb \end{pmatrix} \cdot eb = a\begin{pmatrix} ec \\ cd \end{pmatrix} \cdot cd,$$

$$\therefore \quad \left\{ \frac{a(ce)\cdot d(ea)\cdot b(ad)}{e(da)\cdot c(ae)\cdot f(ec)} \right\} \left\{ \frac{f(eb)\cdot c(af)\cdot e(dc)}{b(af)\cdot d(eb)\cdot a(cd)} \right\} = \text{unity.} \tag{J}$$

In (J) we have, outside the parentheses, the same hexagon as in (I), *adbfce*, and the proposition of (I) is reversed:

If we go half way around a hexagon dropping perpendiculars from each vertex on the side immediately behind, and the remaining way on the second diagonal ahead; and then reverse, repeating, in the opposite direction, the products are equal.

We have previously shown that (I) and (J) are identical.

If we multiply (I) by (J), the product of the first fraction of (J) by the second of (I) is evidently unity.

$$\therefore \quad \frac{a(be)\cdot e(cd)\cdot c(fa)\cdot f(be)\cdot b(dc)\cdot d(ab)}{a(dc)\cdot d(be)\cdot b(fa)\cdot f(cd)\cdot c(eb)\cdot e(af)} = \text{unity.} \tag{K}$$

And (K) is this:

If we go around a hexagon (here aecfbd) dropping perpendiculars on the diagonal (of opposite vertices) just ahead, and then reverse, the product of the perpendiculars in the one circuit is that of the other.

(We might have said "second diagonal ahead," and then this proposition is the same as that for a g-point, replacing the word "side" by "diagonal".)

This theorem can also be readily proven directly. If the hexagon, *adbfce*,

is drawn on a conic, and the perpendiculars expressed in terms of diagonals and sines of angles, identically the same angle relation as expressed in (E') will appear.

Thus not only are Steiner's theorem in a (g) point and Kirkman's theorem in an (h) point capable of expression in the same form, but they represent identically the same relation[1] among sines of angles between the sides and diagonals of the conic, namely (E'),—a relation of equality of sines in the case of the circle (as in (E')).

Another form of Pascal equation is, for $fabcde$ (1),

$$\frac{ab}{ad}\cdot f\begin{pmatrix} ad \\ ab \end{pmatrix} + \frac{cd}{ad}\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix} = 1. \tag{q_1}$$

For (q_1) is evidently a line through the intersection of $ab = 0$ and $ad - cd\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix} = 0$, or, of ab and de, a pair of opposite sides of (1). It also passes through the intersection of $cd = 0$, and $ad - ab\cdot f\begin{pmatrix} ad \\ ab \end{pmatrix} = 0$, or, of cd and af, another pair of opposite sides of (1).

The third line of (g) may be written

$$ebadcf \tag{3}$$

and its Pascal is by (q_1),

$$\frac{ba}{bc}\cdot e\begin{pmatrix} bc \\ ba \end{pmatrix} + \frac{dc}{bc}\cdot f\begin{pmatrix} bc \\ dc \end{pmatrix} = 1. \tag{q_2}$$

Eliminating ab between (q_1), (q_2), the coefficient of dc is

$$f\begin{pmatrix} ad \\ ab \end{pmatrix}\cdot f\begin{pmatrix} bc \\ dc \end{pmatrix} - e\begin{pmatrix} bc \\ ba \end{pmatrix}\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix},$$

which is zero.

For if the conic is

$$ad\cdot bc = m\cdot ab\cdot cd,$$

$$f\begin{pmatrix} ad \\ ab \end{pmatrix}\cdot f\begin{pmatrix} bc \\ dc \end{pmatrix} = m = e\begin{pmatrix} bc \\ ba \end{pmatrix}\cdot e\begin{pmatrix} ad \\ cd \end{pmatrix}.$$

Thus the eliminant is

$$e\begin{pmatrix} ba \\ cb \end{pmatrix}\cdot bc = f\begin{pmatrix} ba \\ ad \end{pmatrix}\cdot ad,$$

which, by (p), is the Pascal of

$$ebcfad, \tag{2}$$

the middle line of (g), which shows concurrency for (g).

By means of equations like (q_1) concurrency for (h) may be shown.

[1] See Veronese, loc. cit.

A SIMPLIFIED PROOF OF THE DISTRIBUTIVE LAW OF MULTIPLICATION GIVEN IN HILBERT'S "THE FOUNDATIONS OF GEOMETRY."

By WILLIAM E. ROTH, University of Illinois.

Professor Hilbert has developed an algebra of segments in his *Foundations of Geometry*[1] on the following assumptions:

I. The axioms of connection:
  1. Two distinct points A and B always completely determine a straight line.
  2. Any two distinct points of a straight line completely determine that line.

II. The axioms of order.

III. The parallel axiom.

IV. A special form of Desargue's theorem.

In the proof which follows, the axioms of order are not used and, therefore, need not be stated; the parallel axiom is that of Euclid; and, of the special form of Desargue's theorem used by Hilbert, only the part given below is applied in the demonstration:

If two triangles are so situated in a plane that the straight lines joining the homologous vertices intersect in a common point, or are parallel to one another, and, furthermore, if two pairs of homologous sides are parallel to each other, then the third sides of the two triangles are parallel to each other.

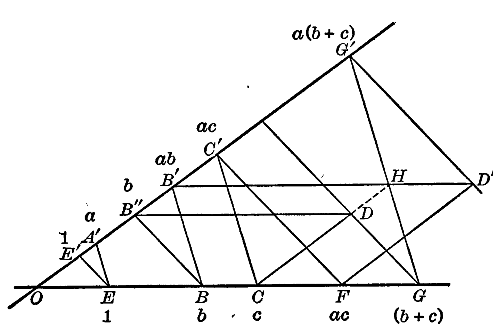
On the basis of the above assumptions, Hilbert defines both addition and multiplication of segments by constructions which will be given below. However, the associative law of multiplication is not established, and so it is desirable to establish the distributive law in two forms, namely

$$a(b + c) = ab + ac,$$

and

$$(a + b)c = ac + bc.$$

The present article aims to present for the first of these a simpler proof than that given by Dr. von Schaper in Hilbert's book.



[1] Translation by E. J. Townsend, pages 79–99.

Constructions: Take on one of two straight lines meeting in O

$$1 = OE, \quad b = OB, \quad c = OC,$$

and on the other

$$1 = OE', \quad a = OA'.$$

Draw EA' and parallel to it draw BB' and CC', which meet OE' in the points B' and C' respectively. Then by Hilbert's definition of multiplication (*l. c.*, page 81),

$$OB' = ab \text{ and } OC' = ac.$$

Draw the unit line EE' and parallel to it draw BB'' and $C'F$, which respectively meet OE' in B'' and OE in F. Then

$$OB'' = b \text{ and } OF = ac.$$

Next construct $B''D$ and $B'D'$ parallel to OE, and CD and FD' parallel to OE'; the points D and D' are uniquely determined. A straight line through D parallel to EE' cuts OE in a point G, such that by the definition of addition of segments (Hilbert, *l. c.*, page 81),

$$OG = b + c.$$

Now take GG' parallel to EA'; so again by the definition of multiplication

$$OG' = a(b + c). \tag{1}$$

If we were to draw through D' a straight line parallel to EE', it would cut OE' in a point defining the segment $ab + ac$, but instead of making this construction, we shall simply draw the straight line $D'G'$ and prove that it has the above property.

Proof: Denote the point of intersection of GG' and $B'D'$ by H. Connect D and H by a straight line. In the triangles $BB''B'$ and GDH, we have the homologous sides BB'' and BB' parallel to GD and GH respectively, and their corresponding vertices are connected by the parallel lines BG, $B''D$, and $B'H$. Then by the part of Desargue's theorem given above, DH is parallel to $B'B''$. But by construction CD is parallel to OE'; so the points C, D and H are collinear because of the parallel axiom. Therefore, in the triangles $CC'F$ and $HG'D'$, the following relations hold: CC' and CF are parallel to HG' and HD' respectively; also the lines joining the corresponding vertices are mutually parallel; consequently $G'D'$ is parallel to $C'F$ by Desargue's theorem. Then $G'D'$ is parallel to the unit line EE'. Then from the definition of addition, we have

$$OG' = ab + ac. \tag{2}$$

Then through equation (1) we get immediately

$$a(b + c) = ab + ac.$$

Thus the first form of the distributive law of multiplication is demonstrated on the basis of the given assumptions.

The above proof is simpler than that of Dr. von Schaper in that it requires only two applications of the second part of Desargue's theorem; whereas, the latter requires three applications of the first part of this theorem and five of the second part to complete it.

THE DERIVATIVE OF THE LOGARITHM.

By M. B. PORTER, University of Texas.

That the problem of deriving the logarithm presents pedagogic difficulties is sufficiently evident to any one who turns the pages of the texts on the calculus. A great many of these content themselves with showing that $[1 + (1/n)]^n$, n positive and integral, approaches $\Sigma(1/n!)$ as a limit as n becomes infinite and hence that, if $\log x \doteq \log e$ as $x \doteq e +$, the right hand incremental ratio approaches the limit $(1/x) \cdot \log_e a$ when Δx approaches zero over a certain denumerable point set.[1] Some show that this limit is the same over any point set to the right or left of x, though all assume the continuity of $\log x$. The mechanism of this proof involves the binomial theorem for positive integral exponents, simple convergence tests, and obvious inequalities. The main criticisms that can be urged against such proofs is that they are incomplete, that the binomial theorem has usually been proved by an incomplete induction, and that the proof involves many different steps. The steps are simple in themselves, but after all almost as much is assumed as is proved.

In the first edition of Vallée-Poussin's *Cours* an interesting proof of these results is obtained by means of the elementary inequality $a^{n+1} > 1 + (n+1)(a-1)$, followed by the substitution of $[1 + \omega/(n + 1)] \div [1 + (\omega/n)]$ for a.[2] Here, while the steps are all elementary, the obvious artificiality of the whole process unfits it for elementary instruction; the substitution is one that the student would never invent for himself or remember. On the other hand, the Davis-Hedrick *Calculus*, frankly recognizing the unconvincing character of elaborate proof as well as its incompleteness, for the immature mind of the average beginner, makes a stronger appeal to intuition and thus obtains a greater vividness of effect by a sharper, quicker attack and produces quite as satisfactory a state of mental *bien être* on the part of the youthful and uncritical student as that obtained by the more tiresome process, thus following the safe pedagogic principle that it is not worth while to bother the student with details of proof which he cannot understand or at least whose necessity he does not appreciate.

The question of the continuity of the logarithm can be treated by constructing the values of the logarithm function by the insertion of a series of arithmetic and geometric means—the method used by Briggs in the calculation of his tables.

To many teachers of the calculus it seems desirable to put in the hands of the student a simple outline of a proof, which he can fill in, whereby the existence of

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[1] Osborne's Calculus, revised ed., pp. 9–10.

[2] See Granville's *Calculus*, p. 31, where the proof is reproduced.

the limit $\lim_{n=\infty} [1 + (1/n)]^n$ is demonstrated. Since the points to be established in all such proofs are the same, the only simplification possible is in the manner in which these points are established, and here it seems evident that the greatest simplification will be obtained if this process is identical for all the points involved. The writer has tried with considerable success the following procedure.

Lemma. Applying the first law of the mean twice to $(1 - x)^n$ where n is rational, we have

$$\begin{aligned}(1 - x)^n &= 1 - nx(1 - x_1)^{n-1} \\ &= 1 - nx[1 - (n - 1)x_1(1 - x_2)^{n-2}] \\ &= 1 - nx + n(n - 1)xx_1(1 - x_2)^{n-2}, \qquad x > x_1 > x_2 > 0.\end{aligned}$$

Step 1°.

$$\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n,$$

n rational and positive. To prove this show that

$$\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} \equiv \left(1 + \frac{1}{n}\right)\left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1} \equiv \left(1 + \frac{1}{n}\right)\left[1 - \frac{1}{(n+1)^2}\right]^{n+1} > 1$$

by applying the lemma to the last bracket.

Step 2.° Show that $[1 + (1/n)]^n$ does not increase indefinitely with n. Apply the lemma to

$$\left(1 + \frac{1}{n}\right)^{-n/2}$$

and thus show that $[1 + (1/n)]^n < 4$.

Step 3°. Show that

$$\lim_{n=+\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n=+\infty} \left(1 - \frac{1}{n}\right)^{-n}$$

by applying the lemma to

$$\frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 - \frac{1}{n}\right)^{-n}} \equiv \left(1 - \frac{1}{n^2}\right)^n.$$

The theorem has now been proved for n rational and the remainder of the proof is filled in as usual by considering the inequality

$$\left(1 + \frac{1}{n+1}\right)^n < \left(1 + \frac{1}{\omega}\right)^\omega < \left(1 + \frac{1}{n}\right)^{n+1}, \qquad n < \omega < n + 1.$$

We can now calculate e by a table of logarithms.

A proof of this character should, of course, not be given until the use of the mean value theorem has been freely illustrated by applications to particular

functions and numerical problems. Such applications are numerous and interesting. We merely note here such as

$$\begin{array}{lll} \sin x = x - \epsilon & \text{where} & \epsilon < x^3, \\ \cos x - 1 = -(x^2/2) + \epsilon & \text{where} & \epsilon < x^4/4, \qquad x < 1, \\ (1+x)^{1/n} = 1 + (x/n) - \epsilon & \text{where} & \epsilon < x^2/n, \\ \tan^{-1} x = x - \epsilon & \text{where} & \epsilon < x^3, \end{array}$$

etc., and the justification of the ordinary rules of interpolation in tables of natural sines, cosines, etc., by means of the double application of this theorem. Other applications to maxima and minima problems, and asymptotes at once suggest themselves, but it is not worth while to enter into further detail.

NEW BOOKS RECEIVED.

Solid Geometry. By William Betz and Harrison E. Webb. Ginn and Company, Boston, 1916. xxii + 178 pages. $0.75.

Solid Geometry. By John H. Williams and Kenneth P. Williams. Lyons and Carnahan, Chicago, 1916. xii + 162 pages. $0.80.

Text-Book of Mechanics, Volume VI. Thermodynamics. By Louis A. Martin, Jr. John Wiley and Sons, New York, 1916. xviii + 313 pages. $1.75.

A Community Arithmetic. By Brenelle Hunt. American Book Company, New York, 1916. vii + 277 pages. $0.60.

Analytic Geometry. By W. A. Wilson and J. I. Tracey. D. C. Heath and Company, Boston, 1915. ix + 212 pages. $1.20.

Theory of Errors and Least Squares. A Textbook for College Students and Research Workers. By LeRoy D. Weld. The Macmillan Company, New York, 1916. xii + 190 pages. $1.25.

Goursat's Mathematical Analysis, Volume II, Part I. Functions of a Complex Variable. By E. R. Hedrick and Otto Dunkel. Ginn & Company, Boston, 1916. x + 259 pages. $2.75.

Five-Figure Mathematical Tables. By E. Chappell. The D. Van Nostrand Company, London, 1915. xvi + 320 pages. $2.00.

BOOK REVIEWS.

Send all communications to W. H. Bussey, University oi Minnesota.

Grundriss der Differential- und Integral-Rechnung. L. Kiepert. I Teil: *Differential-Rechnung.* Zehnte Auflage des gleichnamigen Leitfadens von Dr. Max Stegemann. 1905. II Teil: *Integral-Rechnung.* Neunte Auflage, 1908. Helwingsche Verlagsbuchhandlung. Hannover.

This treatise is probably the most popular text on the calculus now available in Germany. The many merits of the work have been preserved in the new edi-

tion and a number of new subjects have been included. For an interesting review of the eighth edition see E. B. Van Vleck, *Bulletin of the American Mathematical Society*, 1896. The author has taken into account, in the later editions, some of the points raised by Professor Van Vleck.

E. J. W.

Leçons de Mathématiques Générales. By L. ZORETTI. xvi+753 pp. Gauthier-Villars, Paris, 1914. 20 francs.

This book is intended as a text to be used by those students in the French universities who, while not specializing in mathematics, find it necessary to study mathematics in preparing for their future careers. The book is admirably suited to its purpose, and American college teachers will find it interesting to note that, to a very considerable extent, the contents of this book coincide with what they usually present to their students in their courses in analytic geometry and calculus. There are included however a number of topics not usually treated in our American courses, and it would seem to be a question well worthy of serious thought, whether some or all of these subjects might not be as valuable to American as to French students of this class. The book is introduced by a preface written by Professor Appell, which discusses with great lucidity the pedagogic situation involved.

E. J. W.

Historical Introduction to Mathematical Literature. By G. A. MILLER. The Macmillan Company, New York. xiii+302 pp. $1.60.

This, the most recent product of Professor Miller's prolific pen, is a real innovation in mathematical literature. The plan and scope of the volume, its purposes and contents, make it differ in kind from any other book about mathematics with which the reviewer is acquainted.

As the author tells us, the book found its origin in a series of lectures which were intended to supplement the regular mathematical courses. Naturally the book itself has turned out to be something partaking of the nature of a supplement, exhibiting a certain lack of unity and a rather noticeable looseness of connection between its various parts. But each of these parts is itself well bound together, and the author expresses his views on a large number of questions in an interesting and forceful style, which frequently assumes the form of epigram.

Professor Miller feels, as many of us do, that something should be done to widen the perspective of our students of mathematics. He thinks that this can best be accomplished, by supplementing the detailed work, in problems and theorems, of the regular courses, by material of an informational and historical character. This is the need which we attempt to meet by "synoptic and inspirational courses." Professor Miller thinks that his book may serve as a basis for such courses, and also for a first course in the history of mathematics.

In regard to the history of mathematics, the author takes a rather novel and interesting point of view. He thinks that a first course in this subject should

discuss "recent mathematical events and developments" rather than the mathematics of the ancients, and he presents some strong arguments in favor of this view. In fact, it is evident that historical study of any sort is apt to degenerate into mere text-book work, unless the primary sources of information are, at least in part, available for the use of the student. And it is certainly not an easy matter to establish direct connections with the sources of ancient mathematics, on account of the linguistic difficulties involved. By concentrating attention on more recent developments, some genuine historical study becomes possible even for those who know English only. Again, the author points out how very essential such detailed historical investigations and comparisons become, if we wish to arrive at a correct interpretation of even such a simple historical statement, as "Newton discovered the binomial theorem." The greater ease with which this can be done for more recent mathematical developments, as compared with the obscurity which necessarily surrounds the origins of various ideas transmitted to us by the Ancients, makes it fairly evident that we can hardly hope to understand the latter except by means of the light which is thrown upon intellectual origins in general by studies of the former kind.

Of course there are serious difficulties in the way of carrying out a program of this kind for the history of mathematics. The mathematics of the ancients not only came first chronologically, but much of it comes first logically and heuristically, as a prerequisite for later developments. Thus, much of the ancient mathematics is simpler and more easily understood than most of the modern work based upon it. But the simplicity of the ancient mathematics, as contrasted with the modern, is very much exaggerated by our traditional methods of teaching, methods which are being modernized very slowly indeed. As a matter of fact, many notions of the so-called higher mathematics are much simpler, and much more important, than many of the things now taught in every high-school. Nothing could be more helpful in advancing mathematical education than a thoroughgoing revision of the mathematical curriculum of the secondary schools from this point of view. I believe that a combined synoptic and historical course, still largely based on the chronological order, but taking into account only the most essential developments of both ancient and modern times, would furnish the best solution of the educational problem in which Professor Miller is interested, namely, to secure an intelligent appreciation, by educated people in general, of the rôle of mathematics in human thought and in the history of civilization. I feel that such a plan would be even more satisfactory than that outlined by Miller, but nevertheless his point of view is very suggestive.

Professor Miller's book brings together much information of great value which is not easily available elsewhere. Chapter II, which discusses the various kinds of mathematical literature, and the appendix, "Lists of important works," alone are worth the price of the book. Every one who has attempted to direct advanced students, knows how difficult it is for them to find their way through the literature. I feel that, from now on, my troubles in this direction, at least, are at an end. I shall ask them to read Miller's "Introduction."

Chapter VII contains interesting sketches of twenty-five "deceased mathematicians." There are at least two serious omissions in this list, Jacobi and Riemann. Jacobi, to be sure, is mentioned several times in other parts of the book; but even in the sketch of Abel, no mention is made of Jacobi's fundamental contributions to the theory of elliptic functions. Riemann's name does not occur anywhere in the book, so far as I have been able to make out, although the theory of functions is mentioned several times, and although biographical sketches of Weierstrass and Cauchy are included among the twenty-five.

Professor Miller's book is a valuable contribution to popular mathematical literature, and should help to arouse a more general interest in mathematics. It deserves an honorable place in every mathematical library, and it should be read by all those who feel some interest in mathematics, especially if their mathematical education has been limited. They will, at least, find out that mathematics is not dead; that it is growing day by day, and far more rapidly now than in ancient times.

E. J. WILCZYNSKI.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

460. Proposed by J. J. GINSBURG, Student, Cooper Union, New York.

Find the value of $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+}}}}$ to infinity.

461. Proposed by E. T. BELL, Seattle, Washington.

(1) Two events have probabilities p, q, respectively. The events may be either (i) mutually independent; or (ii) mutually exclusive. Assign meanings to the symbol p^q, in terms of the two events, where p^q is written for $p \times p \times \cdots \times p$, ($q$ factors p), in cases (i), (ii), and $p \times p$ has the customary meaning (as a probability).

(2) What relations, if any, other than (i) or (ii) can exist between two events? Upon what postulates is the answer to this based?

462. Proposed by H. S. UHLER, Yale University.

Show how to transform A into S, where these symbols denote the equivalent formulæ for the general case of Calculus Problem No. 363, pages 52 and 54 in the February, 1916, MONTHLY:

$$A \equiv 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R\sin\frac{\pi}{n}}{\sqrt{R^2-a^2}}\right) + 2anR\sin^{-1}\left[\frac{2a\left(\tan\frac{\pi}{n}\right)\left(\sqrt{R^2-a^2\sec^2\frac{\pi}{n}}\right)}{R^2-a^2}\right],$$

$$S \equiv 4nR\left\{a\sin^{-1}\left(\tan\frac{\pi}{n}\cdot\frac{a}{\sqrt{R^2-a^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-a^2\sec^2\frac{\pi}{n}}}{\sqrt{R^2-a^2}}\right]\right\}.$$

GEOMETRY.

492. Proposed by FRANK V. MORLEY, Student, Haverford College.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle, and let the symmedian point of the triangle formed by omitting a_i be s_i. Prove that the four points s_i have the same diagonal triangle as the four points a_i.

493. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore.

Construct three circles each of which shall be tangent to the other two and to two sides of a given triangle.

494. Proposed by DAVID F. BARROW, University of Texas.

Students of geometry are very apt to assume that a theorem, true in general, will hold in all limiting cases. This trustfulness leads to frequent errors. An example is the following: Let C_1, C_2, C_3, C_4 denote four circles, and P_{ij}, P_{ij}' denote the two points in which C_i and C_j intersect. If P_{12}, P_{23}, P_{34}, P_{41} are concyclic on a circle C, then P_{12}', P_{23}', P_{34}', P_{41}' will be concyclic on a circle C'. This is still true if C is very small. Hence we might hastily conclude that: If four circles are concurrent, then their other intersections, taken in pairs in a cyclic order, are concyclic. Why is not this true?

CALCULUS.

410. Proposed by J. A. BULLARD, Worcester, Massachusetts.

(*a*) Find the area of the loop of the curve $x^{2q+1} + y^{2q+1} = (2q + 1)ax^q y^q$. (For $q = 1$, we have the folium of Calculus Problem No. 379.)

(*b*) Find the area between the curve and its asymptote.

[From Johnson's *Integral Calculus*.]

411. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Prove that the volume bounded by the surface $f(x, y, z) = 0$ is $\frac{1}{3}\int\int\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)dxdy$ integrated over the area determined by projecting the surface on the xy-plane.

412. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Given a triangular field of sides a, b, and c. Show how to divide the field into two equal parts by a straight fence so that the cost of the fence is the least.

MECHANICS.

328. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Find the envelope of all possible trajectories when a particle is projected with a constant velocity v from a fixed point at a distance a from the center of attraction under the law of gravitation.

329. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A smooth circular table is surrounded by a smooth vertical rim. Show that the ball, whose coefficient of restitution is e, projected along the table from a point in the rim in a direction making an angle $\tan^{-1} e^{\frac{3}{2}}$ with the radius through the point, will return to the point of projection after three rebounds.

NUMBER THEORY.

246. Proposed by ALBERT A. BENNETT, Princeton University.

Prove that

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n - \left(\frac{a-\sqrt{b}}{2}\right)^n\right]$$

is an integer for every positive integral value of n, whenever a is an odd integer, positive or negative, and $b \equiv 1 \pmod{4}$.

247. Proposed by NORMAN ANNING, Chilliwack, B. C.

To dissect the triangle whose sides are 52, 56, 60 into three Heronian triangles by lines drawn from the vertices to a point within.

The word Heronian is used in the sense of the German Heronische (Wertheim, *Anfangsgründe d. Zahlenlehre*, p. 140) to describe a triangle whose sides and area are integral.

248. Proposed by E. T. BELL, Seattle, Washington.

If $u_{n+2} = 4u_{n+1} - u_n$, with $u_0 = 2$, $u_1 = 4$, prove that the triangle Δ_n, whose sides are $u_n - 1$, u_n, $u_n + 1$, has an integral area; also that all triangles, Δ_n, whose areas are integers, and whose sides are consecutive integers, are given by this process. Hence show that, as n increases, the area of Δ_n approximates $(\sqrt{3}/4)u_n^2$, and find the degree of approximation.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

447. Proposed by ELIJAH SWIFT, University of Vermont.

(*a*) A method is sought of forming an equation such that the first k figures of some root shall be given numbers. For example, form a cubic equation such that one of its roots is 1.918 +.

(*b*) Of all the equations suggested in (*a*), determine that one for which the sum of the absolute values of all the coefficients is least.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Let the degree of the desired equation be n, the nearest integer to the desired root be x_0, and the root itself be $(x_0 + \Delta x)$.

Choose any $\phi(x)$ of degree $(n - 1)$, and form $y = f(x) = (x - x_0)\phi(x)$.

Compute

$$\Delta y = \Delta x\cdot f'(x_0) + \frac{\overline{\Delta x}^2}{\underline{|2}} f''(x_0) + \cdots + \frac{\overline{\Delta x}^n}{\underline{|n}}\cdot f^{(n)}(x_0)$$

from

$$\Delta y = \Delta x\left[\phi(x_0) + \Delta x\cdot f'(x_0) + \cdots + \frac{\overline{\Delta x}^{n-1}}{\underline{|n-1}}\cdot\phi^{(n-1)}(x_0)\right] \tag{1}$$

or from

$$\Delta y = \Delta x\cdot\phi(x_0 + \Delta x). \tag{2}$$

The more convenient of (1) and (2) should be chosen; usually (1) will be preferable. The computation should be carried to at least as many decimals as are required in the root; if it is important that the root should be greater than $(x_0 + \Delta x)$ rather than less, this is readily managed by slightly increasing or decreasing Δy according as $f'(x_0)$ is > 0 or < 0, with due regard to the fact that $\Delta y \doteq \Delta x\cdot\phi(x_0)$. The desired equation is $f(x) - \Delta y = 0$.

If $n = 3$, and $\phi(x) = \alpha_0 x^2 + \alpha_1 x + \alpha_2$,

$$\Delta y = \Delta x[\alpha_0 x_0^2 + \alpha_1 x_0 + \alpha_2 + (2\alpha_0 x_0 + \alpha_1)\Delta x + \alpha_0\overline{\Delta x}^2].$$

To form a cubic equation having the root 1.918 +, let $\phi(x) = x^2 - 2x + 7$; then

$$f(x) = x^3 - 4x^2 + 11x - 14,$$

and

$$\Delta y = -.082[4 - 4 + 7 + (4 - 2)(-.082) + (1)(.082)^2] = -.5610 -.$$

Increase Δy by .001 to insure a root > 1.918; then $-\Delta y = .560$, and the desired equation is

$$f(x) - \Delta y \equiv x^3 - 4x^2 + 11x - 13.44 = 0,$$

which has a root 1.918 +.

(*b*) If, in the preceding discussion,

$$\phi(x) = \alpha_0 x^{n-1} + \alpha_1 x^{n-2} + \cdots + \alpha_{n-1},$$

the sum of the absolute values of all the coefficients of $f(x) - \Delta y \equiv (x - x_0)\phi(x) - \Delta y$ is

$$\Sigma = |\alpha_0| + |\alpha_1 - \alpha_0 x_0| + |\alpha_2 - \alpha_1 x_0| + \cdots$$
$$+ |\alpha_{n-1} - \alpha_{n-2}x_0| + |-\alpha_{n-1}x_0 - \Delta x \cdot \phi(x_0 + \Delta x)|. \quad (3)$$

The α's are arbitrary. Let $\alpha_i = \alpha_{i-1}x_0$, so that $\alpha_i = \alpha_0 x_0^i$; then

$$\Sigma = |\alpha_0| + |-\alpha_0 x_0^n - \Delta x \cdot \phi(x_0 + \Delta x)|.$$

The equation $f(x) - \Delta y = 0$ must, of course, become

$$\alpha_0[x^n - (x_0 + \Delta x)^n] = 0, \quad \text{and} \quad \Sigma = |\alpha_0|(1 + |x_0 + \Delta x|^n).$$

By choosing α_0 small enough, we can make Σ as small as we please.

This type of equation may not be what is desired, though it serves very well for practice in Horner's Method. In any case, the expression (3) for Σ is a useful guide to a happy medium of simplicity.

448. Proposed by W. D. CAIRNS, Oberlin College.

In the *Washington* (D. C.) *Times*, Mr. W. A. Dayton called attention some time ago to a curious repetition of digits in the decimal value of 1/115. If this decimal, which we print in the form $.00\dot{8}6956521739130\ 4347826\dot{0}$, be divided by two, the result is $.00\dot{4}3478260\ 8695652173913\dot{0}$, the fourteen-digit and eight-digit groups having been thus interchanged. A similar result, as he points out, is obtained if the original decimal value is divided by four. Mr. Dayton asks that this curiosity be explained.

SOLUTION BY H. S. BROWN, Hamilton College, N. Y.

The decimal point was misplaced in the original question. The separation of the decimal into two *groups* of digits has no bearing on the problem.

The following theorem, found in almost any algebra that has a chapter on Theory of Numbers, applies directly: "If $1/p$ be converted into a circulating decimal with $p - 1$ figures in its recurring period, p must be prime, and the recurring period being multiplied by 2, 3, $\cdots$, $(p - 1)$ will reproduce its own digits in the same order." (C. Smith's *Treatise on Algebra*, p. 502.) Proof: If

$$1/p = .\dot{a}_1 a_2 a_3 \cdots \dot{a}_{p-1},$$

we have

$$10 = a_1 p + r_1, \quad 10r_1 = a_2 p + r_2, \quad 10r_2 = a_3 p + r_3, \quad \cdots.$$

Now p is prime to 10, for otherwise $1/p$ would not be reducible to a pure circulator; hence p is prime to $r_1, r_2, r_3, \cdots$. But as $r_1, r_2, r_3, \cdots$, are all different and there are $p - 1$ of them, they are the numbers 1, 2, 3, $\cdots$, $(p - 1)$, not of course in order. Also from the above, p is prime. Now since $r_1, r_2, r_3, \cdots$ include all numbers 1, 2, 3, $\cdots$ $(p - 1)$, it follows that when k/p is reduced to a decimal the recurring digits will be the same as before, beginning with that one for which the previous remainder was k.

In the above example, we have

$$\frac{1}{115} = \frac{1}{5 \times 23} = \frac{2}{10 \times 23}, \quad \text{where } k = 2.$$

It is worth noting that if the decimal 1/23 of 22 places is separated into two equal groups and the sum of the digits of the first half added to the sum of the digits of the second half, the result will consist wholly of nines. Thus, $1/23 = .\dot{0}43478260869565217391\dot{3}$.

$$\begin{array}{rlr} 48 & = \text{sum of the first eleven digits,} & 04347826086 \\ \underline{51} & = \text{sum of the second eleven digits,} & \underline{95652173913} \\ 99 & & 99999999999 \end{array}$$

This result is general under the condition that the denominator is prime and the number of figures in the recurring period is even. We note also that twos and fives, as factors in the denominator of a fraction, have no effect on the number of recurring figures when the fraction is reduced to a decimal.

Also solved by E. E. WHITFORD, E. B. ESCOTT, ELIJAH SWIFT, NORMAN ANNING, R. D. BOHANNAN, GEORGE BLANCHARD, R. E. GAINS, and E. F. CANADAY.

449. Proposed by FRANK IRWIN, University of California.

Sum the expression

$$1 + 2\binom{k+1}{k} + 3\binom{k+2}{k} + \cdots + (n-k+1)\binom{n}{k}.$$

Also show how to sum

$$1\cdot 2 + 2\cdot 3\binom{k+1}{k} + 3\cdot 4\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)\binom{n}{k},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4\binom{k+1}{k} + 3\cdot 4\cdot 5\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)(n-k+3)\binom{n}{k},$$

etc., where $\binom{l}{k}$ is used to denote the coefficient of x^k in $(1+x)^l$.

SOLUTION BY A. M. KENYON, Purdue University.

The sum of the first n binomial coefficients in any column of Pascal's triangle is given by the formula,

$$(1) \qquad \sum_{i=0}^{n-1}\binom{k+i}{k} = \binom{k+n}{k+1} = \binom{k+n}{n-1}, \qquad \begin{cases} k = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{cases}$$

as may be verified and established by induction.

Making use of the notation,

$$x^{(n)} = \prod_{i=0}^{n-1}(x-i), \qquad x^{[n]} = \prod_{i=1}^{n}(x+i), \qquad n = 1, 2, 3, \cdots$$

$$x^{(0)} = x^{[0]} = 1,$$

we have

$$i^{(m)}\binom{k+i}{k} = i^{(m)}\binom{k+i}{i} = k^{[m]}\binom{k+i}{k+m}, \qquad m = 0, 1, 2, \cdots$$

and this in (1) gives,

$$(2) \qquad \sum_{i=0}^{n-1} i^{(m)}\binom{k+i}{k} = k^{[m]}\binom{k+n}{k+1+m}, \qquad \begin{cases} k, m = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{cases}$$

From the relations[1] among the coefficients of the polynomial in x which result from expanding $x^{(n)}$ we find,

$$i^{[m]} = \sum_{j=0}^{m}\binom{m}{j}^2 \underline{|m-j}\, i^{(j)}, \qquad m = 0, 1, 2, \cdots$$

whence on making use of (2)

$$(3) \qquad \sum_{i=0}^{n-1} i^{[m]}\binom{k+i}{k} = \sum_{i=0}^{m}\binom{m}{i}^2 \underline{|m-i}\, k^{[i]}\binom{k+n}{k+1+i}, \qquad \begin{cases} k, m = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{cases}$$

Since the problem proposed requires the sum of the first $n-k+1$ terms, we put $n-k+1$ for n,

$$(4) \qquad \sum_{i=0}^{n-k} i^{[m]}\binom{k+i}{k} = \sum_{i=0}^{m}\binom{m}{i}^2 \underline{|m-i}\, k^{[i]}\binom{n+1}{k+1+i}, \qquad \begin{cases} m = 0, 1, 2, \cdots \\ n > k = 0, 1, 2, \cdots \end{cases}$$

Setting $m = 0, 1, 2, 3$, etc., in (4), we get

[1] See, "Some Properties of Binomial Coefficients," *Indiana Academy of Science*, 1914, p. 449.

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1};$$

$$1\binom{k}{k}+2\binom{k+1}{k}+3\binom{k+2}{k}+\cdots+(n-k+1)\binom{n}{k}=\binom{n+1}{k+1}+(k+1)\binom{n+1}{k+2};$$

$$1\cdot 2\binom{k}{k}+2\cdot 3\binom{k+1}{k}+3\cdot 4\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)\binom{n}{k}$$
$$=2\binom{n+1}{k+1}+4k^{[1]}\binom{n+1}{k+2}+k^{[2]}\binom{n+1}{k+3};$$

and

$$1\cdot 2\cdot 3\binom{k}{k}+2\cdot 3\cdot 4\binom{k+1}{k}+3\cdot 4\cdot 5\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)(n-k+3)\binom{n}{k}$$
$$=6\binom{n+1}{k+1}+18k^{[1]}\binom{n+1}{k+2}+9k^{[2]}\binom{n+1}{k+3}+k^{[3]}\binom{n+1}{k+4}.$$

Excellent solutions were also received from E. B. Escott, L. C. Mathewson, Elijah Swift, S. C. Withers, and the Proposer.

GEOMETRY.

A Correction.—Professor R. A. Johnson, Cleveland, Ohio, has called our attention to an error in Professor Clawson's solution of Geometry problem 467, page 50, of the February Monthly.

In lines four and five, Mr. Clawson "refers to two quadrilaterals, using the word in the sense of quadrangle, as being inversely congruent, the corresponding sides being equal and parallel but arranged in opposite orders." Manifestly, two such quadrilaterals cannot be drawn. Mr. Clawson admits the justice of this criticism and says that what he wished to state was the fact that one of the quadrilaterals would have to be turned through two right angles or "inverted" in order to have it similarly placed with the other. Mr. Clawson also says that what he meant by saying in line 5 that the corresponding sides of the quadrilaterals are arranged in opposite order was that the sides are opposite in direction.

Professor Johnson points out that the entire solution can be made rigorous, by deleting the last five words of line 5 and also the word "inversely" wherever it occurs. Editors.

474. Proposed by LAENAS G. WELD, Pullman, Ill.

Upon a fixed and constant base stands a system of co-planar triangles, for each of which the radius of the inscribed circle is to that of the circumscribed circle as 1 : 2. What is the locus of the vertices opposite to the given fixed base?

Solution by J. W. Clawson, Collegeville, Pa.

Denote the length of the base by a, the varying sides by r and t, the base angle opposite the side t by θ, radius of incircle by i, and radius of circumcircle by R.

Now

$$R = t/2\sin\theta \text{ and } 2i = (a + r - t)\tan\theta/2.$$

Hence, if $2i = R$, $t = 4(a + r - t)\sin^2\theta/2$, from which we obtain, $t = \dfrac{2(a+r)(1-\cos\theta)}{3-2\cos\theta}$.

Again,

$$t^2 = a^2 + r^2 - 2ar\cos\theta.$$

Hence,

$$4(a + r)^2(1 - \cos\theta)^2 = (a^2 + r^2 - 2ar\cos\theta)(3 - 2\cos\theta)^2,$$

or

$$(a^2 + r^2)(4\cos\theta - 5) + 2ar(4 + \cos\theta - 8\cos^2\theta + 4\cos^3\theta) = 0,$$

or

$$(a - r)^2(4\cos\theta - 5) = 2ar(1 - \cos\theta)(1 - 2\cos\theta)^2.$$

Now the left side of this equation is essentially negative or zero; and the right side is essentially positive or zero. Hence the only real points on the locus are the points for which both sides are zero. These are the points $r = a$, $\theta = 0$; $r = a$, $\theta = 60°$; $r = a$, $\theta = 300°$.

Hence, the only triangles in the system are the two equilateral triangles on opposite sides of the given base.

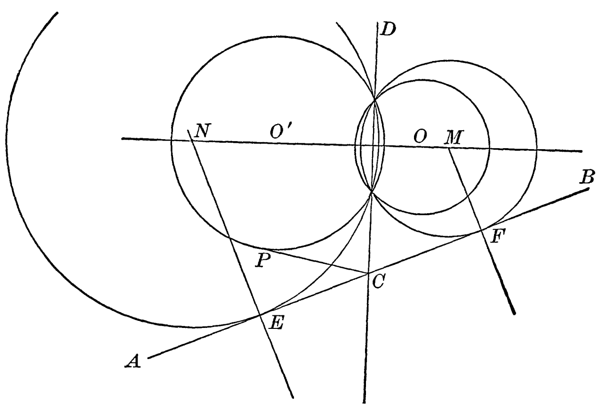
Note.—One other solution, unsigned, was received. This solution was based upon the relation between the radii of the in- and circumcircles and the distance between their centers, namely, $R^2 - d^2 = 2Rr$; since, in the given problem, $r = \frac{1}{2}R$, $d = 0$. Hence, the circles are concentric and the system of triangles reduces to two equilateral triangles on either side of the given line. EDITORS.

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

Let O and O' be the two circles and AB the given straight line. Draw the radical axis CD of the two circles. From the point of intersection C of the axis and line AB, construct a tangent to O or O'. Then with C as a center and the length of this tangent, CP, as a radius lay off a distance CE and CF on AB. E and F will be points of tangency. The points, N and M, where perpendiculars erected to AB at E and F meet the line of centers OO' will be the centers of the required circles.



Since the required circle is coaxial with O and O' its center must be on the line of centers OO'. Because the distance along a tangent drawn from any point on the radical axis to each circle of the pencil is constant E or F must be the required point of tangency. Hence, there are two solutions; one circle with center M and radius MF, the other with center N and radius NE.

Also solved by ROGER JOHNSON and H. L. AGARD.

476. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the locus of the middle points of a set of parallel chords intercepted between an hyperbola and its conjugate is $4b^2x^2y^2 - 4a^2y^4 = a^2b^4$.

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (I) and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ (II) be the equations of the hyperbola and its conjugate respectively. Let m be the slope of the set of parallel chords and $P_1(x_1y_1)$, $P_2(x_2y_2)$ be the points where the chord $y = mx + c$ intersects the primary and conjugate hyperbolas respectively; and $P(xy)$ the middle point between P_1 and P_2. Then

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \text{(I)}, \qquad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = -1 \quad \text{(II)},$$

$$y - y_1 = m(x - x_1) \quad \text{(III)}, \qquad x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2} \quad \text{(IV)}.$$

From II and IV, we have:

$$\frac{(2x - x_1)^2}{a^2} - \frac{(2y - y_1)^2}{b^2} = -1,$$

which becomes

$$\frac{x^2 - xx_1}{a^2} - \frac{y^2 - yy_1}{b^2} = -\frac{1}{2}.$$

Combining this equation with III we get,

$$x - x_1 = \frac{a^2b^2}{2(a^2my - b^2x)};$$

from I and III we get,

$$\frac{x_1^2}{a^2} - \frac{[y - m(x - x_1)]^2}{b^2} = 1.$$

Eliminating x_1 between these two equations we have

$$\frac{1}{a^2}\left[x - \frac{a^2b^2}{2(a^2my - b^2x)}\right]^2 - \frac{1}{b^2}\left[y - \frac{a^2b^2m}{2(a^2my - b^2x)}\right]^2 = 1,$$

which, by I, reduces to

$$4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(\frac{amy}{b} - \frac{bx}{a}\right)^2 = (a^2m^2 - b^2),$$

which is the general equation of the desired locus. As a particular case, if the chords are perpendicular to the x axis, then writing the equation in the form,

$$4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(\frac{ay}{b} - \frac{bx}{am}\right)^2 = a^2 - \frac{b^2}{m^2}$$

and making $m = \infty$, we get

$$4b^2x^2y^2 - 4a^2y^4 = a^2b^4.$$

Also solved by PAUL CAPRON.

CALCULUS.

393. Proposed by LAENAS G. WELD, Pullman, Illinois.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

SOLUTION BY FRANK R. MORRIS, Glendale, Calif.

Let the ellipse be represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then each quadrant will contain one brick. Also let m be the length and n the thickness of a brick. The ellipse must pass through the point (m, n), i. e., the equation

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = 1 \text{ must be true.}$$

From this we get

$$a = \frac{bm}{\sqrt{b^2 - n^2}}.$$

The area of the ellipse, which is easily found by integration, is πab. Substituting the above value of a in this expression we have

$$\frac{\pi b^2 m}{\sqrt{b^2 - n^2}},$$

which is a function, $f(b)$, of the independent variable b. To find the minimum value of $f(b)$ find the values of b which cause the first derivative to vanish.

$$f'(b) = \pi m \frac{2b(b^2 - n^2) - b^3}{(b^2 - n^2)^{3/2}} = 0.$$

Solving this equation we find b to be 0 or $n\sqrt{2}$. The latter value is obviously the desired one. The corresponding value of a is $m\sqrt{2}$.

Hence, the area is $\pi m\sqrt{2} \cdot n\sqrt{2}$ or $2\pi mn$.

From geometrical considerations it does not seem necessary to show that $f''(b)$ is positive and that, therefore, $f(b)$ is a true minimum.

Also solved similarly by H. S. Uhler, H. C. Feemster, George W. Hartwell, and H. L. Agard.

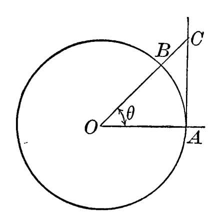
Note.—The above solution assumes that the bricks are laid side by side without mortar, whereas in a "brick wall" they are laid so as to *break joints*. Very possibly the question as answered is the one really intended by the proposer but the ellipse found certainly does not inclose four bricks as they are laid in a *brick wall*. Editors.

394. Proposed by W. W. BURTON, Macon, Ga.

A horse runs 10 miles per hour on a circular race-track in the center of which is an arc-light. How fast will his shadow move along a straight board fence (tangent to the track at the starting point) when he has completed one eighth of the circuit?

Solution by C. E. Horne, Westminster College, Colorado.

Let B and C be the position of the horse and shadow at any time after starting from the point of tangency, A. Let $AB = s$, $AC = y$, angle $AOC = \theta$, and $r =$ the radius of the ring. Then $s = r\theta$ (1) and $y = r \tan \theta$ (2). From (1), $ds/dt = r(d\theta/dt)$, the rate of the horse and from (2), $dy/dt = r \sec^2 \theta(d\theta/dt)$, the rate of the shadow, $= \sec^2 \theta(ds/dt) = 10 \sec^2 \theta = 20$ miles per hour when $\theta = \pi/4$.



Also solved by A. H. Holmes, H. L. Agard, J. A. Caparo, D. Rumble, W. C. Eells, Clifford N. Mills, Horace Olson, H. L. Agard, and George W. Hartwell.

395. Proposed by W. W. BURTON, Mercer University, Macon, Ga.

Into a full conical wine glass whose depth is a and whose angle at the base is 2α there is carefully dropped a spherical ball of such size as to cause the greatest overflow. Show that the radius of the ball is $a \sin \alpha/(\sin \alpha + \cos 2\alpha)$.

From Woods and Bailey's *A Course in Mathematics* (1907), Volume I, page 213.

Solution by H. S. Uhler, Yale University.

Let the sphere touch the inside of the cone near the rim with its center above the plane of the edge. Also, let h and r denote, respectively, the altitude of the submerged spherical segment and the radius of the sphere. Then $h = a + r(1 - \csc \alpha)$. The volume v of the liquid spilled is, of course, equal to the volume of the submerged spherical segment.

The volume of a spherical segment of altitude h, and radius of base r_1 is $\frac{1}{6}\pi h^3 + \frac{1}{2}\pi h r_1^2$. Here,

$$r_1 = \sqrt{2rh - h^2}, \qquad \text{hence} \qquad v = \pi h^2(r - \tfrac{1}{3}h).$$

Now,

$$\frac{dv}{dr} = \pi h\left[h + (2r - h)\frac{dh}{dr}\right], \qquad \text{and} \qquad \frac{dh}{dr} = 1 - \csc\alpha.$$

Therefore,

$$\frac{dv}{dr} = \pi h[a\csc\alpha - r(\csc\alpha - 1)(\csc\alpha + 2)].$$

Obviously, $h = 0$ cannot give a maximum overflow. Consequently,

$$r = \frac{a\csc\alpha}{(\csc\alpha - 1)(\csc\alpha + 2)} = \frac{a\sin\alpha}{(1 - \sin\alpha)(1 + 2\sin\alpha)} = a\sin\alpha/(\sin\alpha + \cos 2\alpha).$$

Furthermore,

$$\frac{d^2v}{dr^2} = -\pi h(\csc\alpha - 1)(\csc\alpha + 2).$$

Since $h \neq 0$ and $\csc\alpha > 1$, d^2v/dr^2 is negative so that the condition for a maximum is satisfied.

Also solved by A. H. Wilson, S. E. Rasor, Elmer Schuyler, George Raynor, C. N. Schmall, H. L. Agard, C. N. Mills, Lewis Clark, L. M. Pickett, C. A. Bergstresser, C. Hornung, L. G. Weld, J. A. Bullard, G. W. Hartwell, J. V. Balch, J. A. Whitted, Elijah Swift, J. C. Rayworth, and J. A. Caparo.

396. Proposed by ELBERT H. CLARKE, Purdue University.

The length of the curve $y = x^n$ from the origin to the point (1, 1) is given by the formula

$$= \int_0^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx.$$

Our geometric intuition would tell us that the limit of this length as n becomes infinite is 2. Give a strict analytic proof that

$$\lim_{n\to\infty} \int_0^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx = 2.$$

Solution by Elijah Swift, University of Vermont.

We can easily show that

(1) $$\lim_{n\to\infty} n^2x^{2n-2} = 0, \qquad 0 \leqq x < 1.$$

Next, we choose a positive number, ϵ, as small as we like. We can then find a value N of n such that for it and all larger values

(2) $$n^2(1 - \epsilon)^{2n-2} < \eta,$$

where η is a positive number as small as we like. This is possible on account of (1).

Let x_n be a value of x such that $n^2x_n^{2n-2} = 2$.

Now write the given integral, I, as a sum of three integrals.

$$I = \int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}}\cdot dx + \int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}}\cdot dx + \int_{x_n}^1 \sqrt{1 + n^2x^{2n-2}}\cdot dx.$$

Considering values of $n > N$, we have

(3) $$\left|\int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}}\cdot dx - 1\right| = \left|\int_0^{1-\epsilon}\{\sqrt{1 + n^2x^{2n-2}} - 1\}dx - \epsilon\right|$$
$$\leqq \left|\int_0^{1-\epsilon}\{\sqrt{1+\eta} - 1\}dx\right| + \epsilon < \frac{\eta}{2} + \epsilon. \quad \text{(See (2).)}$$

This holds if $n \geqq N$.

Now

(4) $$\int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}}\cdot dx < \int_{1-\epsilon}^1 \sqrt{1 + 2}\cdot dx = \sqrt{3}\cdot\epsilon.$$

Finally, since for values of x greater than x_n, $n^2x^{2n-2} > 2$, we may develop $\sqrt{1 + n^2x^{2n-2}}$ into a series in descending powers of n^2x^{2n-2}, and the series will be uniformly and absolutely convergent for values of x between, and including, x_n and 1, and all values of $n \geqq N$, we have

$$\sqrt{1 + n^2x^{2n-2}} = nx^{n-1} + \frac{1}{2} \cdot \frac{1}{nx^{n-1}} - \frac{1}{2 \cdot 4} \cdot \frac{1}{n^3x^{3n-3}} + \cdots.$$

Hence,

$$(5) \qquad \int_{x_n}^{1} \sqrt{1 + n^2x^{2n-2}} \cdot dx = 1 - x_n^n + \left[\frac{1}{2} \cdot \frac{1}{n(-n+2)} x^{-n+2} - + \cdots\right]_{x_n}^{1}.$$

Since,

$$(6) \qquad nx_n^{n-1} = \sqrt{2},$$

$$(7) \qquad \lim_{n \to \infty} x_n^n = 0.$$

We may now obtain the limit of the right hand side of (5) when $n \to \infty$ by taking the limit of each term. This gives, in consequence of (6) and (7), 1 as the limit. In other words, we may choose n so large that

$$(8) \qquad \int_{x_n}^{1} \sqrt{1 + n^2x^{2n-2}} \cdot dx = 1 + \zeta,$$

where ζ is a constant as small numerically as we please for all values of $n > N$.

Then

$$\left|\int_0^1 \sqrt{1 + n^2x^{2n-2}} \cdot dx - 2\right| \leqq \left|\int_0^{1-\epsilon} \sqrt{1 + n^2x^{2n-2}} \cdot dx - 1\right| + \left|\int_{1-\epsilon}^{x_n} \sqrt{1 + n^2x^{2n-2}} \cdot dx\right| + \left|\int_{x_n}^{1} \sqrt{1 + n^2x^{2n-2}} \cdot dx - 1\right| < \frac{\eta}{2} + \epsilon + \sqrt{3} \cdot \epsilon + |\zeta|.$$

Since this may be made as small as we please by a suitable choice in order of ϵ, η, and ζ, we have proved the statement.

Also solved by TOBIAS DANTZIG.

MECHANICS.

312. Proposed by C. N. SCHMALL, New York City.

A ball of elasticity e is projected upward from a point on an inclined plane, so that after its first contact with the plane it rebounds to its starting point. If ϕ be the inclination of the plane to the horizontal, and ψ the angle made by the line of projection with the inclined plane, show that

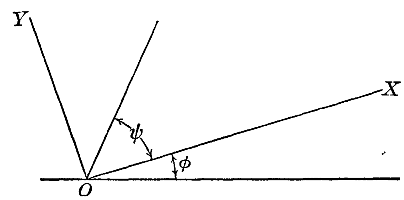
$$\cot \phi \cot \psi = e + 1.$$

I. SOLUTION BY HORACE OLSON, Chicago, Illinois.

In order that the answer given may be true the ball must be thrown in a vertical plane perpendicular to the inclined plane. Taking the axis of coordinates as indicated in the figure, the equations of the trajectory are:

$$x = vt \cos \psi - \frac{gt^2 \sin \phi}{2},$$

$$y = vt \sin \psi - \frac{gt^2 \cos \phi}{2},$$



t being the time from the instant of projection. Then,

$$\frac{dx}{dt} = v \cos \psi - gt \sin \phi \qquad \text{and} \qquad \frac{dy}{dt} = v \sin \psi - gt \cos \phi.$$

When the ball strikes the plane, $y = 0$,

$$t = \frac{2v \sin \psi}{g \cos \phi}, \quad \text{and } x \text{ becomes} \quad x_1 = \frac{2v^2 \sin \psi}{g \cos^2 \phi} (\cos \psi \cos \phi - \sin \psi \sin \phi),$$

whence,

$$\frac{dx_1}{dt} = v \cos x - \frac{2v \sin \psi \sin \phi}{\cos \phi}, \quad \frac{dy_1}{dt} = - v \sin \psi.$$

After the rebound,

$$x = x_1 + vt \cos \psi - \frac{2vt \sin \psi \sin \phi}{\cos \phi} - \frac{gt^2 \sin \phi}{2}, \quad y = evt \sin \psi - \frac{gt^2 \cos \phi}{2},$$

t being here reckoned from the instant of rebound. When the ball again strikes the plane, $y = 0$, and

$$t = \frac{2ev \sin \psi}{g \cos \phi} \text{ and } x \text{ becomes } x_2 = x_1 + \frac{2ev^2 \sin \psi \cos \psi}{g \cos \phi} - \frac{4ev^2 \sin^2 \psi \sin \phi}{g \cos^2 \phi} - \frac{2e^2v^2 \sin^2 \psi \sin \phi}{g \cos^2 \phi},$$

or

$$x_2 = \frac{2v^2 \sin \psi}{g \cos^2 \phi} \{(e + 1)\cos \psi \cos \phi - (e^2 + 2e + 1) \sin \psi \sin \phi\}.$$

By the conditions of the problem, $x_2 = 0$, and, hence, $\cot \phi \cot \psi = e + 1$.

II. Solution by the Proposer.

Let v be the velocity of projection; then $v \sin \psi$ and $v \cos \psi$ are the components respectively perpendicular and parallel to the plane; and the components of gravity perpendicular and parallel to the plane are $g \cos \phi$ and $g \sin \phi$. Considering the perpendicular component, the time of flight is twice the time in which the velocity $v \sin \psi$ would be acquired under the action of the force $g \cos \phi$, *i. e.*, $2v \sin \psi / g \cos \phi$. After rebounding, the ball's velocity perpendicular to the plane is $ev \sin \psi$; hence, the time required in returning to the point of projection is $2ev \sin \psi / g \cos \phi$; and the whole time of flight is

$$2 \frac{v \sin \psi}{g \cos \phi} (e + 1).$$

Now, since the motion parallel to the plane is not affected by the impact, the entire time of flight is equal to twice the time in which the velocity $v \cos \psi$ would be produced by the force $g \sin \phi$, *i. e.*, $2 \dfrac{v \cos \psi}{g \sin \phi}$. Therefore, equating the two expressions found for the time, we have

$$2 \frac{v \cos \psi}{g \sin \phi} = 2 \frac{v \sin \psi}{g \cos \phi} (e + 1);$$

whence,

$$\cot \phi \cot \psi = e + 1.$$

NUMBER THEORY.

An excellent solution of 226 by Mrs. Elizabeth Brown Davis should have been reported in the April issue. Editors.

227. Proposed by R. P. BAKER, University of Iowa.

Show that every rational number can be expressed as a finite sum $\sum_{n=m}^{n=m+k} \frac{a_n}{n}$, where a_n is either 0 or 1 and m is any positive integer.

Solution by Frank Irwin, University of California.

Since the series $\sum_{n=m}^{n=\infty} \frac{1}{n}$ is divergent, we can find a value of r such that our given rational number, N, lies between $1/m + 1/(m + 1) + \cdots + 1/(r - 1)$ and $1/m + 1/(m + 1) + \cdots + 1/r$ (unless indeed N is equal to an expression of this sort, in which case our problem is solved). This reduces the problem to that of expressing $N' = N - [1/m + 1/(m + 1) + \cdots + 1/(r - 1)]$, that is, a (proper) fraction $< 1/r$, as $\sum_{n=r}^{n=m+k} \frac{a_n}{n}$.

Let $N' = c/d$. We shall show that we may express c/d as

$$\frac{1}{l}+\frac{c_1}{d_1},$$

when l is $> r$ (else would c/d be $> 1/r$), c_1/d_1 is $< 1/l$, and $c_1 < c$. If d is divisible by c, the thing is done ($c_1 = 0$). Otherwise, let

$$d = (l-1)c + m, \quad 0 < m < c.$$

Then

$$\frac{c}{d} = \frac{1}{l} + \frac{c-m}{dl}$$

is an expression of the kind desired, since $(c-m)/dl < c/dl < 1/l$ (for c/d is a proper fraction).

Now apply the same process to c_1/d_1, which is $< 1/l$:

$$\frac{c_1}{d_1} = \frac{1}{l_1} + \frac{c_2}{d_2}, \qquad l_1 > l.$$

Continuing in this way, we shall eventually get for N' an expression of the kind we are seeking:

$$N' = \frac{c}{d} = \frac{1}{l} + \frac{1}{l_1} + \cdots + \frac{1}{l_s},$$

where

$$r < l < l_1 < \cdots < l_s;$$

for the process must come to a stop since the numerators $c_1, c_2, \cdots$ continually decrease (the process being accelerated whenever one of the fractions c_i/d_i reduces).

The procedure may be described in a few words: to keep adding to our set of fractions the largest one available (*i. e.*, one that has not already been used and whose denominator is not $< m$) that will not make their sum greater than the given rational number.

It is, perhaps, interesting to illustrate the methods employed by a numerical example. Let it be required to express 7/22 as $\sum_{n=10}^{n=10+k} \frac{a_n}{n}$.

$$\frac{7}{22} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{29}{660}, \qquad \text{where} \qquad \frac{29}{660} < \frac{1}{13};$$

$$\frac{29}{660} = \frac{1}{23} + \frac{7}{15180}; \qquad \frac{7}{15180} = \frac{1}{2169} + \frac{3}{32925420} = \frac{1}{2169} + \frac{1}{10975140};$$

the numerators 29, 7, 3 forming a decreasing series, and the last "remainder," 3/32925420, reducing to lower terms.

Our result then is:

$$\frac{7}{22} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{23} + \frac{1}{2169} + \frac{1}{10975140}.$$

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

In April, 1915, there was published in this department of the MONTHLY (Vol. XXII, pp. 119–121) a summary of remarks made by Commissioner Snedden of Massachusetts at the Cincinnati meeting of the N. E. A., Feb. 25, 1915, on the topic "Can algebra and geometry be reorganized so as to justify their retention for high-school pupils not likely to enter technical schools?" These remarks

were published to call the attention of teachers of college mathematics to a movement having for its purpose the practical elimination of algebra and geometry from the secondary-school curriculum. A question was asked in connection with the summary and the replies received were so few as to indicate that the large majority of our readers are comparatively indifferent to the movement. They may have thought that it was a matter which interested high-school teachers only.

The movement is, however, one of vital concern to teachers of college mathematics for at least two reasons. In the first place, it will not be long until the same attack will be made upon the teaching of college mathematics in other than technical schools. A revaluation and reorganization of college mathematics will be demanded, and teachers of college mathematics will do well to take some observations of the movement now in order that they may the better understand the situation when it confronts them. In the second place, if this movement is successful the teacher of college mathematics must begin his teaching on a foundation including little beyond the four fundamental operations in arithmetic.

The reader who inclines to the view that these possibilities are too remote for serious consideration will do well to read "A Modern School," by Abraham Flexner, issued by the General Education Board (61 Broadway, New York) as Occasional Paper No. 3 (23 pp.), and sent gratis on request.

The following extract from an editorial in the *New Republic* for April 8, 1916 (pp. 249–250), is doubtless somewhat typical of the way Dr. Flexner's paper will be taken by editors of other papers, especially by editors of the more radical educational journals:

> Dr. Abraham Flexner's manifesto of a modern school, issued this week by the General Education Board, cuts out at a blow all the tangled controversy of schoolmen over what shall be taught and why. The modern school would include nothing in its curriculum for which an affirmative case can not now be made out. Grammar, formal history, dead languages, formal mathematics, are in school now because of tradition and assumption. It is useless to ask whether a knowledge of these subjects is valuable. The fact is that children today do not get value from them. That is all a modern school need know. . . . Dr. Flexner's manifesto puts the whole problem as concisely as it can be put. He gives us a standard, upon which progressive educators already agree, and which can be used to test and to judge the individual school. If the teacher is to have an educational creed, this is the creed for the American school of the immediate morrow.

Dr. Flexner's guiding thesis and its bearing on mathematics are thus stated by him (p. 17):

> Modern education will include nothing simply because tradition recommends it or because its inutility has not been conclusively established. It proceeds in precisely the opposite way: *it includes nothing for which an affirmative case can not now be made out.* As has already been intimated, this method of approach would probably result in greatly reducing the time allowed to mathematics, and decidedly changing the form of what is still retained. If, for example, only so much arithmetic is taught as people actually have occasion to use, the subject will sink to modest proportions; and if this reduced amount is taught so as to serve real purposes, the teachers of science, industry and domestic economy will do much of it incidentally. The same policy may be employed in dealing with algebra and geometry. What is taught, when it is taught, and how it is taught will in that event depend altogether on what is needed, when it is needed, and the form in which it is needed.

To those of us who have been accustomed to think of mathematics as the simplest form of abstract reasoning it will seem a bit difficult to reconcile the foregoing with the following statement found on pp. 9–10:

> But in the end, if the Modern School is to be adequate to the need of modern life, this concrete training must produce sheer intellectual power. Abstract thinking has perhaps never played so important a part in life as in this materialistic and scientific world of ours—this world of railroads, automobiles, wireless telegraphy, and international relationships.

Dr. Snedden, Dr. Flexner and others who are attacking mathematics are insisting that the burden of proof to show that it should remain in the curriculum is upon those who advocate its continuance and not upon those who would eliminate it. To the writer it seems that these iconoclasts are placing entirely too much faith in the methods of experimental pedagogy so enthusiastically advocated in schools of education at the present time. Experimental pedagogy is a great improvement, but it should be borne in mind that it is only an improvement. It merely attempts to formulate explicitly experimentation which the race has long carried on informally. Humanity decided from its experience that mathematics was valuable and placed it in the course of study. It would seem that the burden of proof ought to be upon the man who would replace it with something else to demonstrate beyond question that the substitute is better than the original.

If these iconoclasts are right, the college teachers ought to be among the first to discover it and to aid in making proper readjustments. But regardless of whether they are right or wrong, teachers of college mathematics ought to be giving more thought and attention to the purposes of their teaching and the sequence and content of courses. Much remains to be done toward organizing and standardizing the undergraduate mathematical curriculum in colleges and universities. There is a somewhat general agreement as to sequence and content of courses so far as to include elementary differential and integral calculus; but beyond that point there is great variation both in opinion and practice.

It ought not to be that each college or university should be left to work out these problems by itself. There ought to be such coöperation as would enable each school to profit by the experience and counsel of others.

The editors would like to see more use made of this department in attacking these and such kindred problems as the proper differentiation of courses for groups of students specializing along different lines. Questions 2, 7, 11, 16, 27, 30 and 31 serve as a beginning in this direction. We hope that more questions will be sent in and also many more replies to questions 27, 30 and 31.

DISCUSSIONS.

Relating to "The Unification of Freshman Mathematics."

By F. L. Griffin, Reed College, Portland, Ore.

Agreeing heartily with nearly all that Mr. Nyberg says in describing his very interesting and valuable course, in the April issue of the Monthly, I reluctantly feel obliged to record a difference of opinion on certain points.

(1) *As to the comparative difficulty of his course and the one which I outlined.*

The chief difference as regards scope is that in my course elementary integral calculus replaces the more advanced parts of trigonometry, college algebra and analytics in his. It should be remembered that we carefully avoid problems in which the details are very complicated; and that we introduce the calculus gradually, returning to it again and again during the year, and thus driving it home with a minimum of difficulty; also, that elementary integral calculus is much easier for the average student to understand than the more advanced and abstract parts of trigonometric analysis, such as the general treatment of imaginaries and vectors, progressive waves, the solution of cubics, etc., or the study of higher plane curves, hyperbolic functions, etc. The fact is that my course formerly included more of these topics and less integral calculus, and was noticeably more difficult then. Consequently, I am unable to conceive how his course could succeed in any institution where mine would fail.

Indeed I do not understand why anyone should expect my course to fail with the class of students found in our state universities. After having taught the course five years to students whose preparation has varied all the way from a year of algebra and a term of geometry to four years of mathematics, I should expect the course to succeed almost anywhere with good teaching. Our students work well, but their *preparation* is in no way unusual. Their knowledge of the technique of algebra and the facts of geometry is decidedly defective.

(2) *As to the importance of including integral calculus.* It ought to be possible for prospective workers in physical chemistry, biology, statistics, or economics to get in one year the mathematical equipment which they need, including some understanding of the elementary applications of integration. If any university is not interested in providing such equipment (supposing this to be feasible), so much the worse for that university. Moreover, for physicists, integral calculus is many times as useful as differential. Again, the student specializing in mathematics should become familiar with both branches of the calculus as early as possible. Also "the non-specialist who wishes an introduction to mathematics" should by all means see something of integral calculus. The inclusion of this subject tends decidedly to unify the whole, and to give the student a "knowledge of calculus" that will really "compensate" him for the omission of other topics. Why allow the complex variable, and the witch, cissoid, conchoid, and polar equations of conics to shut out the most valuable part of the course?

(3) Finally, my statement that "the more theoretical parts of college algebra and analytics are postponed to the junior year" seems to have been misleading. The analytics as a whole is not postponed, but only the study of the more intricate parts. All the usual type equations are covered—in fact, all the plane analytics needed for the usual work in calculus.

Relating to the Slope-angle of a Curve.

By H. B. Phillips, Massachusetts Institute of Technology.

The recent review of my analytic geometry by Dr. G. R. Clements contains a rather severe criticism of my use of the angle φ from the x-axis to a line. If I

had not thought the matter over so carefully, I should heartily endorse his criticism.

The angle φ as I use it is not "the least positive angle" but any angle, positive or negative, from the positive direction of the x-axis to the line. Precisely which angle is to be represented by φ in a particular case, is determined by convenience. The tangent formulæ are correct for all.

My reason for this is to avoid the discontinuity that otherwise occurs. According to the "least positive angle" definition a line with a small positive slope makes with the x-axis a small angle while a line with a small negative slope makes an angle near 180°. This leads to trouble when an angle passes through zero. For example, the curvature is usually defined as $d\varphi/ds$. I do not know of any calculus text where this formula is considered inexact at points where $\varphi = 0$. Yet that is the case if φ is the least positive angle.

In defining angles, several cases are to be considered. In determining a directed line, angles congruent, modulo 2π, are equivalent. In determining merely the position of a line, angles congruent, modulo π, are equivalent. In three dimensions there is often no distinction between positive and negative angles. In each of these cases it is not only unnecessary but actually inconvenient to fix the angle more precisely.

OFFICIAL ANNOUNCEMENTS OF THE COUNCIL.

During the past three months, the Council has taken official action on pressing matters which seemed to require immediate action. In general, the members of the Association will be informed of pending actions in order that their opinions may reach the Council before decisions are made. These questions did not appear to be of that kind.

The actions submitted by mail and approved thus far are as follows:

(1) The Council authorized the election to membership of all those who signed the original Call for the Columbus meeting, and of those who participated in that meeting.

(2) The Council authorized the Secretary to allow a few days, not to exceed ten days, of grace for mail to reach him after April 1st, so that those desiring to enter as charter members should not be excluded by accident or by unavoidable delay.

(3) The Council ordered that the books and magazines belonging to the Association be deposited in the Oberlin College Library, under direct charge of the Secretary of the Association and the Librarian of Oberlin College. It is understood that these books are to be kept in a separate and safe place, and that they are to be subject to withdrawals and to requests from members for temporary loan, subject to the further wishes of the Association. See also (7) below.

(4) The Council ordered that a summer meeting be held near the time of the summer meeting of the American Mathematical Society, in the buildings

of the Massachusetts Institute of Technology, which institution had formally invited the Association to hold such a meeting. A Committee on the Program and a separate Committee on Arrangements were authorized; and the President was asked to appoint these committees.

(5) The Council authorized the formation of a Committee on Mathematical Requirements, to be appointed by the President.

This Committee has now been appointed as follows: J. W. YOUNG, Dartmouth College, Chairman; A. R. CRATHORNE, University of Illinois; E. H. MOORE, University of Chicago; D. E. SMITH, Columbia University; and OSWALD VEBLEN, Princeton University. It is hoped that a tentative report and announcement of plans may be ready for presentation at the coming summer meeting.

(6) The Council ordered that the Secretary-Treasurer shall pay to the treasurer of each section of the Association all money actually received by him in payment of initiation fees of new members in the territory of that section, provided, however, that this money shall be used exclusively for the payment of the incidental expenses of the section itself, and that in no case shall the fees be returned to the individual initiates.

(7) The Council authorized the appointment of a committee to be known as the Library Committee. This committee will not only have general charge of the books and magazines belonging to the Association but it will have the larger function of stimulating by every feasible means the wider knowledge and use of mathematical books. Some proposed plans have already been referred to in connection with lists of books suitable for college libraries. These and other matters will be taken up at an early date. The committee has not yet been appointed.

(8) The Council authorized the appointment of a committee to have charge of a Bureau of Information. The duties of this committee will consist in answering inquiries aside from those pertaining to the solution of problems. A more complete statement concerning the work of this committee along with the names of the committee will be announced later.

The President has appointed the following committees for the summer meeting, in accordance with the authorization of the Council. In each case, the first named member is the chairman.

Program Committee: H. B. FINE, Princeton University; DUNHAM JACKSON, Harvard University; and C. S. SLICHTER, University of Wisconsin.

Committee on Arrangements: H. W. TYLER, Massachusetts Institute of Technology; H. E. HAWKES, Columbia University; G. D. OLDS, Amherst College; CLARA E. SMITH, Wellesley College; and R. G. D. RICHARDSON, Brown University. This committee later added to its number G. W. EVANS, Charlestown High School; H. D. GAYLORD, Browne and Nichols School; W. R. RANSOM, Tufts College, and MABEL M. YOUNG, Wellesley College.

These committees are authorized to fill any vacancies which may occur and to add to their own membership other members of the Association. They

have power to make all arrangements for the summer meeting. Announcements will be issued by them in due course, in the MONTHLY, and separately through the Secretary of the Association.

The Council has decided to hold the annual meeting in connection with the meeting of the American Association for the Advancement of Science during the Christmas vacation. The meeting this year will be in New York City.

E. R. HEDRICK,
PRESIDENT OF THE ASSOCIATION.

THE SUMMER MEETING OF THE ASSOCIATION.

It is definitely determined that the first summer meeting of the Association will be held at the new buildings of the Massachusetts Institute of Technology in Cambridge, Mass., on Friday and Saturday, September first and second.

The Program Committee has been asked to decide upon the topics to be discussed in the papers to be read at the meeting and to select the members of the Association who are to be invited to prepare these papers. The Committee will be glad to receive and to consider any suggestions which members of the Association may wish to send them relative to this matter. Numerous topics and speakers are under consideration and full announcement will be made in August by letter addressed to each member of the Association. All communications in regard to the program should be addressed to Professor H. B. FINE, Princeton, New Jersey.

The Association headquarters will be at Riverbank Court, Cambridge, where rooms may be reserved in advance. The Committee on Arrangements will also have further announcements to make concerning accommodations in both Cambridge and Boston. In particular, special arrangements will be made for ladies attending the meetings and correspondence on this point may be addressed to Professor CLARA E. SMITH, or to Dr. MABEL M. YOUNG, Wellesley College, Wellesley, Mass.

A regular summer excursion railroad rate of thirty dollars for the round trip from Chicago to Boston, and correspondingly low rates from certain other points will be available. Inquiry should be made of the local ticket agents. Persons arriving in Boston by land or water and desiring to reach the Massachusetts Institute of Technology, or Riverbank Court, should inquire for trolley car for Harvard Bridge, and leave the car at the Cambridge end of the bridge. Kendall Station of the Cambridge subway is five minutes' walk from the Institute. Mail and express may be addressed, at the time of the meeting, to Riverbank Court, Cambridge, or to the Mathematical Department, Massachusetts Institute of Technology.

A dinner for members will be arranged on Friday evening, September first, at Riverbank Court, and luncheon will be served there between sessions on Friday and Saturday. If consistent with the plans of the Program Committee, there will be an automobile excursion to Wellesley, Mass., starting about four o'clock

on Saturday, with a brief visit to the college, optional dinner at the Wellesley Inn, and return by train.

Members expecting to attend the meetings will greatly assist the Committee on Arrangements by notifying the Secretary of the Committee, Mr. H. D. GAYLORD, 104 Hemenway St., Boston, Mass., (*a*) whether they desire a list of convenient boarding places; (*b*) whether they are likely to attend the Association dinner on Friday evening (probable cost $1.50); (*c*) whether they are likely to join in the Wellesley excursion and supper on Saturday (probable cost $1.50).

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

Dr. GEORGE A. PFEIFFER has been appointed instructor in mathematics at Princeton University.

Professor C. J. KEYSER, of Columbia University, and Professor M. W. HASKELL, of the University of California, will exchange chairs for the first half-year 1916–17.

At the University of Oklahoma, Mr. C. T. LEVY, of the University of California, has been appointed instructor in mathematics, to take the position occupied by Mr. E. D. Meacham, who has been granted leave of absence for one year to study at Harvard University.

Mr. R. E. GILMAN, of Princeton University, has been appointed instructor in mathematics at Cornell University.

Professor A. H. NORTON, head of the department of mathematics of Elmira College, N. Y., has been appointed vice-president of that college.

Dr. GEORGE SARTON, formerly of Ghent and editor of *Isis* has been appointed lecturer on the "history of science" at Harvard University. He will give one course next year on "The origin and development of Greek science," and one on "The principles of mathematics historically considered."

Mr. ARCHIE S. MERRILL has been appointed to an assistant professorship of mathematics at the University of Montana. He is a candidate for the doctorate at the University of Chicago at the coming summer convocation.

Professor E. R. HEDRICK gave a series of three mathematical addresses at the University of Iowa on May 26 and 27, on the special invitation of those interested in forming an Iowa Section of the ASSOCIATION.

The quarter-centennial anniversary of the founding of the University of

Chicago was celebrated with due ceremony during the four days of June 3–6, 1916. Among the features of the program was a series of departmental conferences held especially in honor of those who hold the doctorate at the University, now numbering nearly one thousand. The departments of mathematics, mathematical astronomy, and physics joined together in two conferences and a dinner and social gathering. At one of these meetings there were brief reports on research activities by three doctors from the department of physics of the University of Chicago and two from the department of mathematics, namely, Professor OSWALD VEBLEN of Princeton University and Professor ARNOLD DRESDEN of the University of Wisconsin. At the other meeting there were three addresses as follows: "The Problems of Astrophysics," by GEORGE ELLERY HALE, Director of the Solar Observatory at Mount Wilson; "The Relation of Pure Science to Industrial Research," by JOHN J. CARTY, chief engineer of the American Telegraph and Telephone Company; "Current Tendencies in Mathematical Research," by EDWARD B. VAN VLECK, professor of mathematics at the University of Wisconsin. These three were among those on whom was conferred the honorary degree of Doctor of Science at the Convocation exercises on June sixth.

In 1915 a Joint Committee on Classification of Technical Literature was appointed by delegates from thirty-two technical bodies. The representative of the American Mathematical Society on this Committee is Professor E. V. HUNTINGTON, of Harvard University. The Sub-committee of the Society associated with Professor HUNTINGTON consists of Professor R. C. ARCHIBALD of Brown University, Professor T. H. GRONWALL of Princeton University, Professor E. H. MOORE of the University of Chicago, and Professor E. B. WILSON, of the Massachusetts Institute of Technology. This sub-committee is considering a suitable classification of mathematical literature.

Professor WEBSTER WELLS, of the Massachusetts Institute of Technology, died in Boston on May 23, 1916. He graduated from the Institute in 1873 and immediately joined the teaching staff. In 1893 he was made professor of mathematics, a position which he held until 1911 when he retired. He was the author of a series of textbooks in mathematics.

At Brown University three fellowships, each with a stipend of five hundred dollars, have been awarded to students who are to pursue graduate work in mathematics at the university during 1916–1917. The recipients of the awards are: BANCROFT HUNTINGTON BROWN, of Hyde Park, Mass., Grand Army Fellow; MARION ELIZABETH STARK, Norwich, Conn., Lyra Brown Nickerson Fellow; MARIAN MARSH TORREY, of Providence, R. I., Emma Josephine Arnold Fellow. Professor R. G. D. RICHARDSON has been granted leave of absence and he will probably spend the year in Cambridge, Mass.; Dr. W. BURGESS, instructor in mathematics at Cornell University and a Rhodes scholar from Brown, has been appointed instructor in mathematics; Mr. R. L. BLANCHARD has been appointed assistant in mathematics.

The National Academy of Sciences recently elected nine new members, including G. A. BLISS, professor of mathematics at the University of Chicago. Among the newly elected members of the American Philosophical Society are MAXIME BÔCHER, professor of mathematics at Harvard University, and F. R. MOULTON, professor of astronomy at the University of Chicago.

The twelfth annual session of the Association of Ohio Teachers of Mathematics and Science was held at Ohio State University on April 21 and 22. This meeting was really a joint session of the above mentioned association with the Ohio Academy of Science, the Ohio College Association and the Ohio Section of the Mathematical Association of America. General meetings of all the co-operating associations were held on Friday evening and Saturday morning; sectional meetings were held on Friday. Among the papers of interest to readers of the MONTHLY may be mentioned: "The training of science and mathematics teachers," by Professor G. R. TWISS, of Ohio State University; "Mathematics and the college curriculum," by Professor A. D. PITCHER, of Western Reserve University; "Relation of the newly organized Mathematical Association of America to the Association of Ohio Teachers of Mathematics and Science," by Professor C. C. MORRIS, of Ohio State University; "Supervised study in high school mathematics," by Mr. W. B. SKIMMING, of East High School, Columbus; "Correlated secondary mathematics," by Professor O. L. DUSTHEIMER, of Baldwin-Wallace College. A full report of the Ohio Section meeting of the ASSOCIATION is found elsewhere in this issue.

The Association of Mathematics Teachers of New Jersey held its fourth regular session at Princeton University on May 6. The program consisted of the report of the committee on "Courses in Trigonometry," by Professor C. O. GUNTHER, of Stevens Institute; also the following papers: "Euclid's theory of incommensurable magnitudes," by Professor H. B. FINE, of Princeton University; "Ptolemy's theorem," by Mr. E. FLORENCE, of Rutgers College; "An exposition of Napier's principle of logarithms," by Mr. E. S. INGRAM, of Rutgers College; "Certain religious implications of the mathematical infinite," by Rev. F. C. DOAN, of Summit, N. J.; and "The ultimate aim of a course in arithmetic," by Professor J. C. STONE, of Montclair Normal School.

The Mathematics Club of Albion College, Albion, Mich., was organized on Jan. 17, 1911, with fifteen members present. Membership is limited to those who have had at least two years of college mathematics, and who propose to continue their work along mathematical lines. Membership is gained through recommendation of the head of the department of mathematics and by vote of the club. The membership of the club shows a total enrollment of fifty since its organization in 1911. Of this number all but two have either completed their undergraduate course at Albion or are now students in the college; nine have received the M.A. degree, nine have been granted fellowships or scholarships by different universities, and one has received the Ph.D. degree. Eighteen are

teaching in the high schools of Michigan. The Albion Mathematics Club is well organized; it meets for one hour on each Tuesday evening, at which time a regular program is presented as follows: (1) Roll-call, (2) a five-minute talk on some assigned topic, (3) topic of the evening, (4) critic's report, (5) general discussion. The secretary sends a most interesting collection of topics used for response at roll-call, those used for five-minute talks, and those used as the general topic of the evening.

The following courses are announced at summer sessions:

COLUMBIA UNIVERSITY SUMMER SESSION (July 10–August 18).—By Professor M. W. HASKELL: Differential equations, five hours; Modern analytic geometry, five hours.—By Professor JAMES MACLAY: Theory of geometric constructions, five hours.—By Professor EDWARD KASNER: Theory of functions of a real variable, five hours.—By Professor W. B. FITE: Higher algebra, five hours.

CORNELL UNIVERSITY SUMMER SESSION (July 6–August 16).—By Professor VIRGIL SNYDER: Geometric constructions for high school teachers, five hours; Seminar in algebraic geometry.—By Professor W. A. HURWITZ: Mathematical analysis, five hours; Supplementary problems in algebra for high school teachers, five hours; Seminar in integral equations.—By Professor F. W. OWENS: Projective geometry, five hours; Seminar in foundations of geometry.

The Bureau of the Census, Department of Commerce, has just issued a set of "Life Tables," the first of their kind which have ever been prepared by the United States government. These tables, compiled in the division of vital statistics, under the direction of Professor J. W. GLOVER, professor of mathematics and insurance in the University of Michigan, show death rates and expectation of life at all ages for the population of the six New England states, New York, New Jersey, Indiana, Michigan, and the District of Columbia, on the basis of the population in 1910 and the mortality for 1909, 1910, and 1911. The Bulletin includes twenty-five Life Tables covering important classifications of the population, such as white, negro, native white, foreign-born white, city and urban. It is remarkable that the government has taken a census every ten years for a century, and has diligently collected mortality statistics, yet has never reduced this mass of facts to the form of Life Tables. Great credit is due Professor Glover for his labor during the past three years in planning and supervising this work. The first edition of about 25,000 copies will be ready for distribution soon, and may be had by addressing Director S. L. ROGERS, Bureau of the Census, Washington, D. C.

In the December number of *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht* is published an interesting paper by Professor W. LOREY, Leipzig, on the early life of Karl Weierstrass, the noted German mathematician, the one-hundredth anniversary of whose birth was celebrated on October 31, 1915. The paper by Lorey is devoted chiefly to the interesting events of Weierstrass's early life, some of which are here given. Weierstrass was born in the village of

Ostenfelde, on October 31, 1815, and died in Berlin, February 19, 1897. He received his early training in the gymnasium at Paderborn. His interest in mathematics was aroused while in the gymnasium by accidentally finding in the library an uncut number of *Crelle's Journal,* containing some of the beautiful geometrical work of Steiner. His interest for pure geometry, awakened by this article, remained with him throughout his mathematical career. In his Berlin lectures in later life, Weierstrass would frequently turn away from his analytical interests and give a course of lectures on some phase of pure geometry. In 1834 he left the gymnasium at Paderborn and entered the University of Bonn, not as a student of mathematics, but of law. His interest in mathematics was somewhat incidental while at the University; he studied some of the work of Laplace, but was not influenced by the mathematical lectures at Bonn. The first mathematician to make a lasting impression upon Weierstrass was Professor C. Gudermann (1798–1851) of Münster, who had already published a number of researches in spheric geometry, hyperbolic functions, and elliptic functions, in the early volumes of Crelle. Through a fellow student at Bonn, Weierstrass had come into possession of a copy of Gudermann's lectures on elliptic functions; he was so attracted by the subject that he left the University of Bonn and entered the University of Münster, 1839, to carry on work with Gudermann. He remained but one semester in Münster as a student with Gudermann, having written, however, an important memoir on "The development of modular functions," one on pedagogy, and two on philologic-historic problems. His mathematical work on "modular functions" was intended as a dissertation, but as the faculty at Münster did not grant the doctorate he did not receive that distinction until several years later, when the University of Königsberg granted him the degree. Weierstrass qualified as a gymnasium instructor in 1840, and continued in this service for sixteen years, during which time he taught a variety of subjects—science, mathematics, physical training. Remarkable mathematical investigations on algebraic functions carried on during the latter part of this gymnasium period (1848–1856), attracted nation-wide attention to him, so that in 1856 he was called to Berlin, first as instructor in a *Technical Academy* with a minor position in the University of Berlin. These positions he held until 1864, when he was promoted to a full professorship of mathematics at the University, a position in which he continued until his death in 1897.

The career of Weierstrass was very different from that of most brilliant mathematicians. The creative work of such men as Pascal, Lagrange, Abel, Galois, and many others, was done, or at least mapped out, very early in life, from 18 to 30 years of age. Weierstrass, having set out to become a lawyer, at the age of twenty-five became somewhat interested in mathematics, and then spent sixteen years in elementary teaching. He really began his remarkable mathematical investigations at the age of thirty-three, and did not enter upon service in the university until one half of his life had passed. His remarkable work was all accomplished after the age at which many brilliant mathematicians cease their research work.

The next issue of the MONTHLY will be early in September.

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THE AMERICAN MATHEMATICAL MONTHLY

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VOLUME XXIII — SEPTEMBER, 1916 — NUMBER 7

THE REAL FUNCTION DEFINED BY $X^y = Y^x$.

By E. J. MOULTON, Northwestern University.

It is obvious that for powers the commutative law does not hold in general. That is,

$$a^b \neq b^a, \text{ in general.}$$

But in some cases the letters may be interchanged, as for example when $a = b$, or when $a = 2, b = 4$. In looking for other values of the letters for which they may be interchanged, I have made the following discussion of the complete real locus of the equation

$$x^y = y^x. \tag{1}$$

I. THE FIRST QUADRANT.

First, let us consider only positive values of x and y. When an exponent is fractional, as in $x^{1/2}$, or irrational, we shall understand by the symbol the positive real value of the indicated power. Then our equation is equivalent to

$$y \log x = x \log y. \tag{2}$$

To discuss this equation we consider the equation

$$\frac{\log x}{x} = \frac{\log y}{y} = k. \tag{3}$$

The parameter k is real. Now let us plot the equations

$$u = \log t \quad \text{and} \quad u = kt. \tag{4}$$

Suppose these curves intersect at $Q_1(t_1, u_1)$ and $Q_2(t_2, u_2)$. Then we have

$$\frac{\log t_1}{t_1} = \frac{\log t_2}{t_2} = k.$$

Hence we obtain the following four solutions of (3): $x = t_1, y = t_1$; $x = t_1, y = t_2$; $x = t_2, y = t_1$; $x = t_2, y = t_2$; or in a brief notation,

$$P_{11}(t_1, t_1),\ P_{12}(t_1, t_2),\ P_{21}(t_2, t_1),\ P_{22}(t_2, t_2).$$

Thus, by finding the intersections of equations (4) we find solutions of (3). It is also clear that the values of x and y in a solution of (3) are values of t at points of intersection of the curves (4). Hence, we shall find all the solutions of (3) or of (1) by the method indicated.

When $k = 0$ the curves (4) have only one point of intersection Q_1 (see Fig. 1). Hence, corresponding to each value of $k = 0$ there is just one point P_{11} on (1).

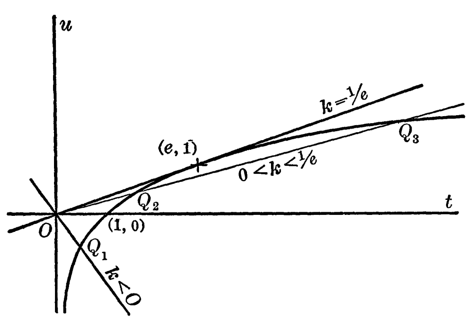


FIG. 1. $u = \log t$ and $u = kt$.

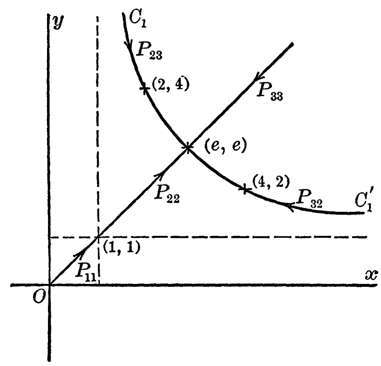


FIG. 2. Part of locus: $x^y = y^x$.

As the parameter k varies from $-\infty$ to 0, t_1 varies from 0 to 1, and hence P_{11} moves along the line $y = x$ from (0, 0) to (1, 1), the former point being excluded, the latter included.

When $0 < k < 1/e$, $e = 2.71828 \cdots$, the curves (4) have two points of intersection, Q_2 and Q_3. Hence, corresponding to each of these values of k there are four points, P_{22}, P_{23}, P_{32}, P_{33}, on (1). As the parameter k varies from 0 to 1, t_2 varies from 1 to e and t_3 from ∞ to e; and hence P_{22} moves along $y = x$ from (1, 1) to (e, e), P_{23} comes from infinity along a curve C_1 asymptotic to $x = 1$, and approaches (e, e), P_{32} comes from infinity along a curve C_1' asymptotic to $y = 1$, and approaches (e, e), and P_{23} comes from infinity along $y = x$ to (e, e). In describing C_1 the point P_{23} moves monotonically to the right and down; in describing C_1' the point P_{32} moves monotonically to the left and up. The point (2, 4) is found to lie on C_1 and (4, 2) on C_1'. See Fig. 2.

When $k = 1/e$ the curves (4) have one point of intersection $(e, 1)$. The corresponding point of (1) is (e, e).

When $k > e$ the curves (4) do not intersect and there is no corresponding point of (1).

II. The General Position.

If either x or y is negative and the other fractional or irrational, it is not easy to describe simply the unique meanings to be attached in general to each of th

expressions x^y and y^x. For example, $(-2)^{3/7}$ is seven-valued, and $(-2)^e$ is infinitely-many-valued, each value being a complex number. Accordingly, we shall say that (a, b) is a point of the locus (1) if by proper choice from the values of a^b and b^a we have $a^b = b^a$.

Obviously the equation is satisfied if $y = x$, except at (0, 0). If $x = 0, y = 0$, we have the symbol 0^0, which is customarily considered meaningless (not many-valued), except as it is associated with the limit of a function. If we take $y = x$ in (1) and let $x = 0$ we obtain the symbolic equation $0^0 = 0^0$, and in this sense the point (0, 0) is on the locus of (1) also.

Equation (1) is equivalent to

$$e^{y \log x} = e^{x \log y}.$$

The logarithmic function is many-valued; this is shown explicitly by writing our equation

$$e^{y[\mathrm{Log}\,|x| + i(\theta + 2p\pi)]} = e^{x[\mathrm{Log}\,|y| + i(\phi + 2q\pi)]}.$$

Here $\mathrm{Log}\,|x|$ is the single-valued real logarithm of the absolute value of x, and $\mathrm{Log}\,|y|$ of y; p and q are positive or negative integers or zero, and θ and ϕ are zero or π according as x and y, respectively, are positive or negative numbers. These two exponentials are equal if and only if the exponents differ by $2n\pi i$, n an integer. Hence (1) is equivalent to

$$y\,\mathrm{Log}\,|x| + i(\theta + 2p\pi)y = x\,\mathrm{Log}\,|y| + i(\phi + 2q\pi)x + 2n\pi i.$$

For real solutions we must have simultaneously

$$y\,\mathrm{Log}\,|x| = x\,\mathrm{Log}\,|y|, \tag{5a}$$

$$(\theta + 2p\pi)y = (\phi + 2q\pi)x + 2n\pi. \tag{5b}$$

The graph of (5a) is obtained by the method used in discussing (2). The figures corresponding to Fig. 1 and Fig. 2 are given in Fig. 3 and Fig. 4. The

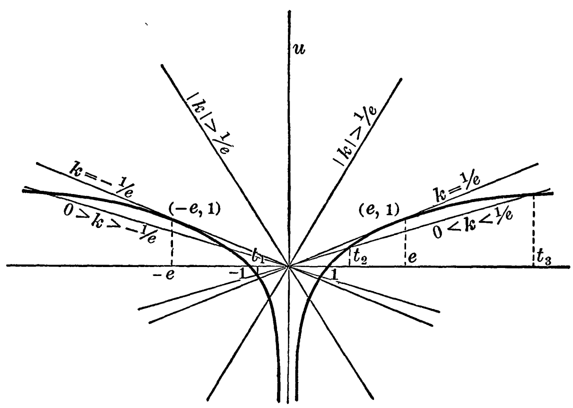


FIG. 3. $u = \log|t|$ and $u = kt$.

points of the locus in Fig. 4 which satisfy (5*b*) for some choice of p, q and n constitute the graph of (1). We have already seen that the line $y = x$ and the curve C_1 are parts of the locus; it remains to consider C_2, C_3 and C_4.

In the second quadrant $\theta = \pi$, $\phi = 0$, and (3*b*) becomes

$$(7) \qquad (2p + 1)y = 2qx + 2n.$$

Not every point of C_2 is on $x^y = y^x$, for p, q, n cannot be chosen so that (7) is satisfied by $(-1, 1)$, since otherwise we would have an odd number equal to

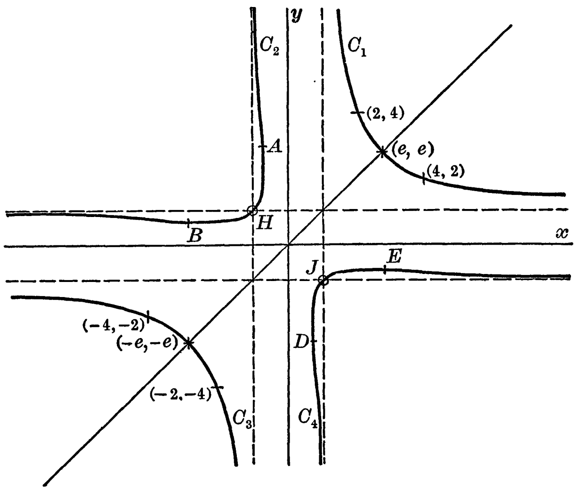


FIG. 4. Locus $y \log |x| = x \log |y|$. Includes locus $x^y = y^x$.

Special points: $A(-.7569\cdots, e)$; $B(-e, .7569\cdots)$; $D(.7569\cdots, -e)$; $E(e, -.7569\cdots)$; $H(-1, 1)$; $J(1, -1)$. B is a minimum and E a maximum of the locus.

an even number. *But the points of C_2 on $x^y = y^x$ in the vicinity of any point of C_2 form a dense point aggregate.*[1] To prove this we let (a, b) be any point on C_2 and show that it is a limit point of a set of intersections of (7) with C_2. We consider only the lines for which $q = 0$,

$$y = \frac{2n}{2p + 1},$$

which are parallel to the x-axis. It is then clearly sufficient to show that b is a lower limit point of a set of numbers of the form $2n/(2p + 1)$. Such a set is formed by taking in succession $p = 1, 2, 3, \cdots$, and choosing n so that

$$(8) \qquad b(2p + 1) < 2n \leqq b(2p + 1) + 2.$$

Since the extremes of these inequalities differ by 2, there is just one integer $2n$ satisfying (8). And since (8) may be written

$$0 < \frac{2n}{2p + 1} - b \leqq \frac{2}{2p + 1}$$

[1] PIERPONT, Theory of Functions of a Real Variable, I, p. 167.

it is seen that in this set are an infinity of numbers greater than b and that they approach b as a limit as p increases indefinitely.

In the fourth quadrant the situation is precisely similar to that in the second.

In the third quadrant $\theta = \pi$, $\phi = \pi$, and (5b) becomes

$$(2p + 1)y = (2q + 1)x + 2n. \tag{9}$$

Hence the points of C_3 on $x^y = y^x$ in the vicinity of any point of C_3 form a dense point aggregate. For in this case we need consider only the lines when $q = p$,

$$y = x + 2n/(2p + 1),$$

which are parallel and have intercepts which are dense on the y-axis. Since C_3 is nowhere parallel to these lines the conclusion is obvious.

The points $(-2, -4)$ and $(-4, -2)$, in particular, lie on $x^y = y^x$.

Conclusion.—Fig. 4 is the graph of $\dfrac{\operatorname{Log}|x|}{x} = \dfrac{\operatorname{Log}|y|}{y}$. It contains all points of the graph of $x^y = y^x$; also it contains other points. Every point on the line $y = x$ and every point on the curve C_1 lies on $x^y = y^x$; the points H and J are not on the locus, but there is an everywhere dense set of points on C_1, C_2, C_3 lying on the locus. I do not know whether there are points on C_3 which are not on $x^y = y^x$, or how many points on C_2 and C_4 are not on $x^y = y^x$.

CONCERNING ROULETTES.

By GOLDIE HORTON, University of Texas.

I. If one curve rolls on another the curve traced by any point in the plane of the rolling curve is called a roulette. The rolling curve is called the moving centrode M and the fixed curve is called the fixed centrode F. To safely apply the methods of infinitesimals we shall suppose that the functions employed in defining the moving and fixed centrodes have continuous first and second derivatives.

(1) As M rolls on F, a line l in the plane of M and a point P on l go into a line l' and a point P' on l', and the point of contact T of M and F moves to T'. The range P on l is congruent to the range P' on l' and hence the pencil PT is projective with the pencil $P'T'$. Since the limiting position of the intersection of PT and $P'T'$ as P' moves back to P is the center of curvature at P of the roulette traced by P, we have

THEOREM 1. *The locus of the centers of curvature of the elements described simultaneously by all the points of a line in the plane of the moving centrode, for an infinitesimal movement, is a conic tangent to the moving centrode and also to the fixed centrode at the instantaneous center of rotation.*

Bresse attributes this theorem to Rivals.[1] It was Mannheim, however, who

[1] See *Journal de l'École Polytechnique*, Cahier 35, p. 112.

first wrote out a proof.[1] He proves it by means of projective properties after using a formula of Savary to prove the equality of certain angles.

(2) In case the moving centrode is a conic and the tracing point P is on the conic, the pencils PT and $P'T'$ are again projective. Hence

THEOREM 2. *If the moving centrode is a conic, the locus of the centers of curvature of the elements described simultaneously by the points of the conic, for an infinitesimal movement, is a conic tangent to the rolling conic and to the fixed centrode at the instantaneous center of rotation.*

This proof of this theorem is given by Mannheim in the paper above referred to.

(3) In case the moving centrode is a circle and the tracing point P is on the circle, the angle between any two rays of the pencil PT equals the angle between the corresponding rays of the pencil $P'T'$, and we have

THEOREM 3. *If the moving centrode is a circle, the locus of the centers of curvature of the elements described simultaneously by the points of the circle, for an infinitesimal movement, is a* CIRCLE *tangent to the rolling circle and to the fixed centrode at the instantaneous center of rotation.*

So far as we know this theorem has not been stated before. It is the purpose of this paper to show its importance in the theory of roulettes.

II. In case both centrodes are circles (we shall note in (6) that they can always be so regarded) there exists a very simple relation between their radii and the maximum radius of curvature of the corresponding epi- or hypocycloid. We now derive this relation and show that from it and Theorem 3 there follows a simple proof of Savary's elegant construction of the center of curvature at any point of a roulette.

(1) THEOREM 4. *Let ρ be the radius of the fixed circle, r that of the rolling circle, and R the radius of curvature of the point P on the rolling circle where the diameter through the rolling point cuts the circle on the opposite side. Then for the epicycloid*

$$\frac{\rho}{\rho + 2r} = \frac{R - 2r}{2r},$$

and for the hypocycloid

$$\frac{\rho}{\rho - 2r} = \frac{R - 2r}{2r}.$$

To establish the first relation consider the arc δ traced by P as the moving circle rolls over an arc δ' of the fixed circle. First, the arc δ is the product of $2r$ and the angle of rotation, which is the sum of the angles subtended by δ' at the centers of the two circles, that is, $\delta'/r + \delta'/\rho$; second, the arc δ is the product of R and the angle between successive normals to the roulette, which is $\delta'/(R - 2r)$. This double expression gives

$$\frac{\rho}{\rho + 2r} = \frac{R - 2r}{2r}$$

[1] Mannheim, *Construction of Centers of Curvature, ibid.*, Cahier 37, p. 187.

For the hypocycloid ρ changes sign, and the relation becomes

$$\frac{\rho}{\rho - 2r} = \frac{R - 2r}{2r}.$$

(2) In view of Theorem 3 it follows, in either of the above cases, that the circle, which is the locus of the centers of curvature of the elements described simultaneously by the points of the rolling circle, for an infinitesimal movement, is of diameter $R - 2r$.

(3) Theorems 3 and 4 give the following construction for the center of curvature of the epicycloid or hypocycloid corresponding to any point.

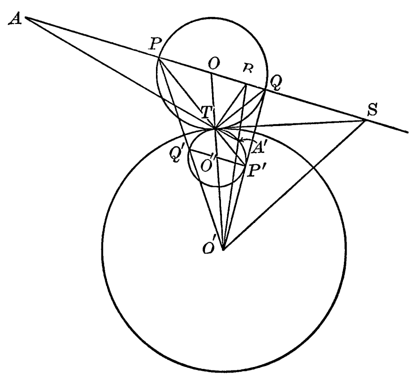


FIG. 1.

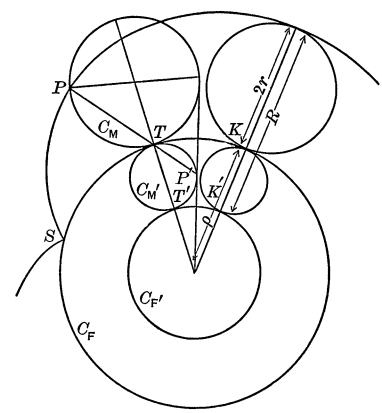


FIG. 2.

Consider the epicycloid. Let P be a point on the rolling circle (see Fig. 1). Draw PT, TQ perpendicular to PT, $O'Q$ and thus determine P'. Determine Q' similarly. Now P' is the center of curvature of the element at P of the epicycloid traced by P, and Q' is the center of curvature of the element at Q of the epicycloid traced simultaneously by Q. That is, P' and Q' are on the circle which is the locus of the centers of curvature of the elements described simultaneously by the points of the moving circle, for an infinitely small movement, and which by (2) above has for diameter $R - 2r$. For, from elementary geometry, O'' is the center of the circle described on $P'Q'$ as diameter. Also

$$\frac{P'Q'}{PQ} = \frac{\rho}{2r + \rho},$$

or,

$$P'Q' = \frac{2\rho r}{2r + \rho},$$

which by Theorem 4 is $R - 2r$; and this proves the construction.

That this construction holds for the hypocycloid may be proved similarly.

(4) We now prove that the center of curvature of the element of a roulette

described by any point A in the plane of the rolling circle is given by the following construction due to Savary. The accompanying figure shows the circles convex to each other. A proof similar to the following shows that the construction holds in the contrary case.

Draw AT, TB (see Fig. 1) perpendicular to AT, BO' and thus determine A', which we prove is the center of curvature of the element of the roulette described, by A for an infinitesimal movement. By Theorem 1 the centers of curvature of the elements of the roulettes traced by the points of the line AO, which we shall call l, lie on a conic touching the fixed and rolling circles at their point of contact T. By (3) above P' and Q' are on this conic; O' is evidently the center of curvature of every point of the roulette traced by O. The conic is therefore determined by P', Q', O' and the point of tangency T. Since the pencils TA and TB are congruent, the ranges A and B on l are projective. Then the pencils (1) and (1′) are projective and hence corresponding rays intersect on a conic. This is the conic already determined for TP and $O'Q$, TO and $O'S$, TQ and $O'P$, and the double rays TS and $O'T$ are corresponding rays of the pencils (1) and (1′).

The construction of the preceding paragraph is a special case of this.

(5) It follows from (4) that for any M and F, that is, for any definition of the movement of the points in the plane of M, the locus of the centers of curvature of the elements described simultaneously by the points on the line at infinity, for an infinitesimal movement, is a circle tangent to the fixed and moving centrodes at their point of contact, for in that case the corresponding pencils are congruent.

If M is a line tangent to the fixed centrode the locus of the centers of curvature of the elements described simultaneously by the points on the line at infinity is a circle having for diameter the radius of curvature of F at the instantaneous center of rotation.

This property is given by Mannheim in the paper already referred to.

(6) It is to be noticed that the above construction gives the center of curvature of the element described by any point in the plane of the moving centrode in any case in which one can construct the centers of curvature of the fixed and moving centrodes at their point of contact.

(7) As an application of Theorem 4 we shall prove the following well-known theorem.

THEOREM 5. *The evolute of any epicycloid is a similar epicycloid.*

By II (2) the centers of curvature of the elements described simultaneously by the points of the moving circle C_M of radius r, for an infinitesimal movement (see Fig. 2), lie on a circle $C_{M'}$ of radius $\frac{1}{2}(R - 2r)$, which by Theorem 4 may be written $\rho r/(2r + \rho)$. The ratio of the radius of $C_{M'}$ to that of C_M is therefore $\rho/(2r + \rho)$. The locus of the centers of curvature of the high points of C_M as it rolls is clearly a circle $C_{F'}$ concentric with the fixed circle C_F and is of radius $\rho - (R - 2r)$, which may be written $\rho^2/(2r + \rho)$. Then the ratio of the radius of $C_{F'}$ to that of C_F is $\rho/(2r + \rho)$. From elementary geometry we may write

$$\text{arc } TP' = \frac{\rho}{2r+\rho}\text{arc } PT = \text{arc } ST,$$

$$\text{arc } P'T' = \text{arc } TP'T' - \text{arc } TP' = \frac{\rho}{2r+\rho}\text{arc } SK^* - \frac{\rho}{2r+\rho}\text{arc } PT$$

$$= \frac{\rho}{2r+\rho}\text{arc } TK = \text{arc } T'K'.$$

This proves that as C_M rolls on C_F, $C_{M'}$ rolls on $C_{F'}$, that is, as P traces an epicycloid so does P'. Now the ratio of the radii of $C_{M'}$ and $C_{F'}$ is

$$\frac{\rho r}{2r+\rho} \div \frac{\rho^2}{2r+\rho} = \frac{r}{\rho},$$

which is the ratio of the radii of C_M and C_F. Therefore the epicycloid traced by P is similar to that traced by P'.

AN ELEMENTARY THEORY OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By EDWARD V. HUNTINGTON, Harvard University.

In most textbooks on the calculus, the proofs of the formulas for differentiating the logarithmic and exponential functions are either confessedly incomplete, or are made to depend on a preliminary study of the complicated function, $y = \lim_{x \doteq \infty} (1 + 1/x)^x$. This function represents one of the most difficult of the indeterminate forms, the study of which would seem more properly to come late in the course, instead of at the beginning. Moreover, the usual treatment passes over altogether too lightly the questions connected with the existence and meaning of the function a^x for irrational values of x—questions which the student can hardly be supposed to have solved satisfactorily in his previous course in algebra.

The present paper is an attempt to develop the theory of logarithms and exponents, including existence theorems and rules for differentiation, in a new way, which it is hoped will prove not only rigorous but teachable. The discussion is confined to the case of the real variable, and *no knowledge of algebra beyond positive integral exponents is pre-supposed.*

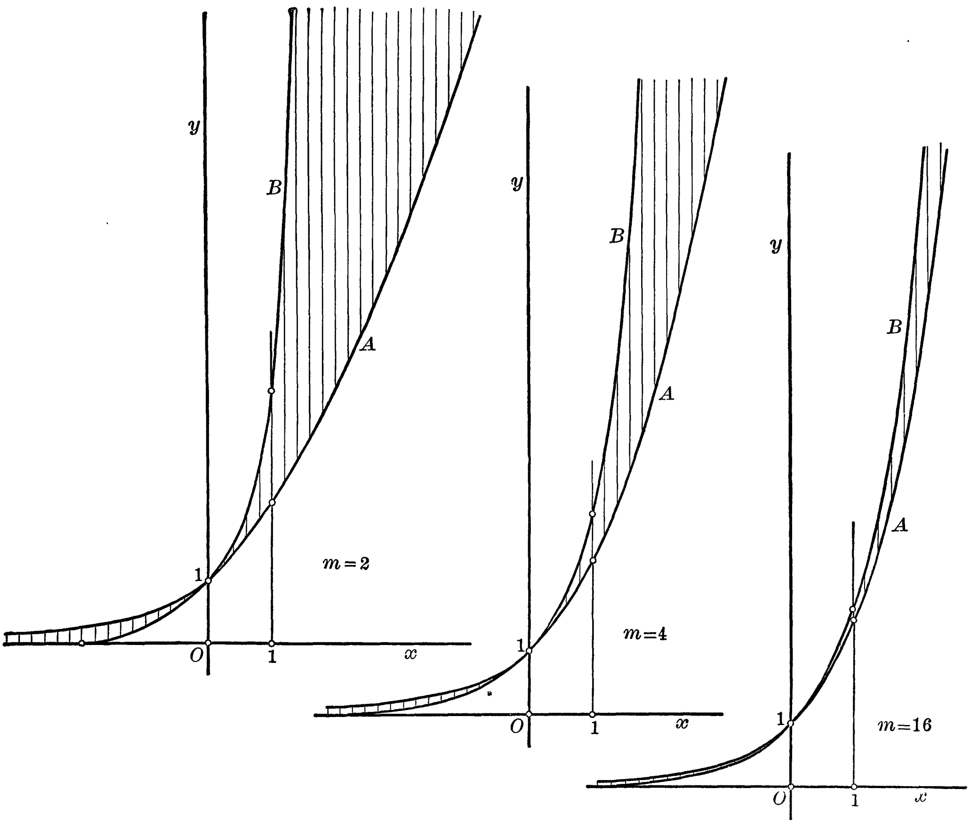
Definition of the Exponential Curve. Let us begin by supposing that in the process of plotting a variety of different curves, some one hit upon the idea of plotting the family of curves representing the following simple algebraic functions:

$$(A) \qquad (1 + x/2)^2,\ (1 + x/4)^4,\ (1 + x/16)^{16},\ \cdots (1 + x/m)^m,\ \cdots$$

$$(B) \qquad \frac{1}{(1 - x/2)^2},\ \frac{1}{(1 - x/4)^4},\ \frac{1}{(1 - x/16)^{16}},\ \cdots \frac{1}{(1 - x/m)^m},\ \cdots,$$

* Since SK is equal to a semicircumference of C_M.

where m runs through the sequence of numbers 2, 4, 16, 256, $\cdots$, each of which is the square of the preceding. It is to be understood, however, that only those values of x are considered which lie between $-m$ and $+m$. Under this restriction, the quantities in parentheses will always be positive. A few of these curves are shown in the accompanying figure.



Each of these functions has the value 1 when $x = 0$.

Fixing now our attention upon any given value of x (not zero), we proceed to prove the following facts, in corroboration of what is apparent from the figure.

(*a*) For any given value of x the "A" curves in the figure (beginning with the first value of m which is $> | x |$) form an ascending sequence, while

(*b*) the "B" curves form a descending sequence. Moreover,

(*c*) each "A" value is less than the corresponding "B" value, and

(*d*) the difference between corresponding values of the "B" and "A" curves approaches zero when m increases.

Hence, for a given value of x, the "A" sequence and the "B" sequence converge toward a common limit; and this limit is what we shall take as the definition

of the *exponential of* x, denoted by $\exp x$. The curve for $y = \exp x$ forms a boundary line between the "A" curves below it and the "B" curves above it.

Proofs of Statements (a)–(d). In order to prove the foregoing statements, (a)–(d), we first establish the following inequality:

$$(1 + d)^m > 1 + md, \tag{1}$$

where m is a positive integer and d is positive or lies between -1 and 0. The proof consists of two parts.

First, when d is positive, the inequality is obvious from the binomial theorem for positive integral exponents.

Secondly, when d lies between -1 and 0, let $b = 1 + d$ (so that $0 < b < 1$) and note that

$$1 - b^m = (1 + b + b^2 + \cdots + b^{m-1})(1 - b).$$

Hence, since $0 < b < 1$,

$$1 - b^m < (1 + 1 + 1 + \cdots + 1)(1 - b).$$

Therefore, $b^m > 1 - m(1 - b)$, or $(1 + d)^m > 1 + md$.

Having established this inequality, we now prove (a)–(d) as follows:

(a) By (1), $(1 + x/m^2)^m > 1 + x/m$ since $|x/m| < 1$ and therefore $|x/m^2| < 1$. Hence, $(1 + x/m^2)^{m^2} > (1 + x/m)^m$; that is, each of the "$A$" values is greater than the preceding.

(b) By (1), $(1 - x/m^2)^m > 1 - x/m$, since $|x/m| < 1$ and therefore $|x/m^2| < 1$. Hence, $(1 - x/m^2)^{m^2} > (1 - x/m)^m$ or $1/(1 - x/m^2)^{m^2} < 1/(1 - x/m)^m$; that is, each of the "$B$" values is less than the preceding.

(c) To prove (c), suppose that $(1 + x/m)^m$ were greater than $1/(1 - x/m)^m$; then we should have $(1 - x^2/m^2)^m$ greater than 1, which is impossible, since $x^2/m^2 < 1$. Therefore, $(1 + x/m)^m < 1/(1 - x/m)^m$.

(d) To prove (d), note that the difference in question is

$$\frac{1}{(1 - x/m)^m} - (1 + x/m)^m = \left[\frac{1}{(1 - x/m)^m}\right][1 - (1 - x^2/m^2)^m].$$

By (b), as soon as m is large enough to make $|x/m| < 1$ the first factor decreases. But by (1), as soon as m is large enough to make $x^2/m^2 < 1$, we have $(1 - x^2/m^2)^m > 1 - x^2/m$, so that from this point on, the second factor is less than x^2/m. Hence, the second factor approaches zero, while the first factor remains less than a finite quantity; therefore, their product approaches zero.

We have thus established the existence of the function $\exp x$ for every real value of x, positive, negative, or zero. In particular $\exp 0 = 1$, and $\exp 1 = 2.718 \cdots$. This latter value can be computed to any desired degree of approximation by evaluating either $(1 + 1/m)^m$, or $1/(1 - 1/m)^m$, for larger and larger values of m, and is here taken as the definition of e:

$$e = \exp 1 = 2.718 \cdots.$$

Fundamental Properties of the Exponential Function. From this definition of $\exp x$ we now proceed to deduce the fundamental theorem:

$$\exp(x + y) = (\exp x)(\exp y).$$

To prove this we show that the ratio $(\exp x)(\exp y)/\exp(x + y) = 1$, as follows:

First, using the "A" approximations, $(\exp x)(\exp y)/\exp(x + y)$ is the limit of

$$\frac{[1 + x/m]^m[1 + y/m]^m}{[1 + (x + y)/m]^m}$$

as m runs through its sequence of increasing values. This expression reduces to $\left[1 + \frac{1}{m}\cdot\frac{xy}{m + (x + y)}\right]^m$, which, by (1), is $> 1 + \frac{xy}{m + (x + y)}$ when m is sufficiently large; hence, the limit in question must be $\geqq 1$.

Secondly, using the "B" approximations, $(\exp x)(\exp y)/\exp(x + y)$ is the limit of

$$\frac{1/[1 - x/m]^m \cdot 1/[1 - y/m]^m}{1/[1 - (x + y)/m]^m}.$$

This expression reduces to $1 \div \left[1 + \frac{1}{m}\cdot\frac{xy}{m - (x + y)}\right]^m$, which, by (1), is $< 1 \div \left[1 + \frac{xy}{m - (x + y)}\right]$; hence, the limit in question must be $\leqq 1$.

Comparing these two results, we see that the limit must be precisely 1. That is, $(\exp x)(\exp y)/\exp(x + y) = 1$, or $\exp(x + y) = (\exp x)(\exp y)$.

From this fundamental addition theorem, the following theorems are easily derived:

$$\exp 0 = 1, \qquad \exp(-x) = \frac{1}{\exp x}, \qquad \exp(x - y) = \frac{\exp x}{\exp y};$$

also,

$$\exp n = e^n, \qquad \exp\frac{1}{n} = \sqrt[n]{e} = e^{1/n}, \qquad \exp\frac{m}{n} = \sqrt[n]{e^m} = e^{m/n},$$

where m and n are positive integers.

These last results show that whenever x is a rational number, $\exp x$ is precisely equal to what would be called in algebra e^x, that is, the xth power of $2.718\cdots$. On account of our familiarity with the exponential notation, it is usually convenient to write e^x instead of $\exp x$, even when x is not a rational number; we may therefore use the equation,

$$e^x = \exp x$$

as the definition of e^x in the general case.

In this notation, the fundamental theorem assumes the familiar form

$$e^{x+y} = e^x e^y.$$

The notation $\exp x$ may be retained, however, in cases where x is a long expression which cannot conveniently be printed as an exponent.

It is easily seen from the figure that e^x is > 1 when x is positive; hence, whenever $x > y$ (say $x = y + c$), we have $e^x = e^{y+c} = e^y e^c > e^y$, so that e^x is an increasing function.

Moreover, as x increases positively, e^x increases indefinitely, and as x increases negatively, e^x approaches zero.

Derivative of e^x. To prove the existence of the derivative of the function $y = e^x$ and to find its value at any point we form the difference-quotient in the usual way as follows:

$$\frac{\Delta y}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x} = \frac{e^x e^{\Delta x} - e^x}{\Delta x} = e^x \cdot \left[\frac{e^{\Delta x} - 1}{\Delta x}\right].$$

The first factor does not depend on Δx. To find the limit of the second factor, we proceed as follows: By (*a*) and (*b*), and (1), when Δx is small enough to make $-1 < \Delta x < 0$ or $0 < \Delta x < 1$, we have

$$1 + \Delta x < \left(1 + \frac{\Delta x}{m}\right)^m < e^{\Delta x} < 1 \div \left(1 - \frac{\Delta x}{m}\right)^m < \frac{1}{1 - \Delta x},$$

and hence after subtracting 1 and dividing through by Δx;

$$1 < (e^{\Delta x} - 1)/\Delta x < 1/(1 - \Delta x), \text{ if } \Delta x \text{ is positive,}$$

or

$$1 > (e^{\Delta x} - 1)/\Delta x > 1/(1 - \Delta x), \text{ if } \Delta x \text{ is negative.}$$

The left-hand member of this inequality is constantly 1, and the right-hand member approaches 1 as a limit, as Δx approaches zero through any sequence of values whatever; hence the limit of the middle member must also be 1; that is,

$$\lim_{\Delta x \doteq 0} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right] = 1.$$

Therefore the limit of $\Delta y/\Delta x$ is e^x; that is,

$$\frac{dy}{dx} = e^x, \qquad \text{or} \qquad d(e^x) = e^x dx.$$

The formula for differentiation is thus proved at one stroke for all real values of x, positive, negative, or zero.

Definition of Natural Logarithms. Having thus established the well-known shape of the exponential curve, and proved the existence of a tangent at every point, we can at once infer that for every positive quantity N there is a real quantity z such that $e^z = N$. This quantity is then, by definition, the natural logarithm of N:

$$z = \log_e N.$$

The familiar properties of the logarithm follow immediately from the definition, in the usual way.

Derivative of $\log_e x$. The curve for $y = \log_e x$ is the same as the curve for $y = e^x$, reflected in the 45° line of the first quadrant. Hence, the slope of the curve $y = \log_e x$ at any point $x = a$ is the reciprocal of the slope of the curve $y = e^x$ at the corresponding point, $y = a$. From this it follows immediately that

$$\left[\frac{d \log_e x}{dx}\right]_{x=a} = \frac{1}{a}.$$

Or, we may write $y = \log_e x$ in the form $e^y = x$, whence $e^y dy = dx$, or $dy = (1/e^y)dx$, or $dy = (1/x)dx$.

Definition of a^x (a positive). Finally, we define a^x (where a is positive) by means of the general equation

$$\log_e (a^x) = x \log_e a, \qquad \text{or} \qquad a^x = e^{x \log_e a}.$$

When x is a positive integer (or a positive or negative rational number) this definition of a^x reduces at once to the forms which are familiar from elementary algebra.

By means of this definition, all the usual properties of a^x follow immediately from the corresponding properties of e^x; in particular, the inverse of the function 10^x gives immediately the logarithm to the base 10 with all its properties. In terms of $\log_{10}$, the definition of a^x may be written in the form

$$\log_{10} (a^x) = x \log_{10} a, \qquad \text{or} \qquad a^x = 10^{x \log_{10} a};$$

which is the form most convenient for numerical computation.

Conclusion. The method of presentation here suggested will be found to be very much shorter and simpler than any of the older methods that give the complete results (see, for example, Chrystal's *Algebra*, or Stolz's *Allgemeine Arithmetik*), and practically as short as many of the methods of the current textbooks, which give the results only for the rational case.

ELEMENTARY PROOF OF A THEOREM DUE TO F. MORLEY.

By TOBIAS DANTZIG, Indiana University.

In a paper read before the Columbus meeting of the American Mathematical Society, December 30, 1915, Professor H. S. White mentioned a theorem due to Professor F. Morley, and first given by him in a memoir entitled: "On Reflexive Geometry."[1] The theorem follows:

If a ring of five circles be formed, the center of each upon a fixed circle and each

[1] *Transactions of the American Mathematical Society*, Vol. 8, 1907, pp. 23–24.

circle of the ring intersecting the next on this fixed circle, the five other intersections when joined in succession will form a pentacle whose vertices lie one upon each of the five circles.[1] See Fig. 2.

Professor Morley's proof of this theorem is based on considerations of synthetic geometry. It is hoped that the very simple elementary proof here given will be of interest; especially as a few other remarkable properties of the same configuration immediately follow from the method used.

1. Let us first recall a theorem of elementary geometry which may not be very well known:

If C_1, C_2 are centers of two circles intersecting in A and B (Fig. 1) *and if the*

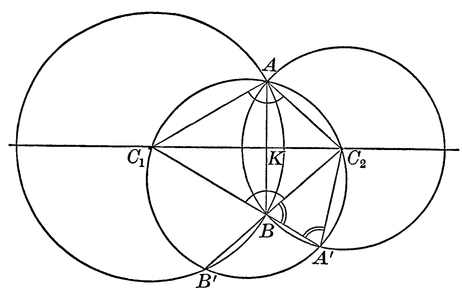


FIG. 1.

two lines C_1B and C_2B be drawn to meet the circles in A' and B', respectively, the five points A, A', B', C_1 and C_2 will lie on a circle.

For $\angle C_1AC_2 = \angle C_1BC_2$ and $\angle C_2BA' = \angle C_2A'B$; hence, we have $\angle C_1AC_2 + \angle C_2A'B = \pi$, and A' is concyclic with A, C_1 and C_2.

Conversely: *If a circle be drawn through the centers of two given circles, C_1 and C_2, and through one of their intersections say A, and if A', B' be the two other points where this circle meets the two given circles; then will the three points C_2, B, B' lie on a straight line.*

2. With the aid of this theorem Professor Morley's proposition is easily proved. Let $C_1, C_2, \cdots$ (Fig. 2) be the centers of the five circles in question all lying on the circumference O. The circles C intersect in five points $A_{12}, A_{23}, \cdots$ on the circumference O, and in five other points $B_{12}, B_{23}, \cdots$. The pentacle constructed by five joins of the points B has for vertices $M_1, M_2, \cdots$. We are to prove that the points M are on the circumferences C, say M_1 is on C_1, M_2 on C_2 and so on.

Draw C_1P and C_1Q parallel to the sides of the pentacle M_1M_4 and M_1M_3, respectively. Then

$$\angle QC_1B_{12} = \angle M_1B_{12}C_1 = \angle B_{23}B_{12}A_{23} = \angle B_{23}A_{12}A_{23},$$

and

$$\angle PC_1B_{51} = \angle M_1B_{51}C_1 = \angle B_{45}B_{51}A_{45} = \angle B_{45}A_{51}A_{45},$$

[1] This theorem can be considered as the converse of a proposition due to Auguste Miquel and generalized by Clifford. See A. Miquel. Mémoire de Géométrie, *Journal de Liouville,* Vol. X (1844), page 347. Also: Clifford, Collected Mathematical Papers, page 38.

using parallels and inscribed angles subtending equal arcs. Hence C_1P and C_1Q pass through C_4 and C_3, respectively.

Since arc $C_4C_3 = \frac{1}{2}$ arc $A_{45}A_{23}$ it follows that

$$\angle\, C_4C_1C_3 = \angle\, M_1 = \tfrac{1}{2}\,\angle\, B_{51}C_1B_{12}.$$

Now angle $B_{51}C_1B_{12}$ is central in the circle C_1 and angle M_1, subtending the same

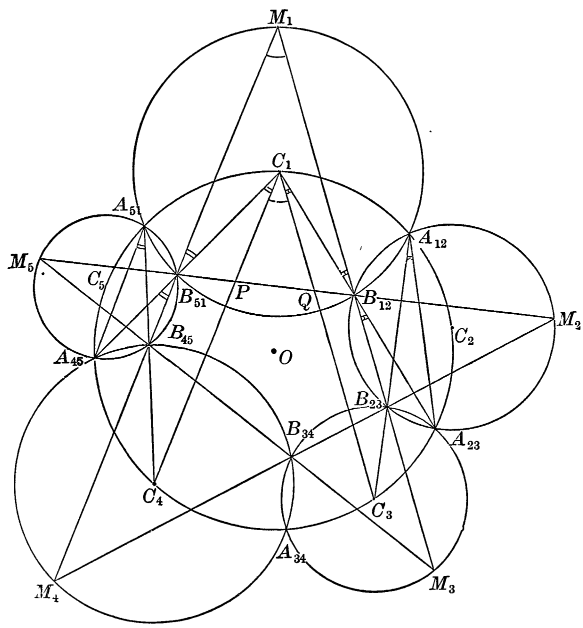


FIG. 2.

arc as the central angle and equal to one half of it, must be inscribed in the same circle. Hence the theorem.

3. By the method used the following results are readily derived:

(1) *The pentacle built on the five centers C has its sides respectively parallel to the pentacle M.*

(2) *The radical axes AB of the five circles C are respectively perpendicular to the sides of the pentacle M.*

(3) *If from any one of the points A we drop perpendiculars on the four adjacent sides of the pentacle M, the feet of the perpendiculars are on one straight line (the Simpson Line). The radical axis through A is perpendicular to the Simpson line. The five angles thus formed at A by the five perpendiculars are respectively equal to the five angles of the pentacle at M, and the Simpson lines are parallel to the sides of the pentacle M.*

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Theory of Errors and Least Squares. By LEROY D. WELD. New York: The Macmillan Company, 1916. 12mo, pp. xii + 190, with eleven figures in the text. $1.25 (weight 18 oz.).

This very useful and handy little text-book embodies the material used by the author as lecture notes during twelve years. It presents the theory of errors and least squares in such a simple and concise form as to be suitable as a text-book for a brief undergraduate course, or as a convenient reference which any research worker with a little preparation in mathematics can read in a few hours and put into immediate practice.

Though certain more or less standard developments in the method of least squares are omitted, such as Chauvenet's criterion and the elegant method by Gauss of successive substitutions for the solution of normal equations, with its useful system of notation and checks, the author covers quite adequately the essentials of the subject. In the eight chapters he discusses in order: measurement, errors, probabilities, the error equation and the principle of least squares, the adjustment of indirect observations, empirical formulas, weighted observations, and precision.

Of the 190 pages in the book about forty-five are given over to exercises; perhaps twenty more pages are devoted to well-chosen illustrations of the applications of the developments discussed; and fourteen pages are used for an appendix containing, in addition to a few of the more complicated mathematical discussions, a collection of definitions, theorems, rules and formulas for convenient reference. Only through experience could one judge whether such an extensive complement of auxiliary matter is desirable in a text and reference book in which brevity seems to have been a prominent object. However, a glance at the illustrative examples and problems will show that they are drawn from various branches of science to suggest a wide range of useful application and in a small measure to exemplify the variety of ways in which ideas relating to the theory of errors and the method of least squares adapt themselves to the daily needs of the scientist in many fields of investigation. Much of the appendix also is designed to provide a convenient means of reference for the scientist who desires to apply least squares.

Although it may cause the scientist no serious inconvenience to find non-uniformity of notation in different textbooks of least squares in reference to standard quantities such as residuals, weights and probable errors, it would be an advantage if such notation could be standardized universally enough for use in headings and margins of tabulations where brevity combined with general intelligibility is demanded. Whereas Professor Weld employs the symbols ϵ and w to denote the much used quantities, probable error and weight, in eight well known textbooks which I find at hand discussing least squares in English

these quantities are designated by r and p respectively. Probably the chance of interesting investigators in the use of the method of least squares would be increased if confusion due to non-uniformity of notation could be avoided.

Every intelligent observer desires some concrete expression of the quality of his observations; the computer who has to combine the results of different series of observations should have some knowledge of their relative accuracy in order to assign to each series its proper weight; and the investigator engaged in a complicated series of experiments desires some criterion by which to estimate the relative errors of the several parts of his work and to apportion properly his care among them. Experienced judgment will go a great way, but the working method of least squares, which has been developed on the basis of experience and analysis into its present form by a succession of thinkers, presents clearly the safest means of obtaining the result of highest probability from a given set of observations and provides methods for appraising the accuracy of such a result and for expressing this accuracy vividly to others. The present volume is planned with the purpose of making the elements of the theory of errors and least squares easily attainable both by students and by research workers. It will prove useful in the class room, in reference libraries and also on the desk.

R. H. CURTISS.

Plane Trigonometry, with Tables. By A. M. HARDING and J. S. TURNER. G. P. Putnam's Sons, New York, 1915. 158 + 51 pages. $1.10 net.

It would be decidedly interesting to know how many trigonometries have been written since Nasir Eddin (or whoever it was) delivered the subject from its bondage to astronomy. It would be much more interesting to know how many authors have, more or less unconsciously, recast the material in the mould of an older form.

At present, in the writing of American trigonometries, we seem to be passing through a period which may, with some appropriateness, be called a "reversion to type." The trigonometry under review, like the Kenyon and Ingold (published 1913) and the Wilczynski and Slaught (published 1914), makes the discussion and solution of triangles the first consideration and admits no diversion until this problem has been completely solved. In the preface we find:

"During the last few years great modifications have been made in the method of presenting plane trigonometry. Formerly the student was introduced to the trigonometric functions without any explanation of their practical utility, and spent three-fourths of his time groping in the dark with trigonometric formulas and identities. The result was that the average student found the subject repulsive.

"In accordance with the modern tendency we have introduced the ratios a few at a time, and then proceeded as soon as possible to the solution of triangles, leaving the more difficult theoretical parts of the subject until the last."

And yet the Davies' *Legendre,* almost universally used in this country for a considerable period more than fifty years ago, made the solution of triangles

the first consideration and, after having completely disposed of that, took up the general properties and relations of the trigonometric functions (functions of any angle, fundamental relations, addition and factoring formulas, etc.) under the title "Analytical Trigonometry." The same order is to be found in Loomis's *Trigonometry* (copyright 1858) of which more than 60,000 copies were printed before the first revised edition was issued.

The comprehensive and well-written textbook of Chauvenet (copyright 1850) placed the treatment of the general properties and relations of the trigonometric functions before any solutions of triangles. This arrangement was adopted by the popular textbooks of Olney (copyright 1870), Wheeler (copyright 1876), Oliver, Wait and Jones (copyright 1881) and Wells (copyright 1883).

Wentworth's trigonometry appeared in 1882 with an order of treatment that seemed to be a kind of compromise between that of Davies' *Legendre* and Chauvenet. The solution of right triangles was placed near the beginning; between that and the solution of oblique triangles was inserted, under the caption "Goniometry," a discussion of the general properties and relations of the trigonometric functions whose content was slightly greater than the "Analytic Trigonometry" which followed the solution of triangles in the Davies' *Legendre* text, and nearly equivalent to three of the chapters (II, III and IV) which preceded the solution of any triangles in the Chauvenet. Simon Newcomb's trigonometry appeared about the same time with a somewhat similar arrangement, but with more of the analytic material preceding the chapter on right triangles.

Wentworth's trigonometry may not have been the first of its type but its great success undoubtedly accounts for the fact that one can readily name ten or fifteen trigonometries published since 1890 which have adopted this hybrid order of placing the "analytical trigonometry" or "trigonometric analysis" between the discussion of right triangles and the discussion of oblique triangles. The tendency at present, of which the book under review is representative, seems to be to put the treatment of oblique triangles back to its original position.

The introductory chapter of the book under review is distinctive. At least, the reviewer knows of no other trigonometry beginning with a similar chapter intended to serve as a connecting link between plane geometry and plane trigonometry. This chapter consists of 13 pages devoted to the measurement of lines, areas and angles (in the sexagesimal system only), the proofs of five theorems on similar triangles, and 19 exercises. At least a part of this material would not be new to a college freshman, but the authors seem to take no cognizance of this fact. The reviewer wonders if students reading the chapter might not be puzzled to understand why proofs are given of some of the theorems they have already proved in elementary geometry and why, at the same time, the validity of other theorems is assumed in making these proofs.

A distinctive feature of Chapter II is that the trigonometric functions are introduced two at a time. After defining the sine and cosine of an angle, several exercises are introduced to fix these concepts before introducing the tangent and

cotangent. Similar sets of exercises are given after each of the other two pairs. The second chapter also includes eight of the fundamental relations of the functions, the functions of the special angles 30°, 45°, 60°, and of complementary angles, and an explanation of the use of a table of natural functions. The explanation of the table is well written and the reviewer believes its introduction at this place is a better plan than to put it back with the tables.

Chapter III treats of right-angled triangles and the experience of many teachers attests that the authors are wise in making no use of logarithms throughout the chapter.

The next five chapters (IV to VIII inclusive) treat of logarithms, oblique triangles, trigonometric functions of any angle, trigonometric functions of two or more angles, circular measure, inverse functions and trigonometric equations, in the order named. While there are some slight changes in these chapters which may be considered improvements, the reviewer has not discovered anything startlingly different from other customary treatments.

The closing chapter (IX) treats of some applications of trigonometry to geometry. This chapter again suggests the older (as well as various English) trigonometries, since the indebtedness to Chauvenet and Loney, acknowledged in the preface, is here most in evidence. Such a chapter, exhibiting the value and power of trigonometry in dealing with old forms, will probably contribute to increased interest on the part of the student. About 600 exercises are included and the answers to part of them are collected after Chapter IX.

The mechanical make-up of the book leaves much to be desired. The figures (cf. Figs. 45, 53) and pages of formulas (cf. pp. 98–100) suffer somewhat in comparison with others which readily come to mind. The use of disproportionately small exponents (cf. pp. 112–113), roman θ and italic ϕ in the same formula (cf. pp. 108, 122) and of a clumsy sign for $\mp$ (cf. pp. 69, 114) suggests lack of the best equipment for mathematical printing.

While one may not agree entirely with the selection or order of subject-matter, one feels immediately that this book was written by men who know something of the processes through which a freshman's mind passes in studying trigonometry. One feels confidence that the book is usable—that it "will teach," as it is sometimes expressed.

A notable omission, the absence of any use of coördinates, also intimates a turning toward older methods. But, after all, the biologists assure us that "reversion to type" is a true test of species.

U. G. Mitchell.

University of Kansas.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

463. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the sum of the series

$$\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n},$$

where $\binom{n}{r}$ denotes the coefficient of x^r in the expansion of $(1 + x)^n$.

464. Proposed by GEORGE Y. SOSNOW, Newark, New Jersey.

Find the general term and the sum of n terms of the series 1, 4, 15, 56, $\cdots$, where

$$U_n = 4U_{n-1} - U_{n-2}.$$

GEOMETRY.

495. Proposed by N. P. PANDYA, Sojitra, India.

A point P moves so that the quadrilateral $PBCD$ is half of a given quadrilateral $ABCD$. Find the locus of P.

496. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find all the lines such that the pairs of tangent planes to a given sphere (ellipsoid) passing through them, shall be orthogonal.

CALCULUS.

413. Proposed by OSCAR S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.

Determine a function of x independent of b such that $\int_b^{b+1} f(x)dx = \frac{1}{b+1}$, the real part of b being positive.

414. Proposed by C. N. SCHMALL, New York City.

Among spherical triangles having the same base and equal altitudes, show that the isosceles triangle has the greatest vertical angle.

Show that this is also true for plane triangles.

MECHANICS.

330. Proposed by PAUL CAPRON, U. S. Naval Academy.

A barker's mill operates under a head of h feet; the linear speed of the orifice is u feet per second; the speed of the water relative to the orifice is v feet per second; and the coefficient of discharge is c, so that $v^2 = c^2(2gh + u^2)$. Given that the work done by the water on the mill is $u(v - u)/g$ foot-pounds per second per pound of water used, find the values of u and v such that the water-power may be most economically used, and find what part of the power is so used.

331. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A cyclist is riding due west at a speed of 12 miles per hour, and the wind is at the time blowing from the southeast with a speed of $5\frac{1}{2}$ miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist need to ride if the flag is to fly due north?

NUMBER THEORY.

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number if $2^n - 1$ is a prime number. Prove the formula.

250. Proposed by JOSEPH E. ROWE, State College, Pa.

Show by comparatively elementary means that the equation $x^{2n} + y^{2n} = z^{2n}$ is impossible of solution in positive integers x, y, z, and n, unless at least one of the integers $x, y, z \equiv 0 \pmod{3.}$. In particular, consider the case $n = 1$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation, $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

SOLUTION BY J. A. BULLARD, Worcester, Mass.

Let

$$A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \equiv (ax^2 + 2bx + c)(a'x^2 + 2b'x + c') = 0.$$

Then

$$a_0 = a'a, \quad 2a_1 = a'b + ab', \quad 6a_2 = a'c + 4b'b + ac', \quad 2a_3 = b'c + bc' \quad \text{and} \quad a_4 = c'c;$$

whence,

$$a_0a_4 = a'ac'c,$$
$$-4a_1a_3 = -ab'^2c - a'b^2c' - a'b'bc - abb'c',$$
$$3a_2^2 = \tfrac{1}{12}(a'c)^2 + \tfrac{4}{3}(b'b)^2 + \tfrac{1}{12}(ac')^2 + \tfrac{2}{3}a'b'bc + \tfrac{2}{3}abb'c' + \tfrac{1}{6}aa'cc'.$$

Adding these equations and simplifying, we have

$$\text{(I)} \qquad B \equiv (b^2 - ac)(b'^2 - a'c') + \tfrac{1}{3}[bb' - \tfrac{1}{2}(a'c + ac')]^2.$$

If the original coefficients are real numbers then a, b, c, a', b', c' can always be taken so as to have real values. If $B = 0$ and the roots are all distinct the product of the discriminants $b^2 - ac$ and $b'^2 - a'c'$ is negative and hence one discriminant is negative and the other positive. Thus two of the four roots of $A = 0$ are the real and unequal roots of one quadratic and the other two roots are the imaginary roots of the second quadratic.

If $B = 0$ and two roots are equal let us assume them to be the roots of $ax^2 + 2bx + c = 0$, that is, $b^2 - ac = 0$. Then from (I) it follows that $bb' - \frac{1}{2}(a'c + ac') = 0$. Solving the quadratics and substituting from the relations just stated, we find the roots to be $-\dfrac{b}{a}$, $-\dfrac{b}{a}$, $-\dfrac{b}{a}$, $-\dfrac{bc'}{a'c}$. Thus, if $B = 0$ and two roots are equal, three roots must be equal.

A further examination of (I) shows that

If $B < 0$, two roots are imaginary and two are real and unequal.

If $B = 0$, two roots are imaginary and two are real and unequal, or three are equal, or all are equal.

If $B > 0$, two are imaginary and two are equal, or all are imaginary, or all are real. In this case we may have two double roots.

Also solved by GEORGE W. HARTWELL and the PROPOSER.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

SOLUTION BY W. E. MILNE, Bowdoin College.

Applying successively the formula $\sin 2\theta = 2\cos\theta\sin\theta$ we obtain

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}, \quad = 4\cos\frac{x}{2}\cos\frac{x}{4}\sin\frac{x}{4}, \quad = 8\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\sin\frac{x}{8}, \quad \cdots,$$

and in general

$$(1) \qquad \frac{\sin x}{2^n \sin (x/2^n)} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}.$$

Now $\lim_{n=\infty} 2^n \sin(x/2^n) = x$. Since equation (1) holds for every positive integral value of n, and the left-hand side approaches a limit $\frac{\sin x}{x}$ as n becomes infinite, it follows that the right-hand side also approaches a limit, and the limits are equal. Therefore,

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

Also solved by E. B. WILSON, PAUL CAPRON, HORACE OLSON, C. A. NICKLE, A. H. HOLMES, H. S. BROWN, E. H. VANCE, NORMAN ANNING, J. A. BULLARD, J. J. GINSBURG, C. K. ROBBINS, H. L. AGARD, H. R. HOWARD, O. S. ADAMS, I. A. BENNETT, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

479. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the locus of the point whose polars (polar planes) with respect to two given conics (quadrics), are perpendicular to each other.

SOLUTION BY THE PROPOSER.

A. POLARS WITH RESPECT TO CONICS.

If P is a point of the required locus, its polars p_1, p_2 with respect to the two given conics (Σ_1), (Σ_2) meet the line at infinity in two points Q_1, Q_2 separated harmonically by the cyclical points $\mathfrak{J}$, $\mathfrak{J}'$ at infinity. The pole, with respect to (Σ_1), of any line passing through Q_1 lies on the polar q_1 of Q_1 with respect to (Σ_1); and the pole, with respect to (Σ_2), of any line passing through Q_2 lies on the polar q_2 of Q_2 with respect to (Σ_2). When Q_1 varies on the line at infinity, q_1 turns about the center C_1 of (Σ_1), the range (Q_1) and the pencil (q_1) being projective. Similarly for the range (Q_2) and the pencil (q_2). The points Q_1, Q_2 being a couple of conjugate elements in the involution whose double elements are the cyclical points $\mathfrak{J}$, $\mathfrak{J}'$, we have

$$(q_1\cdots) \barwedge (Q_1\cdots) \barwedge (Q_2\cdots) \barwedge (q_2\cdots)$$

Hence: *The locus of the point, whose polars, with respect to two given conics, are perpendicular to each other, is, in general, a conic passing through the centers of the given curves.*

I. Two central conics. If (Σ_1) and (Σ_2) are both central conics, let (C) be the circle having for diameter the line C_1C_2 of their centers. The couples of conjugate diameters of (Σ_1) determine an involution (I_1) of points on (C). Let O_1 denote the pole of this involution. The conic (Σ_2) gives rise to a similar involution (I_2) with O_2 as its pole. Let M be any point of (C). If M_1, M_2 are the two other points of intersection of (C) with the lines O_1M, O_2M, respectively, the point of intersection P of the diameters C_1M_1, C_2M_2 will belong to the required locus (Σ), because the polars of P with respect to (Σ_1) and (Σ_2) are parallel to the diameters C_1M and C_2M, respectively conjugate to C_1P, C_2P in the two given conics, and the two former are perpendicular to each other. This affords an expeditious way for the construction of (Σ).

The points of intersection N, N' of the circle (C) with the line O_1O_2 belong to the required locus (Σ), as is readily seen by making the point M coincide with each one of them in turn. These two points will necessarily be real, if at least one of the points O_1, O_2 lies inside of (C).

The double elements of (I_1) on (C) are the points of contact of the tangents to (C) drawn from O_1. The lines joining C_1 to these points of contact are the double elements of the involution of

conjugate diameters of (Σ_1), *i. e.*, the asymptotes of (Σ_1). Hence O_1 will lie outside of (C) if (Σ_1) is an hyperbola, and inside of (C) if it is an ellipse or a circle. Similarly for O_2.

The conic (Σ) will have a real point at infinity if the lines C_1M_1, C_2M_2 are parallel. The line C_1C_2 being a diameter of (C), the quadrilateral $C_1M_1C_2M_2$ inscribed in (C) is a rectangle, and M_1M_2 is a diameter, hence $\angle M_1MM_2 = \angle O_1MO_2 = 90°$. The point M is therefore common to (C) and to the circle having O_1O_2 for diameter. The points of intersection of these two circles are necessarily real and distinct when one of the points O_1, O_2 is inside and the other outside of (C). They may be real, imaginary, or coincident, when O_1, O_2 are both inside, or both outside of (C). Consequently: *The conic (Σ) is, in general, an hyperbola, if one of the given conics is an hyperbola and the other an ellipse. Otherwise (Σ) may be an ellipse, an hyperbola, or a parabola.*

Since the conic (Σ) passes through the four points C_1, C_2, N, N' of (C), the conic can have no other point in common with (C) and remain distinct from this circle. Therefore, if (Σ) is to be a circle distinct from (C) this is only possible, if the points of intersection of O_1O_2 with (C) are the cyclical points at infinity, *i. e.*, O_1O_2 must itself be the line at infinity, and hence the points O_1, O_2 are points at infinity. But the tangents to (C) drawn from O_1 at infinity touch (C) at two diametrically opposite points, the asymptotes of (Σ_1) are therefore real and orthogonal. Similarly for (Σ_2). Hence: *The locus of the point whose polars, with respect to two given equilateral hyperbolas, are perpendicular to each other, is, in general, a circle distinct from the circle having for diameter the line of centers of the given conics.*

The reader may prove the following statements: If an asymptote of (Σ_1) is perpendicular to an asymptote of (Σ_2), the conic (Σ) is tangent to (C) at the point of intersection X of these lines. If X coincides, say, with C_1, (C) is the osculating circle of (Σ) at this point. If C_1C_2 is an axis for each of the given conics, (Σ) has a double contact with (C) at C_1, C_2.

If O_1C_2, O_2C_1 meet on (C), to the line C_1C_2 of the pencil (q_1) at C_1 will correspond the line C_2C_1 of the pencil (q_2) at C_2, so that the line C_1C_2 will be a part of the locus (Σ), the other part being the line O_1O_2. Hence: *If the diameters conjugate in the two given conics, respectively, to the line of their centers, are perpendicular to each other, the locus degenerates into two straight lines, one of them being the line of their centers.*

The conic (Σ) will be identical with (C), if for any point M of (C) the points M_1, M_2 coincide; but then the lines O_1MM_1 and O_2MM_2 will coincide, which makes it in turn necessary for O_1, O_2 to coincide, say in O. It is evident that this last condition is also sufficient for the coincidence of (Σ) with (C). If the involutions (I_1) and (I_2) have the same pole O, the diameters C_1M, C_1M_1 of (Σ_1) which are perpendicular to any couple of conjugate diameters C_2M, C_2M_2 of (Σ_2), are themselves conjugate in (Σ_1). Consequently: *If the couples of conjugate diameters of one of the given conics are the perpendiculars dropped from its center upon the couples of conjugate diameters of the other conic, the required locus is the circle (C).* This will take place, for instance, when the two given conics are both circles, or hyperbolas with mutually perpendicular asymptotes.

If the given conics are concentric, the two projective pencils (q_1), (q_2) are superposed, and their double elements constitute the required locus. These lines may conveniently be constructed by drawing any circle (C) through the common center C of the given conics and determining the points O_1 and O_2 as above; the lines projecting from C the points of intersection of (C) with the circle having O_1O_2 for its diameter, are the required lines. They are always real when one of the given conics is an ellipse (a circle) and the other an hyperbola. When the points O_1, O_2 coincide, *i. e.*, when the given conics have the same involution of conjugate diameters, the required locus has no real points.

N. B. In the above discussion of the central conics (Σ_1), (Σ_2), only their involutions of conjugate diameters were considered. The locus (Σ) would therefore not change, if (Σ_1) would be replaced by any other conic of the pencil having the same involution of conjugate diameters as (Σ_1). Similarly for (Σ_2). For instance, an hyperbola may be replaced by any hyperbola having the same asymptotes: a circle, by any other concentric circle.

II. A parabola and a central conic. The locus (Σ) will pass through the center C_1 of (Σ_1) and through the point at infinity of the parabola (Σ_2). If q_1, q_1' are any two conjugate diameters of (Σ_1), t_2 the tangent to (Σ_2) perpendicular to q_1', the diameter q_2 of (Σ_2) conjugate to t_2 will meet q_1 in a point of (Σ). If d_1, d_1' are the two conjugate diameters of (Σ_1), of which one, say d_1', is perpendicular to the axis a_2 of (Σ_2), the point at infinity of d_1 belongs to (Σ), as may be readily seen. Hence: *The required locus is, in general, an hyperbola passing through C_1, whose asymptotic directions are a_2 and d_1.*

The two asymptotic directions of (Σ) will be perpendicular to each other, when and only when the conjugate diameter d_1 of d_1' coincides with d_1', *i. e.*, when d_1' is an asymptote of (Σ_1). Hence:

The locus will be an equilateral hyperbola, if the axis of the parabola is perpendicular to an asymptote of the other conic.

The two points at infinity of the locus (Σ) will coincide, if d_1 is parallel to a_2, *i. e.*, if d_1' is an axis of (Σ_1). Hence: *The required locus will be a parabola, if the axis of the given parabola is parallel to one of the axes of the other given conic.*

If the central conic is a circle, the locus is always a parabola.

In the construction above, if for the diameter q_2 is taken the one n_2 passing through C_1, the corresponding diameter n_1 of (Σ_1) is the tangent to (Σ) at this point. The diameter n_1 will coincide with n_2, if the conjugate directions of n_2 in (Σ_1) and (Σ_2), respectively, are perpendicular to each other. The line n_2 is then a part of (Σ). Hence: *If the directions conjugate in the two given conics to the line of their centers, are perpendicular to each other, the locus degenerates into two straight lines, one of which is their line of centers.*

The reader may discuss the case when the axis of the parabola passes through the center of the other given conic.

III. Two parabolas. The required locus passes through the points at infinity of the given parabolas (Σ_1), (Σ_2), the conic (Σ) is therefore always an hyperbola. Draw the tangent to (Σ_2) which is perpendicular to the axis of (Σ_1). The diameter of (Σ_1) which passes through the point of contact, is one of the asymptotes of (Σ). The other asymptote is found in the same way starting with (Σ_1). The hyperbola (Σ) will be equilateral, if the axes of the parabolas are perpendicular. If these axes are parallel, their common point at infinity is the center of the two superposed projective pencils (q_1), (q_2), whose double elements constitute the required locus.

B. Polar Planes with respect to Quadrics.

If P is a point of the required locus, its polar planes π_1, π_2 with respect to the two given quadrics (Σ_1), (Σ_2), meet the plane at infinity in two lines p_1, p_2, which are conjugate with respect to the imaginary circle at infinity (γ), *i. e.*, p_2 passes through the pole M of p_1 with respect to (γ). The poles, with respect to (Σ_1) of all the planes passing through p_1, are on the line m_1 which is the conjugate to p_1 with respect to (Σ_1); the poles, with respect to (Σ_2), of all the planes passing through M, are in the polar plane μ_2 of M with respect to (Σ_2). The point P is therefore the point of intersection of the line m_1 with the plane μ_2. If p varies in the plane at infinity, m_1 describes the bundle of lines (m_1) having for its center the center of (Σ_1), and the two forms (p_1), (m_1) are reciprocal; if the point M varies in the plane at infinity, μ_2 describes a bundle of planes (μ_{21}), having for its center the center of (Σ_2), and the two forms (M) and (μ_2) are reciprocal. The two plane forms (M), (p_1) being reciprocal with respect to (γ), the bundle of lines (m_1) is reciprocal to the bundle of planes (μ_2). Hence: *The locus of the point whose polar planes, with respect to two given quadrics, are perpendicular to each other, is, in general, a quadric passing through the centers of the given surfaces.*

Also solved analytically by Florence P. Lewis.

480. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Of equal quadrilaterals on the same base, that which has the least perimeter must have the angles not adjacent to the base equal to each other.

Solution by Elijah Swift, University of Vermont.

We shall first prove a different theorem and then show it is equivalent to the above. "Of isoperimetric quadrilaterals on the same base, that which has the greatest area must have the angles not adjacent to the base equal to each other." Call the vertices $ABCD$, the given base being AD. The maximum quadrilateral must have $AB = BC = CD$, since, if the triangle ABC is a maximum, it is isosceles. Again, it must be inscriptible in a circle and hence is an isosceles trapezoid with angle B equal to angle C.

Now suppose we could find another quadrilateral, $AB'C'D$, having the same area as $ABCD$, but a smaller perimeter. It is clear that we can then construct a third quadrilateral, *e. g.*, by moving the vertex B' to B'', so that $AB''C'D$ has a larger area than $AB'C'D$ or $ABCD$, but the same perimeter as $ABCD$. But this is impossible. Also, there can be no other quadrilateral having the same area and perimeter. Hence, $ABCD$ has the least perimeter, for the given area, as well as the most area for a given perimeter.

CALCULUS.

397. Proposed by C. N. SCHMALL, New York City.

On the radii vectores of one loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ as diameters, circles are described passing through the pole. Find the locus of their points of intersection, and show that the area is twice that of the loop.

SOLUTION BY THEODORE HOWARD, New Haven, Conn.

Let ϕ be the angle between the polar axis and the radius vector of the circle described on the radius vector of the lemniscate as a diameter. Let r be the radius vector of the circle, the origin and polar axis being the same as for the lemniscate. Then the equation of the circle is $r = \rho \cos (\phi - \theta)$. Or $r^2 = \rho^2 \cos^2 (\phi - \theta) = a^2 \cos 2\theta \cos^2 (\phi - \theta)$, substituting from the equation of the lemniscate. In this equation, θ is the variable parameter. Taking the derivative of this equation with respect to θ, we have

$$0 = 2a^2 \cos (\phi - \theta)[\sin (\phi - \theta) \cos 2\theta - \sin 2\theta \cos (\phi - \theta)],$$
$$= 2a^2 \cos (\phi - \theta) \sin (\phi - 3\theta).$$

Hence, $\theta = \dfrac{\phi}{3}$, or $\theta = \phi - \dfrac{\pi}{2}$ an extraneous value. Substituting $\dfrac{\phi}{3}$ for θ in the preceding equation, we have $r^2 = a^2 \cos^3 \dfrac{2\phi}{3}$, the equation of the envelope. The area of one loop of the envelope is $A = \dfrac{a^2}{2} \int_b^a \cos^3 \dfrac{2\phi}{3} d\phi = 3a^2 \int_0^{\pi/4} \cos^3 2\theta d\theta = a^2$. The area of one loop of the lemniscate is

$$A' = a^2 \int_0^{\pi/4} \cos 2\theta d\theta = \frac{a^2}{2}.$$

Hence,

$$A = 2A'.$$

Also solved by C. E. HORNE, PAUL CAPRON, H. L. AGARD, and NORMAN ANNING.

398. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$$

SOLUTION BY W. W. BEMAN, University of Michigan.

Assume $z = \phi(y + mx)$. The equation in m is $2m^2 - 3m - 2 = 0$. Hence,

$$z = \phi(y + 2x) + \psi(2y - x).$$

Also solved by ELIJAH SWIFT, J. L. RILEY, and the PROPOSER.

399. Proposed by B. J. BROWN, Victor, Colorado.

A cow is tethered by a perfectly smooth rope, a slip noose in the rope being thrown over a large square post. If the cow pulls the rope taut in the direction shown in the figure, at what angle will the rope leave the post?

From Granville's *Diff. and Int. Calculus*, p. 120, Prob. 55.

SOLUTION BY H. L. AGARD, Williams College.

The rope leaves the post in such a manner that the knot at the end of the noose is in front of the middle of one face of the post. Let a be the thickness of the post, φ the angle at which the rope leaves the post, b the distance from the knot to the corner of the post, and c the perpendicular distance from the knot to the post.

Then

$$b = \frac{a}{2} \sec \varphi \qquad \text{and} \qquad c = \frac{a}{2} \tan \varphi.$$

If l is the total length of the rope and x is the distance of the cow from the post,

$$x = l - 3a - 2b + c = l - 3a - a \sec \varphi + \frac{a}{2} \tan \varphi. \tag{1}$$

Taking the derivative of x with respect to φ and setting the result equal to 0, we have,

$$\frac{dx}{a\varphi} = -a \sec \varphi \cdot \tan \varphi + \frac{a}{2} \sec^2 \varphi = 0.$$

Hence, $\sec \varphi = 0$ and $\sec \varphi - 2 \tan \varphi = 0$. Whence, φ from the last equation $= \pi/6$. This value of ϕ in $d^2x/d\varphi^2$ gives a negative value. Hence, for $\varphi = \pi/6$, x is a maximum.

Also solved by J. A. BULLARD.

NOTE. It seems to us, that the statement of this problem is misleading. As stated, it suggests a problem in mechanics and as such it is not a problem of maxima and minima except incidentally.

When the rope is taut, the tension in the three parts of the rope about the knot are equal and the knot (better a smooth ring) is then in stable equilibrium. From this fact, it follows that the rope leaves the post at an angle of 30°, and it no more involves the idea of maxima and minima than does the fact that a cube of homogeneous density has its center of gravity in the lowest position when a face of the cube is in contact with a horizontal plane.

However, it turns out in this problem that if the length of the rope is greater than $\frac{1}{3}(9 + 2\sqrt{3})$ times the length of a side of a right section of the post, the cow will be at a maximum distance from the side of the post, when the rope is taut.

To establish the reason why these two facts should coexist transcends the ability of the average student of elementary calculus.

The problem may be stated in definite form as follows: An inextensible weightless string of length $l > na$, where n is an integer greater than unity, is to be cut into two parts. The ends of one part of the string are fastened to the points A and B located in the same horizontal line, the distance between them being a. To one end of the other part of the string is attached a small weight while the other end of this part of the string is attached to the middle point of the suspended part, and the parts and weight are then allowed to hang freely. What angle will the string make with line AB when it is so cut that the weight is at a maximum distance from the line AB?

It may be argued in defense of the problem as given, that it leaves the student to discover for himself the particular quantity that becomes a maximum, instead of calling his attention to that quantity, as the proposed modification does, which is an argument worthy of consideration. But the statement of this problem leads the student by suggestion away from the discovery of the quantity which is to be made a maximum rather than directs him towards such discovery, which seems to us to be objectionable. EDITOR FINKEL.

MECHANICS.

314. Proposed by C. N. SCHMALL, New York City.

A rectangular box of height h, and having a plane mirror for its bottom, contains a quantity of water of unknown height x. In the lid are two small apertures distant $2a$ from each other. A ray of light entering one aperture with an angle of incidence φ, emerges, after refraction and reflection, through the other aperture. If μ be the index of refraction of water, show that the height of the water is

$$x = \frac{h \tan \varphi - a}{\tan \varphi - \dfrac{\sin \varphi}{(\mu^2 - \sin^2 \varphi)^{1/2}}}.$$

SOLUTION BY FRANK IRWIN, University of California.

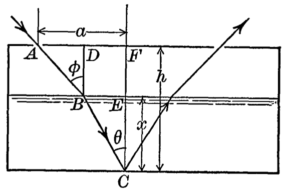
We are given, in the figure, $AF = a$, $CF = h$, $CE = x$. Then $BD = h - x$. $a = AD + BE = (h - x) \tan \varphi + x \tan \theta$. Whence $x = (h \tan \varphi - a)/(\tan \varphi - \tan \theta)$.

Now $\mu = \sin \varphi / \sin \theta$, or $\mu \sin \theta = \sin \varphi$; so that

$$\tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\sin \varphi}{\sqrt{\mu^2 - \sin^2 \varphi}}.$$

Substituting this value of $\tan\theta$ in the value of x obtained above gives us

$$x = \frac{h\tan\varphi - a}{\tan\varphi - \dfrac{\sin\varphi}{(\mu^2 - \sin^2\varphi)^{1/2}}}.$$



Also solved by H. L. AGARD, J. A. CAPARO, HORACE OLSON, H. C. FEEMSTER, L. G. WELD, F. R. MORRIS, H. N. CARLETON, H. S. UHLER, and the PROPOSER.

316. Proposed by C. N. SCHMALL, New York, N. Y.

A body at rest at a point R begins to move toward a center of force F. The distance $RF = d$, and the force varies inversely as the distance. Two intermediate points in the path are P and Q, such that $FP = kd$, and $FQ = k^n d$. Show that the body will traverse the distance QP in a *maximum* of time if $k = 1/n^{1/2(n-1)}$.

SOLUTION BY H. S. UHLER, Yale University.

Since P is a point intermediate with respect to F and R, then $d > kd > 0$ or $1 > k > 0$. Again, since Q is a point between F and R, then $d > k^n d > 0$ or $1 > k^n > 0$. Hence, since k must be positive and less than unity, two possibilities arise, namely;

(a) $n > 1$, and (b) $1 > n > 0$.

Case (a) $n > 1$.

Then $\overline{FQ} < \overline{FP}$. Taking F as origin and defining x as positive from F toward R, we have

$$\frac{d^2x}{dt^2} = -\frac{a}{x},$$

where a is a positive constant. Writing $v \equiv \dfrac{dx}{dt}$, and noting that $\dfrac{d^2x}{dt^2} = v\dfrac{dv}{dx}$, we obtain

$$vdv = -a\frac{dx}{x},$$

whence

$$\tfrac{1}{2}v^2 = c - a\log x.$$

When $x = d$, $v = 0$, therefore $c = a\log d$ and

$$\frac{dx}{dt} = -\left(2a\log\frac{d}{x}\right)^{\frac{1}{2}},$$

the negative sign indicating that the displacement decreases as time increases. Letting T denote the time required to traverse the distance from P to Q we have

$$\int_{kd}^{k^n d}\left(2a\log\frac{d}{x}\right)^{-1/2}dx = -\int_0^T dt, \qquad T = -\int_{kd}^{k^n d}\left(2a\log\frac{d}{x}\right)^{-1/2}dx.$$

Since T is to be tested for maxima and minima we now apply the ordinary method for differentiating a definite integral and obtain

$$\frac{dT}{dk} = \frac{d}{\sqrt{2a}}[(\log k^{-1})^{-1/2} - nk^{n-1}(\log k^{-n})^{-1/2}].$$

The expression within the brackets vanishes when $nk^{2(n-1)} = 1$.

It must next be shown that this value of k corresponds to a maximum of T. The second derivative of T with respect to k reduces to

$$\frac{d}{2k\sqrt{2a}}\{(\log k^{-1})^{-3/2} - nk^{n-1}(\log k^{-n})^{-3/2}[n + 2(n-1)(\log k^{-n})]\}.$$

Substituting from the condition $nk^{2(n-1)} = 1$ we find

$$\frac{d^2T}{dk^2} = -\frac{d(n-1)}{k}\left(\frac{n-1}{a\log n}\right)^{1/2}.$$

Since this result is negative $(n > 1)$ a true maximum obtains.

Case (b). $1 > n > 0$.

Then $\overline{FP} < \overline{FQ}$. Proceeding as before we get

$$T' = -\int_{k^nd}^{kd}\left(2a\log\frac{d}{x}\right)^{-1/2}dx,$$

where T' symbolizes the interval of time consumed in going from Q to P. Consequently $nk^{2(n-1)} = 1$ and

$$\frac{d^2T'}{dk^2} = -\frac{d(1-n)}{k}\left(\frac{1-n}{-a\log n}\right)^{1/2}.$$

Since $n-1$ is now negative, the second derivative is also negative, and hence the condition for a maximum is again fulfilled.

Remark: In making the final reductions in case (b) attention has to be paid to the fact that $\log n$ is negative as well as $n-1$.

Also solved by A. H. Wilson, H. N. Carleton, H. Polish, J. A. Caparo, Elijah Swift, Paul Capron, and the Proposer.

NUMBER THEORY.

209. (March, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Prove that the difference of the sixth powers of two integers cannot be the square of an integer.

Solution by Elijah Swift, University of Vermont.

In Carmichael's Diophantine Analysis, pp. 70, 71, the impossibility of the equation $x^3 + y^3 = 2^n z^3$ is proved. (The proof is valid whether y is positive or negative.)

We are to prove the impossibility of the equation $a^6 - b^6 = c^2$. We know that if $x^2 + y^2 = z^2$, then $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$. (Loc. cit., p. 10.) We assume, of course, that a, b, c are all prime to each other; so are x, y, z also. Two cases present themselves:

$$\text{(I)}\qquad a^3 = m^2 + n^2,\quad b^3 = m^2 - n^2,\quad c = 2mn.$$

$$\text{(II)}\qquad a^3 = m^2 + n^2,\quad b^3 = 2mn,\qquad c = m^2 - n^2.$$

Case I. Of the two integers, m and n, one is odd and the other even. Also these numbers are prime to each other. Consequently $m + n$ and $m - n$ are prime to each other, and since their product is a cube, each of them must be a cube also. If we set them equal to α^3 and β^3 respectively

$$m + n = \alpha^3,\quad m - n = \beta^3,\quad a^3 = m^2 + n^2 = \frac{\alpha^6 + \beta^6}{2},\quad\text{or}\quad \alpha^6 + \beta^6 = 2a^3,$$

the impossibility of which was stated in the first paragraph.

Case II. Assume m even; the proof will apply equally well to the other case, n even. Since $2mn$ is a cube, $2m$ and n must each of them be a cube. Setting them equal to $8\alpha^3$ and β^3, we have

$$m = 4\alpha^3,\qquad n = \beta^3,\qquad a^3 = 16\alpha^6 + \beta^6\qquad\text{or}\qquad 2^4\alpha^6 = a^3 - \beta^6,$$

an equation which was proved to be impossible. Hence the equation $a^6 - b^6 = c^2$ cannot hold.

219. (June, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

Solution by Elijah Swift, University of Vermont.

If the number of sides of a polygon is n, the number of its diagonals is $n(n-3)/2$, which by the conditions of the problem must be a perfect fourth power.

The fourth power of any number is divisible by 16, if the number is even, by 16 with remainder 1 if the number is odd.

We consider two cases: (I) n not divisible by 3, (II) n divisible by 3.

Case I. If n is even, $n - 3$ is odd, and the two are prime to each other. Since their product is twice a fourth power, $n - 3$ is a fourth power. Consequently $n - 3$ is divisible by 16 with remainder 1, by n with remainder 4, and by $n/2$ with remainder 2 or 10; in neither case can $n/2$ be a fourth power, so that $n(n - 3)/2$ cannot be a fourth power either.

If n is odd, it must be a fourth power by the same reasoning as before. $n - 3$ has the remainder 14, and $(n - 3)/2$, has the remainder 7 or 15, when divided by 16; so that $(n - 3)/2$ cannot be a fourth power, nor can $n(n - 3)/2$.

Case II. If n is divisible by 3, $n - 3$ is also, and so is the fourth power, so that we have the equation $n(n - 3) = 2\cdot 3^4\cdot k^4$. Either n or $n - 3$ is divisible by 3^3 and the other one by 3. Calling the one $3^3\beta$ the other will be $3^3\beta \pm 3$, the sign depending on which is divisible by 3^3. Our equation takes the form $3^3\beta(3^3\beta \pm 1) = 2\cdot 3^4\cdot k^4$, or $\beta(3^2\beta \pm 1) = 2\cdot k^4$.

If β is odd, it must be a fourth power, and $(3^2\beta \pm 1)$ twice a fourth power. But if $\beta \equiv 1 \pmod{16}$, $3^2\beta \pm 1 \equiv 8$ or $10 \pmod{16}$, so that it could not be twice a fourth power.

If β is even, $\beta/2$ and $3^2\beta \pm 1$ must both be fourth powers. Here, again, two cases are possible: $\beta/2$ even and $\beta/2$ odd. In the latter case $\beta/2 \equiv 1 \pmod{16}$, or $\beta \equiv 2 \pmod{16}$, so that $3^2\beta \pm 1 \equiv 3$ or $1 \pmod{16}$, from which we deduce at once that only the $-$ sign is admissible. But $3^2\beta - 1 =$ a fourth power, and any fourth power if divided by 3 must yield the remainder $+1$ or 0. Consequently $3^2\beta - 1$ cannot be a fourth power, if $\beta/2$ is odd.

It remains to consider the case $\beta/2$ even. Just as before, we see that $\beta/2$ and $3^2\beta \pm 1$ must be fourth powers. Calling these $16b^4$ and a^4 respectively, (since $\beta/2$ is even, it must be the fourth power of an even integer) we derive the equations $288b^4 \pm 1 = a^4$. Since $a^4 \equiv 1 \pmod{16}$. the $+$ sign only is admissible. We now proceed to show the impossibility of this last equation,

Writing it in the form $288b^4 = a^4 - 1$, we see that for the right hand side to be divisible by 32, a must be of the form $8\lambda \pm 1$. Substituting this value for a and factoring the right-hand side, the equation becomes

$$288b^4 = (64\lambda^2 \pm 16\lambda + 2)(8\lambda \pm 2)8\lambda,$$

or, after division by 32,

$$9b^4 = (32\lambda^2 \pm 8\lambda + 1)(4\lambda \pm 1)\lambda.$$

It is easy to see that the three factors on the right are prime to each other: in fact

$$(32\lambda^2 \pm 8\lambda + 1) - (4\lambda \pm 1)^2 = 16\lambda^2,$$

so that any common factor of the first two must be a factor of $16\lambda^2$. Since the product of these three is a perfect square, each of them must be a square, and we have the three equations

$$\lambda = c^2,$$
$$4\lambda^2 \pm 1 = d^2,$$
$$32\lambda^2 \pm 8\lambda + 1 = e^2.$$

From the first two of these we deduce

$$4c^2 \pm 1 = d^2,$$

which is impossible, for the squares of two integers cannot differ by unity.

As we have examined all possible cases, we conclude that no polygon can have the number of its diagonals equal to a perfect fourth power.

222. (**October, 1914.**) **Proposed by A. H. HOLMES, Brunswick, Me.**

Find rational values for m and n such that $(m^2 + 1)/m^2 + (n^2 + 1)/n^2$ may be the square of an integer.

Solution by Elijah Swift, University of Vermont.

Let $m = b/a$, $n = d/c$, where a, b, c, d, are all integers, and where a is prime to b, and c to d. Substituting these values, $\dfrac{b^2 + a^2}{b^2} + \dfrac{c^2 + d^2}{d^2}$ must be the square of an integer. If it is to be an integer at all, clearly b must equal d, so that we have the equation

$$\frac{a^2 + c^2 + 2b^2}{b^2} = \text{a perfect square},$$

or

$$a^2 + c^2 = b^2 \text{ [a perfect square } - 2] = b^2 k,$$

where k must have one of the values 2 (*i. e.*, $2^2 - 2$), 7, 14, 23, 34, 47, 62, 79, 98, $\cdots$.

If k has one of the values 2, 34, 98, this equation possesses a solution, but not for any of the other values given. To show this and the method of obtaining the solution, it will be sufficient to treat the values $k = 2$, and $k = 7$. If $k = 7$, a and c must be prime to each other, and also to 7, as otherwise they would not be prime to b. Consequently each must be of the form $7\lambda \pm 1$, 2, or 3. But the sum of their squares cannot be divisible by 7 in that case, and the equation is impossible. If $k = 2$, let $x = a/b$ and $y = c/b$, so that the equation takes the form $x^2 + y^2 = 2$. An obvious solution is $x = 1$, $y = 1$. Any line through the point (1, 1) with rational slope will cut the circle $x^2 + y^2 = 2$ in a second rational point, and, conversely, taking the slope to be α/β, where α and β are integers, and solving for x and y, we get the general solution of our equation to be

$$x = \frac{a}{b} = \frac{\alpha^2 - 2\alpha\beta - \beta^2}{\alpha^2 + \beta^2}, \qquad y = \frac{c}{b} = \frac{\beta^2 - 2\alpha\beta - \alpha^2}{\alpha^2 + \beta^2},$$

where α and β are any integers. m and n are the reciprocals of x and y respectively.

A similar solution can readily be found in the other possible cases.

QUESTIONS AND DISCUSSIONS.

[Send all Communications to U. G. MITCHELL, University of Kansas, Lawrence, Kans.]

DISCUSSIONS.

I. Relating to Some Determinants Connected with the Bernoulli Numbers.

By K. P. Williams, Indiana University.

It is well known that the Bernoulli numbers $B_1, B_2, B_3, \cdots$ can be obtained from either of the following recurrence relations,

$${}_{2n+1}C_{2n}B_n - {}_{2n+1}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}_{2n+1}C_2B_1 = (-1)^{n-1}(n - \tfrac{1}{2}),$$

or

$${}_{2n+2}C_{2n}B_n - {}_{2n+2}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}_{2n+2}C_2B_1 = (-1)^{n-1}n,$$

where ${}_nC_r$ denotes the combinations of n things r at a time.[1] If we write out several of the equations we obtain the two series of relations

$${}_3C_2B_1 = \tfrac{1}{2}, \qquad {}_4C_2B_1 = 1,$$

$${}_5C_2B_1 - {}_5C_4B_2 = \tfrac{3}{2}, \qquad {}_6C_2B_1 - {}_6C_4B_2 = 2,$$

$${}_7C_2B_1 - {}_7C_4B_2 + {}_7C_6B_3 = \tfrac{5}{2}, \qquad {}_8C_2B_1 - {}_8C_4B_2 + {}_8C_6B_3 = 3,$$

$${}_9C_2B_1 - {}_9C_4B_2 + {}_9C_6B_3 - {}_9C_8B_4 = \tfrac{7}{2}, \qquad {}_{10}C_2B_1 - {}_{10}C_4B_2 + {}_{10}C_6B_3 - {}_{10}C_8B_4 = 4,$$

. . . .

Since either set will determine $B_1, B_2, B_3, \cdots$ it is obvious that the first n equations in one set and the nth equation of the other set will form a system of $n + 1$ consistent equations in the n quantities $B_1, B_2, \cdots, B_n$; thus the augmented determinant will be zero. For instance, if $n = 4$, we have

[1] Chrystal, *Algebra*, Part II, p. 207 (first edition), or p. 231 (second edition, 1900).

$$\begin{vmatrix} {}_3C_2 & 0 & 0 & 0 & 1 \\ {}_5C_2 & {}_5C_4 & 0 & 0 & 3 \\ {}_7C_2 & {}_7C_4 & {}_7C_6 & 0 & 5 \\ {}_9C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 & 7 \\ {}_{10}C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 & 8 \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} {}_4C_2 & 0 & 0 & 0 & 2 \\ {}_6C_2 & {}_6C_4 & 0 & 0 & 4 \\ {}_8C_2 & {}_8C_4 & {}_8C_6 & 0 & 6 \\ {}_{10}C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 & 8 \\ {}_9C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 & 7 \end{vmatrix} = 0,$$

which show at once the scheme for writing out the corresponding determinants of order $n + 1$.

Let us consider the determinant on the left. Subtract three times the last column from the first. The first element in the first column becomes zero, and the other elements take the form

$${}_rC_2 - 3(r - 2) = \frac{r^2 - 7r + 12}{1 \cdot 2} = \frac{(r - 3)(r - 4)}{1 \cdot 2} = {}_{r-3}C_2.$$

When we expand with reference to the first row we obtain

$$\begin{vmatrix} {}_2C_2 & {}_5C_4 & 0 & 0 \\ {}_4C_2 & {}_7C_4 & {}_7C_6 & 0 \\ {}_6C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 \\ {}_7C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 \end{vmatrix} = 0.$$

A corresponding reduction of the other determinant can be made.

In the first determinant multiply the first four columns by 2, 4, 6, 8, respectively, and divide the rows by 3, 5, 7, 9, 10, respectively, and we obtain

$$\begin{vmatrix} {}_2C_1 & 0 & 0 & 0 & \frac{1}{3} \\ {}_4C_1 & {}_4C_3 & 0 & 0 & \frac{3}{5} \\ {}_6C_1 & {}_6C_3 & {}_6C_5 & 0 & \frac{5}{7} \\ {}_8C_1 & {}_8C_3 & {}_8C_5 & {}_8C_7 & \frac{7}{9} \\ {}_9C_1 & {}_9C_3 & {}_9C_5 & {}_9C_7 & \frac{8}{10} \end{vmatrix} = 0,$$

a similar transformation being possible for the second determinant.

Various other transformations of the original determinant are of course possible. A direct proof of the vanishing of some of these determinants might be of interest.

II. Relating to the Teaching of Logarithms.

By T. M. Simpson, University of Wisconsin.

The present note is the result of observations made in teaching logarithms to a large number of students. In departing somewhat from current textbook

practice, I have secured such uniformly good results that I feel there is merit in the method.

By way of introduction the class is drilled, if necessary, on the two laws of exponents which apply in logarithmic work:

$$10^m \times 10^n = 10^{m+n}, \text{ and } (10^m)^n = 10^{mn},$$

where the numbers, m and n, are positive or negative, integral or fractional, or zero.

The point of departure is now the equation,

$$10^L = A.$$

It is stated that this is also written, $L = \log A$, read "L is the logarithm of A." It is pointed out that A is necessarily positive. With no further preparation the class can now easily translate such forms as

$$10^2 = 100, \qquad \frac{10^2 \times 10^{-3}}{10^5} = 10^{2-3-5} = 10^{-6},$$

$$10^{-3} = .001, \qquad 10^{3.2} \times 10^{2.1} = 10^{5.3},$$

into the language of logarithms. Thus they get clearly in mind the concept of a logarithm as an exponent.

The proofs of the fundamental theorems may be treated as exercises in translating statements of exponent laws into logarithmic form. At first, however, the proofs are passed over rather lightly but the facts are insisted upon strongly.

Next, by a few concrete examples, it is shown that the logarithms of numbers having the same sequence of figures differ only by integers.

Take, for illustration, the equation, $10^L = 382$; multiply both members by 10, 10^2, 10^{-3}. The logarithm of 382 being L, the logarithms of 3820, 38200, .382, are seen to be, respectively, $L + 1$, $L + 2$, $L - 3$. Such examples as this lead directly to the observation that any number a can be written in the form $a' \times 10^n$, where a' is between 1 and 10, and n is a positive or negative integer or zero. This statement is not given to the student in the form written above. The situation is made clear to him and the idea is so simple that he may safely be left to formulate the statement for himself.

n is then the characteristic of the logarithm of a. No other rules for characteristics are given. This is a distinct gain over the usual method where there are four rules—two for direct, and two for inverse use of the tables. That the advantage is real is proved by the fact that students quickly see that the characteristic is given by the number of places they must move the decimal point from units' place to put the number in the form $a' \times 10^n$, and that they have little trouble in working both directly and inversely with the tables.

In computing with logarithms, I question the advantage of writing the characteristic -1 in the form $9. - 10$, and similarly for other negative characteristics. Why not say that if a characteristic is positive it can of course be

written before the mantissa, but if negative it must be placed after it? Thus, log .2 is .3010−1, log .02 is .3010−2, etc. In this way the actual characteristic appears with the logarithm, and labor in computation is somewhat cut down.

If it is required to divide .3010 − 1 by 3, change the logarithm to 2.3010 − 3. The student is made to see that he has an expression of the form $c + m - c'$ in which c and c' may be changed at pleasure, provided their difference remains constant. This leads naturally to the anticipation of another difficulty of the student, which is illustrated by the following problem. It is required to solve the equation, $.2 = .3^x$. The student easily obtains the result

$$x = \frac{\log .2}{\log .3} = \frac{9.3010 - 10}{9.4771 - 10},$$

and is unable to proceed. Perhaps I should say that three quarters of the class subtract the denominator from the numerator while, of the remainder, some know enough not to subtract but do not know what to do. My experience shows the curious fact that if the above result is written $\frac{.3010 - 1}{.4771 - 1}$ many more members of the class will see the correct procedure from this point. Most of the others will handle the fraction correctly if, just before the assignment of this type of problem, the instructor states that, when it is required to multiply or divide by a negative logarithm, the characteristic and mantissa are no longer separated (as always in addition or subtraction) but are combined. So if we wish to add or subtract log .2, we write it .3010 − 1, but if we wish to use it as a multiplier or divisor, we combine the parts and write it − .6990. This fact is seldom explicitly stated in the books and is such a stumbling block to the student that it seems wise to state it.

CORRESPONDENCE.

NOTE. A correspondent asks to be told something about descriptive geometry as a course of more than professional engineering concern. The Editors have asked Professor W. H. ROEVER, of Washington University, to frame a reply to this question, and we print it in full below since we believe it to be of interest to MONTHLY readers in general.

TO THE EDITORS OF THE MONTHLY:

In accordance with your request I submit the following reply to the question of your correspondent:

From my conversations and correspondence with various mathematicians in this country I have been led to the conclusion that many do not have a very clear notion of what descriptive geometry really is. Hence, I will first attempt to state precisely and concisely the nature and object of this branch of applied mathematics.

It is evident that drawing is done in a plane or on a surface. Hence graphical processes are executed in a plane (or on a surface). On the other hand the applied sciences frequently demand a graphical solution for the problems of space

of three (and of higher) dimensions. It, therefore, becomes necessary to represent the objects of space by means of figures in a plane and in such a way that the correspondence between the space object and its plane representative is unique and unambiguous, *i. e.*, it must be possible to pass from the space object to the plane representative and also to pass back again from the plane representative to the space object without ambiguity. If, therefore, we can find such plane representatives for the elements (points, lines, planes, etc.) of space, a problem of space is replaced by a problem in the plane. The plane problem may be solvable by graphical methods (*i. e.*, by a process of drawing) and then the solution of this plane problem is the plane representative of the solution of the given space problem. Hence, the space problem is solved by a graphical process.

Now, the object of descriptive geometry is to represent space objects by corresponding plane figures in such a way that the correspondence is unique and unambiguous and also to solve the problems of space by means of the solutions of the corresponding plane problems, the solutions of which are the plane representatives of the solutions of the given space problems.

In the remarks made above I have stressed the notion of graphical solutions of space problems. There are many branches of descriptive geometry which serve well this purpose. The most familiar branch is that known as the mongean method. By this method a point of space is represented by two points in the plane which lie on the same perpendicular to a line of the plane (called the ground line). A line of space is represented by two lines in the plane and a plane is represented by two lines which intersect on the ground line. As an example of how this method solves space problems let us consider the simple problem:
To find the line x of space which connects the two points A and B of space.

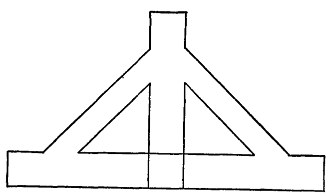
The point A of space is represented by the two points A', A'' of the plane and B is represented by B', B''. The line x' which connects A' and B' and the line x'' which connects A'' and B'', together form the pair of lines (x', x'') which represent the solution x of the given space problem. While the mongean method is a good method for the graphical solution of space problems, the plane representatives or pictures which it furnishes do not in general convey to the mind as adequate a notion of the space forms as do some other methods of descriptive geometry. For instance, the mongean representative (consisting of Figs. 1 and 2, page 268) of a certain bracket-shaped object, does not give as clear a notion of the space form of this object as does the picture in Fig. 3.

It thus appears that certain branches of descriptive geometry enable us to make pictures which help to develop the *power of space visualization*, and this power, as every mathematician knows, is a great asset, at least in some branches of mathematics.

An examination of the books on pure and applied mathematics shows clearly that the authors of mathematical works pay very little attention to the proper drawing of figures. Some persons take the point of view that figures are not necessary, but I think every one will admit that if figures are resorted to they should be properly constructed. In order that a figure such as Fig. 3 below

may serve to give the dimensions of the object which it represents as well as to convey to the mind an adequate notion of its form, it is essential to know the scales on the three axes $O'X'$, $O'Y'$, $O'Z'$ (which are the representatives of the three mutually perpendicular axes OX, OY, OZ, of space). If the figure is an orthographic projection on the plane of the paper, *either* the directions of the axes $O'X'$, $O'Y'$, $O'Z'$ may be chosen (*i. e.*, the angles $\xi = \sphericalangle\, Y'O'Z'$, $\eta = \sphericalangle\, Z'O'X'$,

Fig. 1.



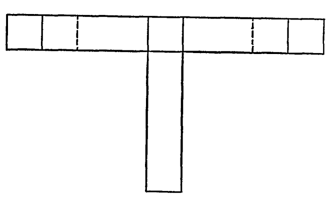


Fig. 2.

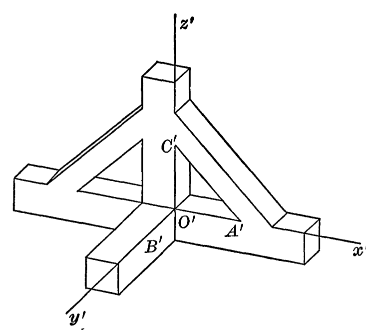


Fig. 3.

$\zeta = \sphericalangle\, X'O'Y'$ all greater than 90°), *or* the scales on these axes may be chosen ($l = O'A'$, $m = O'B'$, $n = O'C'$, $l^2 + m^2 > n^2$, $m^2 + n^2 > l^2$, $n^2 + l^2 > m^2$) but not *both*. The question then is, what is the relation between the axes and the scales? A theorem by Schwarz or a theorem by Gauss gives us information concerning this point. If, on the other hand, the projection is parallel and oblique the theorem of Pohlke tells us that both the angles and the scales may be chosen at random.

I believe that most mathematicians are not familiar with these fundamental theorems or with the fact that most of the representations which they attempt to make are a form of parallel projection known as Cavalier perspective, where $\sphericalangle\, X'O'Z' = 90°$. How properly to represent in figures of the type of Fig. 3 lines and planes which are perpendicular are also problems with which the average mathematician is not familiar.

Thus, it appears that if we wish adequately to represent and to plot space figures, some knowledge of *axonometry* (that branch of descriptive geometry which enables us to draw figures of the type of Fig. 3) is essential.

Hoping that these remarks will answer your question at least in part, I am

Sincerely yours,

WM. H. ROEVER.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO.

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University, Bloomington, Ind.

At Dartmouth College, DR. F. J. MCMACKIN, of Columbia University, has been appointed instructor in mathematics.

Mr. R. W. DICKEY has been appointed associate professor of mathematics at Washington and Lee University.

At Vassar College, Dr. ELIZABETH B. COWLEY has been promoted from an assistant professorship to an associate professorship in mathematics.

Mr. E. F. CANADAY has been appointed to an instructorship in mathematics in the University of South Dakota for the coming year.

Mr. W. L. LORD becomes master in mathematics in Woodberry Forest School, Virginia, with the beginning of the new school year.

Mr. W. C. EELLS, who for the past three years has been instructor in mathematics at the U. S. Naval Academy, returns to his alma mater, Whitman College, as professor of applied mathematics. During the past summer he has been instructor in surveying at the Harvard University Engineering Camp at Ashland, New Hampshire.

At Yale University, Dr. G. I. TRACY has been promoted to an assistant professorship of mathematics.

Mr. F. S. NOWLAN, of Columbia University, has been appointed instructor in mathematics at the Carnegie School of Technology, Pittsburgh.

Professor ELLEN HAYES, of Wellesley College, will retire from active service at the end of the present academic year.

At Brown University, Mr. C. H. CURRIER has been promoted from instructor to an assistant professorship in mathematics.

Mr. J. H. HILL has been appointed professor of mathematics at Ohio Northern University.

Associate Professor L. C. EMMONS, of the Michigan Agricultural College, has been granted leave of absence for study at Harvard University. His place has been filled by the appointment of Mr. VERN JAMES of Indiana University.

In *Rendiconti della Accademia dei Lincei*, Rome, March, 1916, appears a paper by Professor ERNEST PASCAL, of the University of Rome, on "The mechanical solution of the general linear differential equation of the second order."

The Smith prizes, offered by the University of Cambridge for the year 1916, have been awarded as follows: To H. M. GARNER, of St. John's College, for his essay, "On orbital oscillation about the equatorial triangular configuration in the problem of three bodies"; to G. P. THOMPSON, of Corpus Christi College, for his essay, "On aëroplane problems."

The Rayleigh prize for 1916 has been awarded to W. M. SMART, of Trinity College, for his essay, "Libration in the Trojan planets."

A meeting of mathematicians from Sweden, Denmark, Finland and Norway was to be held at Stockholm from August 30 to September 2. It will be recalled that the sixth International Congress of Mathematicians was to have been held there at this time, and the fact that such a meeting was impossible doubtless led to this reunion.

A copy of De Morgan's "Differential and Integral Calculus" 1842, about 800 pages, is offered for sale by Mr. A. C. ANDREWS, Manual Training High School, Kansas City, Mo.

In the announcement concerning the Quarter-Centennial conferences at the University of Chicago printed in the June issue, mention should have been made of a report by Professor J. A. MILLER, Director of the Observatory at Swarthmore College, on the topic: "Choice of fields of comparison stars in parallax determinations."

The petition for the establishment of a section made by the members of the Association in the state of Iowa has been granted through the committee of the Council which has in charge the authorizing of sections. The Iowa Section is, therefore, duly constituted as the fourth section of the Mathematical Association of America.

Volume I, No. 3, April, 1916, of "The Texas Mathematics Teachers' Bulletin" published by the University of Texas, contains a number of articles of interest to teachers of elementary mathematics, among which may be enumerated: "Calculation with logarithms," by Professor M. B. PORTER; "Literal arithmetic," by Professor C. D. RICE; "What great men say about mathematics," "Is mathematics worth while in the high school," by Professor C. N. MOORE; "The mathematics of investment," by Professor E. L. DODD.

There has recently been issued in pamphlet form an extract from the last will and testament of Professor MITTAG-LEFFLER and his wife, this will having been made on March 16, 1916, Professor Mittag-Leffler's seventieth birthday. It seems that by the terms of this will the testators have established a Mathematical Institute to bear the name of the donors. This Institute will be housed in the villa now occupied by Professor Mittag-Leffler at Djursholm, Stockholm. The object is to establish a Foundation which shall assist in the development of pure mathematics in the four Scandinavian countries of Sweden, Denmark, Finland, and Norway, but more especially of Sweden. The terms of the will provide for awarding financial aid to students who give particular promise in the field of pure mathematics, and also to award medals and prizes for noteworthy achievements. So far as possible the prizes will consist of a gold medal, and of sets of the *Acta Mathematica*. These will be bestowed personally at the Institute, and will be the occasion of a formal ceremony. There has been issued also a sumptuous volume giving a complete catalogue of the magnificent library

of Professor Mittag-Leffler, which library will be deposited in the Institute and become available for students from all parts of the world. There has appeared hardly anything so sumptuous in the way of a mathematical bibliography, and the volume will be sought for by all mathematical bibliophiles. The founding of this Institute is one of the most noteworthy acts for the encouragement of mathematics that has yet been recorded in the history of the subject.

Dr. Emory McClintock, president of the New York Mathematical Society, 1890–1892, and president of the American Mathematical Society during the first two years of its existence, 1892–1894, died on July 10, 1916, at his home in Bay Head, N. J. Dr. McClintock was seventy-six years old at the time of his death. He was graduated from Columbia College in 1859; he was assistant professor of mathematics at Columbia from 1859 to 1860; he served as consular agent at Bradford, England, 1863–1866. His life work began in 1867, when he became Actuary of the Asbury Life Insurance Company, a position which he filled for four years; he then became actuary for the Northwestern Mutual Life Insurance Company, in which position he remained for seventeen years; he then became actuary and trustee for the Mutual Life Insurance Company of New York, positions which he held until his retirement from active service in 1911. Dr. McClintock was an honorary fellow of the American Academy of Arts and Sciences, a fellow of the Institute of Actuaries, London, a member and past president of the American Mathematical Society, and a charter member of the newly organized Mathematical Association of America. He made numerous contributions to mathematical periodicals, and during his long and active career was very much interested in the advancement of higher mathematics.

Professor W. H. Metzler, of Syracuse University, calls attention to the fact that the statement made in an article by Professor E. R. Hedrick, in School and Society, March 11, 1916, would seem to him to lead to the impression that the scope of the Mathematics Teacher and the scope of the Association of Teachers of Mathematics of the Middle States and Maryland are restricted to the secondary field. It appears that while the actual activities of this organization and of its official journal have been, until the present year, almost exclusively in the secondary field, yet neither one has ever fixed any limits to its scope. Professor Hedrick requests that this means be taken to correct any misimpression which may have arisen from his article or from any other published statements in the Monthly or elsewhere in this connection.

In connection with the formation of sections of the Mathematical Association of America it will be of interest and value to take note of the geographical distribution of the charter membership, which is shown by the preliminary list published in the April number of the Monthly. An examination of that list shows that New York State has 120 members; the New England States have 125 members; the Middle States and Maryland, aside from New York, 134. The total for the North Atlantic States is 379; for the South Atlantic and other Southern States, aside from Missouri, 130; for the Far Western States, 102. Or,

making the summation in a slightly different way, in all the Atlantic States there are 429; in the Middle West including Missouri, 415 and in the remaining portions of the United States, 182; total for the United States, 1026; for foreign countries, 20; total individual members, 1046. Total institutional members, 52. Grand total, 1098.

Professor F. E. MILLER, of Otterbein University, Westerville, Ohio, desires to secure a copy of the October, 1913, issue of the MONTHLY. He will be glad to purchase it or to exchange for it a copy of the November, 1913, issue, of which he has two numbers. With this single exception he has a complete file of the MONTHLY from its beginning. This request is an example of recent activity on the part of very many in completing MONTHLY files. Any one who may have extra copies will confer a favor by reporting the same to the Managing Editor.

As this issue goes to press the first summer meeting of the Mathematical Association of America is about to assemble in the new buildings of the Massachusetts Institute of Technology. The preliminary announcement of the Committee on program shows that great care has been exercised in selecting the topics and speakers. Every omen seems to indicate an important series of papers on timely topics such as "The teaching of elementary dynamics," by Professor EDWARD V. HUNTINGTON of Harvard University; "The history of mathematical recreations," by Professor DAVID EUGENE SMITH of Columbia University; "Combined courses in mathematics for college freshmen," by Professor JOHN N. VAN DER VRIES of the University of Kansas; and "Combined courses in mathematics for freshmen in schools of technology," by Professor F. S. WOODS of the Massachusetts Institute of Technology.

It is understood that speakers were invited to lead in the formal discussion of these topics and that ample opportunity would be provided for informal discussion. Provision was also made for meetings of the Council, a meeting of Institutional Delegates, and for reports of committees, two or three of which had promised at least preliminary announcements of plans and progress.

On the next following pages are reprinted the Constitution and By-Laws of the Mathematical Association of America, together with the list of officers and members of the Council for 1916. Special attention is called to Article III of the Constitution and Section 2 of the By-Laws (concerning officers, tenure of office, and election of officers) by way of preparation for the first application of these regulations to be made during the autumn. The provision for nomination of officers through open primaries was intended to emphasize the opportunity presented to every member for active participation in the affairs of the Association.

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

ARTICLE I—NAME AND PURPOSE.

1. This organization shall be known as THE MATHEMATICAL ASSOCIATION OF AMERICA.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

ARTICLE III—OFFICERS.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

ARTICLE IV—MEETINGS.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

ARTICLE V—SECTIONS.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

ARTICLE VI—OFFICIAL JOURNAL.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

ARTICLE VII—DUES.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

ARTICLE VIII—AMENDMENTS.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

Following are the Officers of the Association for 1916

For President, E. R. HEDRICK, University of Missouri;

For Vice-Presidents, E. V. HUNTINGTON, Harvard University, and
G. A. MILLER, University of Illinois;

For Secretary-Treasurer, W. D. CAIRNS, Oberlin College;

For additional members of the Executive Council:

To serve for one year

D. N. LEHMER, University of California
R. E. MORITZ, University of Washington
K. D. SWARTZEL, Ohio State University
OSWALD VEBLEN, Princeton University

To serve for two years

R. C. ARCHIBALD, Brown University
FLORIAN CAJORI, Colorado College
M. B. PORTER, University of Texas
J. W. YOUNG, Dartmouth College

To serve for three years

B. F. HINKEL, Drury College
E. H. MOORE, University of Chicago
J. N. VAN DER VRIES, University of Kansas
ALEXANDER ZIWET, University of Michigan

VOLUME XXIII OCTOBER, 1916 NUMBER 8

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

PUBLISHED BY THE ASSOCIATION

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EDITORIAL CORRESPONDENCE should be addressed to the Managing Editor, H. E. Slaught. 5548 Kenwood Avenue, Chicago, Ill.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 5465 Greenwood Ave., Chicago, Ill.

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FIRST SUMMER MEETING OF THE ASSOCIATION.

The first summer meeting of the Mathematical Association of America was held by invitation at the Massachusetts Institute of Technology in Cambridge, Mass., on Friday and Saturday, September 1, 2, 1916. There were 126 persons in attendance at the various sessions, including the following 111 members of the Association:

L. D. Ames, University of Missouri, Columbia, Mo.
R. C. Archibald, Brown University, Providence, R. I.
G. N. Armstrong, Ohio Wesleyan University, Delaware, O.
C. S. Atchison, Washington and Jefferson College, Washington, Pa.

Clara L. Bacon, Goucher College, Baltimore, Md.
A. A. Bennett, University of Texas, Austin, Tex.
Grace E. Berry, Pomona College, Claremont, Cal.
C. L. Bouton, Harvard University, Cambridge, Mass.
H. C. Bradley, Massachusetts Institute of Technology, Cambridge, Mass.
L. A. Brigham, Boston University, Boston, Mass.
H. S. Brown, Hamilton College, Clinton, N. Y.
T. H. Brown, Brown University, Providence, R. I.
R. E. Bruce, Boston University, Boston, Mass.
E. S. Bryant, High School, Everett, Mass.

W. D. Cairns, Oberlin College, Oberlin, O.
Paul Capron, U. S. Naval Academy, Annapolis, Md.
H. E. Cobb, Lewis Institute, Chicago, Ill.
Julia T. Colpitts, Iowa State College, Ames, Iowa.
J. L. Coolidge, Harvard University, Cambridge, Mass.
A. R. Crathorne, University of Illinois, Urbana, Ill.
C. H. Currier, Brown University, Providence, R. I.

E. W. Davis, University of Nebraska, Lincoln, Neb.
C. E. Dimick, U. S. Coast Guard Academy, New London, Conn.
Arnold Dresden, University of Wisconsin, Madison, Wis.
W. P. Durfee, Hobart College, Geneva, N. Y.

L. P. Eisenhart, Princeton University, Princeton, N. J.
H. J. Ettlinger, University of Texas, Austin, Tex.
G. W. Evans, Charlestown High School, Boston, Mass.

F. E. Fash, Fall River, Mass.
H. B. Fine, Princeton University, Princeton, N. J.
W. J. Fisher, Fellow in Clark University, Worcester, Mass.
T. S. Fiske, Columbia University, New York, N. Y.
T. M. Focke, Case School of Applied Science, Cleveland, Ohio.

M. G. Gaba, Cornell University, Ithaca, N.Y.
C. A. Garabedian, State College, Durham, N. H.
H. D. Gaylord, Browne and Nichols School, Boston, Mass.
C. C. Grove, Columbia University, New York, N. Y.

C. E. Haigler, Wentworth Institute, Boston, Mass.
J. N. Hart, University of Maine, Orono, Me.
M. W. Haskell, University of California, Berkeley, Cal.
E. R. Hedrick, University of Missouri, Columbia, Mo.
P. W. Hill, Wabash College, Crawfordsville, Ind.
F. H. Hodge, Franklin College, Franklin, Ind.
H. L. Hodgkins, George Washington University, Washington, D. C.
L. M. Hoskins, Stanford University, Palo Alto, Cal.
E. V. Huntington, Harvard University, Cambridge, Mass.

Dunham Jackson, Harvard University, Cambridge, Mass.
W. W. Johnson, U. S. Naval Academy, Annapolis, Md.

William Kent, Consulting Engineer, Montclair, N. J.
Carl King, Wentworth Institute, Boston, Mass.
Edward Kircher, Harvard University, Cambridge, Mass.

A. E. Landry, Catholic University of America, Washington, D. C.
B. B. Libby, Massachusetts Institute of Technology.
Louis Lindsey, Syracuse University, Syracuse, N. Y.
Joseph Lipka, Massachusetts Institute of Technology, Cambridge, Mass.
U. J. Lupien, Lowell Textile School, Lowell, Mass.

Barry MacNutt, Lehigh University, South Bethlehem, Pa.
H. P. Manning, Brown University, Providence, R. I.
J. L. Markley, University of Michigan, Ann Arbor, Mich.
Helen A. Merrill, Wellesley College, Wellesley, Mass.
Mansfield Merriman, Consulting Engineer, New York, N. Y.

J. F. Messick, Alabama Polytechnic Institute, Auburn, Ala.
H. B. Mitchell, Columbia University, New York, N. Y.
U. G. Mitchell, University of Kansas, Lawrence, Kans.
C. L. E. Moore, Massachusetts Institute of Technology, Cambridge, Mass.
C. N. Moore, University of Cincinnati, Cincinnati, Ohio.
F. C. Moore, State College, Durham, N. H.
M. M. S. Moriarty, High School, Holyoke, Mass.

A. H. Norton, Elmira College, Elmira, N. Y.

G. D. Olds, Amherst College, Amherst, Mass.

L. R. Perkins, Middlebury College, Middlebury, Vt.
Winona M. Perry, Judson College, Marion, Ala.
H. B. Phillips, Massachusetts Institute of Technology, Cambridge, Mass.

W. R. Ransom, Tufts College, Mass.
C. N. Reynolds, Jr., Wesleyan University, Middletown, Conn.
R. G. D. Richardson, Brown University, Providence, R. I.
Maria M. Roberts, Iowa State College, Ames, Iowa.
George Rutledge, Massachusetts Institute of Technology, Cambridge, Mass.

F. H. Safford, University of Pennsylvania, Philadelphia, Pa.
A. R. Schweitzer, Chicago, Ill.
H. M. Sheffer, College of the City of New York, New York, N. Y.
L. L. Silverman, Cornell University, Ithaca, N. Y.
T. M. Simpson, University of Wisconsin, Madison, Wis.
H. E. Slaught, University of Chicago, Chicago, Ill.
Clara E. Smith, Wellesley College, Wellesley, Mass.
D. E. Smith, Columbia University, New York, N. Y.
P. F. Smith, Yale University, New Haven, Conn.
Sarah E. Smith, Mt. Holyoke College, South Hadley, Mass.
E. W. Stanton, Iowa State College, Ames, Iowa.
R. P. Stephens, University of Georgia, Athens, Ga.
E. B. Stouffer, University of Kansas, Lawrence, Kans.
Mary C. Suffa, Student at the University of Chicago, Chicago, Ill.

Edith V. Thompson, Wilkes-Barre Institute, Wilkes-Barre, Pa.
H. W. Tyler, Massachusetts Institute of Technology, Cambridge, Mass.

J. N. Van der Vries, University of Kansas, Lawrence, Kans.
Oswald Veblen, Princeton University, Princeton, N. J.
Roxana H. Vivian, Wellesley College, Wellesley, Mass.

C. A. Waldo, Washington University, St. Louis, Mo.
C. B. Walsh, Ethical Culture High School, New York, N. Y.
A. H. Wheeler, North High School, Worcester, Mass.
C. E. Wilder, Northwestern University, Evanston, Ill.
W. O. Wiley, with John Wiley & Sons, New York City, N. Y.

F. B. Williams, Clark University, Worcester, Mass.
E. B. Wilson, Massachusetts Institute of Technology.
F. S. Woods, Massachusetts Institute of Technology.
Euphemia A. Worthington, Wellesley College, Wellesley, Mass.
W. C. Wright, Consulting Actuary, Medford, Mass.

C. H. Yeaton, Northwestern University, Evanston, Ill.
Mabel M. Young, Wellesley College, Wellesley, Mass.
J. W. Young, Dartmouth College, Hanover, N. H.

Alexander Ziwet, University of Michigan, Ann Arbor, Mich.

The Association was honored in being the first scientific body invited to meet in the new buildings of the Massachusetts Institute of Technology. This privilege was keenly appreciated and the opportunity offered to inspect these magnificent buildings embraced by many members. The committee on arrangements, under the chairmanship of Professor H. W. Tyler, had made such thorough provisions for anticipating every need of those in attendance that nothing was left to be desired. Special mention should be made of the unique scheme of registration devised by the local committee whereby the names and geographical distribution of all in attendance were exhibited in such a way as to effectively promote mutual acquaintance and friendly intercourse. The opening of the Freshman dormitories of Harvard University for the accommodation of the Association also contributed greatly to the comfort and convenience of the members. An expression of appreciation in regard to all these matters was embodied in a resolution which was unanimously voted at the close of the final session.

The dinner on Friday evening at the University Club in Boston was attended by eighty-one persons and was followed by a number of short addresses voicing the felicitations which were evidently in the minds of all with respect to the remarkable progress made by the new Association during the eight months of its history. The speakers were: Professor H. W. Tyler, Massachusetts Institute of Technology; Professor M. W. Haskell, University of California; Professor H. B. Fine, Princeton University; Professor T. S. Fiske, Columbia University; Professor J. L. Coolidge, Harvard University; Professor G. D. Olds, Amherst College; Professor Helen A. Merrill, Wellesley College; Professor Alexander Ziwet, University of Michigan; and Professor H. E. Slaught, University of Chicago. The remarks of Professor Fiske on the relations between the American Mathematical Society and the Mathematical Association of America are printed elsewhere in this issue.

An interesting feature at the dinner was the reading of the following cablegram from Professor Mittag-Leffler, dated Stockholm, Sweden, August 29, 1916: "Scandinavian mathematicians assembled at Stockholm send fraternal greetings to their American brother mathematicians." In reply, Secretary Cairns was authorized to send the following message: "Members of the Mathematical Association of America reciprocate the hearty greetings of the Scandinavian

mathematicians." It was explained by Professor D. E. Smith that in a recent letter to Professor Mittag-Leffler he had mentioned the fact that the Association was meeting at the Massachusetts Institute of Technology on September first and second, and that the Society would be in session at Harvard University during the week following, and that this cablegram, though addressed to Professor Moore at the Institute, was evidently intended for both gatherings.

On Saturday afternoon thirty-four members accepted the invitation to join in a trip to Wellesley for a visit to the college, dinner at the Wellesley Inn, and return by train in the evening. Other trips were offered on Sunday for those who were to remain over for the meetings of the American Mathematical Society.

The following papers and reports were presented, in accordance with the program arranged by the committee under the chairmanship of Professor H. B. Fine of Princeton University:

Friday Morning:

(1) "The Teaching of Elementary Dynamics." PROFESSOR E. V. HUNTINGTON, Harvard University.

(2) Discussion, led by PROFESSOR L. M. HOSKINS, Stanford University.

Friday Afternoon:

(3) "The History of Mathematical Recreations." PROFESSOR DAVID EUGENE SMITH, Columbia University.

Saturday Morning:

(4) "Combined Courses in Mathematics for College Freshmen." PROFESSOR J. N. VAN DER VRIES, University of Kansas.

(5) "Combined Courses in Mathematics for Freshmen in Technical Schools." PROFESSOR F. S. WOODS, Massachusetts Institute of Technology.

(6) Discussion, led by PROFESSOR W. D. CAIRNS, Oberlin College and PROFESSOR F. L. GRIFFIN, Reed College.

(8) "Report of the Committee on Mathematical Requirements." PROFESSOR J. W. YOUNG, Chairman, Dartmouth College.

(9) "Report of the Committee on Bureau of Information." PROFESSOR J. B. SHAW, Chairman, University of Illinois.

The opportunity for general discussion of Professor Huntington's paper on Friday morning was so limited that a special session was held at the close of the afternoon program, when numerous members took occasion to express their views. Altogether, the interest in this topic was manifest and indicative of a desire for more discussions of this kind.

The paper by Professor Smith aroused keen enthusiasm and brought out many expressions of appreciation both at the time and later at the dinner in the evening. Historical topics in connection with mathematics have already won a favorable place in the estimation of members of the Association as desirable program material.

The discussions of Saturday morning on "Combined Courses in Mathematics" proved exceedingly interesting and fruitful. It seems certain that some form of combined courses for the first year, or first two years, of college work is a

slogan that is steadily gaining ground, and hence careful consideration at this time is opportune. Many members participated in the general discussion which was continued as long as the time would permit.

Abstracts of all the papers and formal discussions are given below, pending fuller reports which may be published later. They are numbered to correspond with the numbers on the program above.

Preliminary reports of two important committees were made and these also are given in abstract, in order to indicate some of the lines of work which are already planned for the Association.

Abstracts of Papers.

(1) In his paper on the teaching of elementary mechanics, Professor Huntington attempted to analyze the logical structure of the science by presenting a list of fundamental concepts in terms of which all the concepts employed in dynamics can be defined, and a list of fundamental theorems from which all the theorems used in dynamics can be deduced.

The fundamental concepts are these: (1) Length and time, as measured by a meter stick and a clock; (2) forces, as measured by a spring balance; (3) inert lumps of matter, on which the forces act.

The fundamental propositions are the following: (1) The principle of force and acceleration, as expressed in the equation $F/F' = a/a'$; (2) the principle of the vector addition of forces; (3) the principle of the independence of two perpendicular forces; (4) the principle of action and reaction.

In terms of supporting force, W, and acceleration of falling, g, the fundamental equation assumes the convenient form $F = (W/g)a$, where W/g may be called the inertia of the body.

Among the theorems deducible from these fundamental principles, the most important are the theorem of the motion of the center of mass and the theorem of rotation, by means of which all problems in the plane motion of a rigid body can be solved. Next in importance are the theorem of work and kinetic energy, and the theorems of impulse and momentum (linear and angular).

(2) In discussing Professor Huntington's paper, Professor Hoskins spoke in outline as follows: The principles of dynamics may be formulated in various ways, each logically consistent. The effective presentation to the beginner must, however, concern itself not with mere logic; even more important is the physical basis. How can the ordinary experience of the learner be appealed to most effectively in explaining the meaning of the quantities dealt with in the scientific study of motion?

In the matter of logical consistency little or no exception can be taken to the procedure of Professor Huntington's paper. As regards the physical concepts of force and mass (or "inertia," as it is called in his paper) the treatment seems to be open to certain objections.

There is a traditional treatment of force which encourages the notion that

forces "manifest themselves" in some vague way independently of matter. It is to be feared that the "anthropomorphic" language used in the paper is calculated to perpetuate this vague metaphysical notion. The best way to remove this vagueness is to emphasize from the outset the fact that a force is merely a push or a pull exerted *upon* one body *by* another body, and the further fact that *every* force is one of an action-reaction pair which concerns two bodies and only two. Professor Huntington's treatment could very easily be amended in this respect without changing the scheme as a whole.

The treatment of mass is open to an objection which is more firmly woven into the scheme as outlined. That mass (or inertia as defined in the paper) is an additive property of all matter is a fact so simple as well as so important that it ought to be explicitly recognized as a part of the physical data of the science. The fact that the "inertia" of a body is the sum of the "inertias" of its parts is the key to an understanding of many simple practical problems. Professor Huntington's method does not even recognize the fact that adding matter to a body increases its "inertia" until after an analysis involving internal forces and the law of vector composition is given.

The definition of units involves no serious difficulty when the quantities whose units are to be defined are understood. Difficulties of this kind are neither greater nor less in Professor Huntington's scheme than in other schemes to which he objects.

(3) The early history of mathematical recreations has not yet been seriously considered. The sources of the problems in the Greek Anthology are practically unknown, and the history of the collection itself offers an interesting field for further study. The same may be said of the *Propositiones* attributed to Alcuin, for we can hardly assert that a final study has been made of the internal evidence for or against this supposed authorship. Of the printed books on the subject we have no satisfactory bibliography. The Lucas list was the first serious attempt, but that is very imperfect and its imperfections were copied along with the correct portions in the Ahrens bibliography. Both Ball and Schubert have valuable notes, but neither makes any claim to completeness.

Professor Smith explained the nature of the bibliographical work on which he is engaged. He then laid down several principles upon which these recreations are founded, one of them being that certain problems started as practical ones in the lives of the people and maintained their standing because of the interesting variants which have arisen from age to age. He then traced the history of four of these stock problems, namely, the pipes and the cistern, the testament, the couriers, and the Turks and Christians. As examples of the bibliographical question he considered the curious history of the work of Leurechon, with which are connected the names of Van Etten (pseudonym), Mydorge, Ens, and several others; the history of Bachet de Méziriac's classical work, and that of Ozanam with the elaborate revisions of Montucla and Hutton.

(4) Professor Van der Vries in his paper on "Combined Courses in Mathematics for College Students" considered three questions:

I. The advisability of a rearrangement in the order of the content of the high school mathematical course, especially with respect to the first two units; and the probability of such a rearrangement at an early date.

II. The content and the possible order of a unified course in mathematics for college freshmen based on a rearranged two units of high school mathematics.

III. The advantages and the disadvantages of parallel courses in mathematics for college freshmen.

In Section I was discussed the dissatisfaction with mathematics in general prevalent among public school men and the antagonism on their part, in particular, to the present arrangement of secondary mathematics. The Mathematical Association is in a position to lend a hand in the initiation of needed reforms. A rearrangement in the order of the content of high school mathematics is suggested as the first step. This rearrangement need not change the sum total of mathematics at the disposal of the high school student who desires more than two units, or who plans to enter an institution requiring more than two units for entrance. It can be accomplished

(*a*) by the postponement of the more complicated problems in algebra, especially those which require much detail and special gifts of ingenuity for their solution, and of those problems in geometry which are not necessary for further work, and in which the reasoning is ultra-abstract;

(*b*) by the introduction of a closer union between plane and solid geometry.

The committee of nine of the National Educational Association recommended in 1911 that the requirements in mathematics of a well-planned high school should not exceed two units on a basis of fifteen units for graduation. The 1916 catalogs of one hundred leading colleges and universities of the United States show that forty per cent. require not exceeding two units of high-school mathematics for entrance, and this percentage is rapidly increasing. A rearranged two units, as outlined above, will not only better prepare the students entering these institutions than is now the case but will be of much greater value to those students whose mathematical training ends with their two units of high-school work. In rearranging the high-school curricula, there should be kept in mind constantly the three objectives outlined by Professor Gutzmer, of Halle, at the Rome Congress; namely,

A. The strengthening of space perception.

B. The introduction of *intelligent* applications and of correlation with practical work.

C. Above all, training in the habit of functional thinking.

In Section II was presented a unified course based on the rearranged two units of high-school mathematics suggested in Section I. The three objectives, *A*, *B*, *C*, above are again constantly kept in mind. Lack of space allows the writer to give here only the merest skeleton. In filling out the gaps in the outline, "the notions of limit, variability, rate, function and graph are so constantly borne in mind that when the calculus as such is introduced in the last section, these notions are met as familiar friends." The main divisions are:

(*a*) Introduction and Fundamental Principles.
(*b*) The Linear Function.
(*c*) The Quadratic Function.
(*d*) Algebraic Functions of Degree Higher than Two.
(*e*) Logarithmic and Exponential Functions.
(*f*) Circular Functions, Periodic Functions.
(*g*) Parametric Representation, Implicit Functions, etc.
(*h*) Introduction to the Differential Calculus.

The course is planned for five hours per week for one year.

In section III was discussed a parallel system in which algebra and trigonometry are studied for one semester, algebra on Monday, Wednesday and Friday, and trigonometry on Tuesday and Thursday. This work is followed by parallel courses in analytics and calculus for two semesters, analytics two days per week and calculus the remaining three days per week. This system makes provision for a correlation between the parallel courses and for the earlier introduction of the methods of the calculus both of which are desiderata of a unified course. These ends are, however, accomplished without disturbing the lines of demarkation between the courses, a very desirable feature from the administrative point of view in the accrediting of courses. The parallel system is thus an intermediate step between the customary courses and unified courses. The advantages and disadvantages of the parallel system were enumerated in the paper.

(5) In presenting his outline of a combined course, Professor Woods said that two considerations affect vitally the planning of the first year course in mathematics in a technical school. First, the course is only the beginning of a required course of two years or more in length, and, secondly, it must be so planned and taught that the students may learn to use their mathematics.

The course in trigonometry may well be retained as a distinct course. This is because in some engineering schools it is required for admission, and in others it is voluntarily offered as an admission subject by an increasing number of candidates. Of the work usually given under the titles of advanced algebra and analytic geometry, however, much may be dropped without loss, and the rest placed in the combined course. The result is to make it possible to begin the calculus earlier.

On these lines, Professor Woods presented a course of 90 class exercises without trigonometry, or 120 exercises with trigonometry, with an ample allowance of time for problem work. It begins with the definition of Cartesian coördinates and is for a while geometrical. It passes then to the calculus through a discussion of slopes and areas and closes with a thorough treatment of differentiation of a function of one variable. The student is introduced to the concept of a definite integral as a limit of a sum and does some simple integration, but a systematic study of integration is postponed to the second year. The discussion of functions of two or more variables and of solid analytic geometry is also postponed.

(6) Professor Cairns compared the two types of courses proposed by Pro-

fessors Van der Vries and Woods with those which had been previously proposed and published in the MONTHLY in December, 1915, and April, 1916, the former by Professor F. L. Griffin, and the latter by Mr. J. A. Nyberg. He spoke of the lethargy among college teachers with respect to considerations of both content and method. He emphasized the desirability of open-minded and critical examination not only of arrangement of courses but also methods of presentation.

Professor Cairns would have more concrete teaching at the outset in freshman courses and would introduce the notion of "rate of change" earlier than is done in the outline proposed by the two speakers mentioned above. He would displace the more advanced parts of trigonometry, college algebra, and analytic geometry by the inclusion of the simpler parts of integral as well as differential calculus. This last is demanded by the manifold applications which teachers of chemistry, economics, statistics, biology, etc., desire to make in advanced college courses. He pointed out that more than fifty per cent. of colleges and universities offer only three-hour courses in the freshman year, but that even within the field thus narrowed the same choice of subjects and treatment is possible and that the reports of successful experiment along this line are encouraging.

(7) In the absence of Professor Griffin a communication from him was read by Professor Dunham Jackson of the program committee. He mentioned certain disadvantages involved in the plan of teaching trigonometry, college algebra, analytic geometry, and calculus separately:

(1) The relation of these subjects to each other as parts of a unified whole cannot be seen until several successive courses have been taken. This is entirely lost to the student who can take only one year's work.

(2) Students of the natural and social sciences, who need increasingly an elementary general knowledge of mathematical analysis, are unable to get it early enough.

(3) Even the students specializing in mathematics do not gain facility in drawing upon one subject for help in another.

The four-hour course for one year described in the MONTHLY for December, 1915 (the type advocated by Professor Cairns), gives students a fair command of the essential principles and simpler processes of the subjects named. Plane and solid geometry and algebra through quadratics form a prerequisite, although many students have carried the course successfully with less preparation; indeed solid geometry is a luxury rather than a necessity. Trigonometry should be included in the combination course because trigonometric analysis will mean more there and because this will avoid duplication. Such a course, which may well be available even for the fourth year in the stronger high schools, will meet our obligation to the non-specialist students, and will also be advantageous to the specialists.

Preliminary Report of the Committee on Mathematical Requirements.

The Committee on Requirements consists of Professor J. W. Young, Chairman, and Professors A. R. Crathorne, E. H. Moore, D. E. Smith, and Oswald Veblen. The purpose and scope of the work of this committee is indicated in the following list of questions tentatively adopted by the committee as a basis of procedure:

I. What general educational values (utilitarian, disciplinary, cultural) can actually be secured by the study of mathematics?

II. What should be the primary purposes of mathematical instruction?

III. What topics and what treatment of these topics will best serve to realize the values and purposes under I and II?

IV. How much of the content included under III should be required (*a*) of all students in secondary schools; (*b*) for college entrance; (*c*) of all students in college?

V. What should be the preparation of teachers in secondary schools and in colleges?

For the purpose of the discussion as to secondary school mathematics the committee has voted in favor of a national joint committee as requested by the New England Association of Teachers of Mathematics. The specific suggestions of Professor Tyler, who was chairman of the committee on secondary school mathematics of the New England Association, as to the method of formation of such a joint committee were also adopted. In accordance with these suggestions, the Committee on Requirements will request the New England Association, the Association of Teachers of Mathematics of the Middle States and Maryland, and the Central Association of Science and Mathematics Teachers each to appoint one representative (who shall be a secondary school teacher), these three to join with the Committee on Requirements of the Mathematical Association of America to form the national joint committee for the discussion of mathematical requirements in secondary schools.

The committee voted further to ask Professor H. W. Tyler, of the Massachusetts Institute of Technology, to serve as a member of the committee, and Professor Tyler has consented.

The committee also voted to ask the Council to authorize it to still further enlarge its personnel from outside the membership of the Association, if in its opinion such action would serve the best interests of the committee. This authority was subsequently granted by the Council.

Preliminary Report of the Committee on Bureau of Information.

Since this is a new venture in the history of mathematical societies, only a tentative program can be laid down. The Bureau is the outcome of a feeling, on the part of some who have had the opportunity of visiting many teachers of mathematics, that a large part of the profession is handicapped by very inadequate sources of information, and that many questions arise in their work, which can scarcely be answered save through the use of adequately equipped libraries

which they are not likely to possess. A hesitancy to burden men in the larger institutions generally leaves these questions completely unanswered. Consequently it was felt that if a regular Bureau of Information were established to which questions of any character regarding mathematics might be sent (with certain specified exceptions mentioned below) there would ultimately be a great gain to American mathematics.

This Bureau has now been established, with the following members, who have volunteered to undertake the labor: Professor J. B. Shaw, Chairman, and Professors J. L. Coolidge, L. P. Eisenhart, W. B. Fite, M. W. Haskell, and W. A. Hurwitz. All questions should be sent directly to Professor J. B. Shaw, University of Illinois, Urbana, Illinois, whence they will be distributed to the committee. The answers will be published if of general interest, otherwise they will be sent by mail to the questioners.

The following classes of questions should be sent elsewhere as indicated:

(*a*) All problems, which should go to the problem department.

(*b*) All questions which are merely subjects for discussion, as for instance see page 221 of volume XXIII of the Monthly; these should go to the department of Questions and Discussions.

(*c*) Questions whose answer must be largely only personal opinion; as for instance, "What is the best freshman trigonometry?" See class 7 below.

(*d*) Questions as to courses of reading, lists of new books, and the like, which should go to the Library Committee.

(*e*) Questions of a purely pedagogical character; as, for instance, "When should vector analysis be introduced?" See class 5 below.

The following classification of questions proper to be sent to the Bureau is tentative and subject to revision as experience shows the desirability:

(1) Explanations of terms used and not generally found in the elementary texts; as, adjunct equation, Heine-Borel theorem, Mathieu function.

(2) Explanations of new branches of mathematics, their significance, relation to older branches, references to elementary treatments, references to applications; as orthogonal functions, difference equations, general analysis.

(3) Inquiries as to supposed theorems; as, Cauchy's theorem, Green's theorem, Goldbach's theorem.

(4) References to treatments of specific topics; as, Abelian groups, theta functions, topology, Latin squares, crinkly curves, linkages, non-euclidean sphero-conics.

(5) Bibliographical references. The committee will indicate where bibliographies of general or special topics may be found, will complete imperfect references, will indicate the more important memoirs to be consulted with reference to given subjects of investigation, and will furnish any other assistance which does not demand too much expenditure of time.

(6) Historical references will be indicated so far as possible, although the committee will not undertake categorically to assign priority anywhere.

(7) Book reviews will be indicated, in order that inquirers may know where to find some information as to a book.
(8) The location of journals and other books in the libraries of the country will be indicated, particularly in those libraries which maintain an inter-library loan system. Most institutions of good standing can borrow books for a period of a month by paying the transportation charges.
(9) Questions as to the courses offered by the mathematical departments of American and foreign universities will be answered; as for instance, lectures to be given, the prerequisites for admission, conditions to be fulfilled for a degree, method of obtaining scholarships and fellowships, and any other attainable information.
(10) Information as to mathematical societies, congresses, prizes, etc.

The Bureau will follow as its standard in references and other matters of form the Encyclopédie des Sciences Mathématiques.

Meeting of Institutional Delegates.

A meeting of delegates from institutions holding institutional membership in the Association was held on Friday morning. Eighteen institutions participated through their officially appointed delegates as follows:

Amherst College, Dean G. D. Olds;
Case School of Applied Science, Professor T. M. Focke;
University of Chicago, Professor H. E. Slaught;
Dartmouth College, Professor J. W. Young;
Elmira College, Professor A. H. Norton;
George Washington University, Professor H. L. Hodgkins;
University of Georgia, Professor R. P. Stephens;
Iowa State College, Professor E. W. Stanton;
University of Kansas, Professor J. N. Van der Vries;
University of Maine, Professor J. N. Hart;
University of Michigan, Professor J. L. Markley;
Middlebury College, Professor L. R. Perkins;
University of Missouri, Professor E. R. Hedrick;
Oberlin College, Professor W. D. Cairns;
Ohio Wesleyan University, Professor G. N. Armstrong;
Washington University, Professor C. A. Waldo;
Wellesley College, Professor Helen A. Merrill;
Wesleyan University, Professor B. H. Camp.

Inasmuch as the purpose of the meeting was to consider the possible questions of an institutional character which such a body of delegates might well discuss, and since many other institutions are considering the desirability of becoming institutional members, it was deemed advisable to invite to this meeting any representatives in attendance from other institutions who might be interested. In this way the following additional institutions were represented: University of Alabama, Boston University, Brown University, Catholic University of America,

Columbia University, Franklin College, Harvard University, Massachusetts Institute of Technology, University of Nebraska, University of Pennsylvania, Princeton University, Stanford University, Syracuse University, University of Texas, Tufts College, and Yale University.

On the basis of suggestions from various members, the following may be stated as some of the reasons why institutional membership is desirable and why the delegates from such institutions might well hold a separate meeting for the discussion of questions of an institutional character:

(1) Institutional membership secures two copies of the MONTHLY, one of which can be used in the departmental library or by the local mathematical club for the stimulation of interest among the advanced students and for the dissemination of intelligence concerning mathematical activities throughout the country.

(2) Institutional members are entitled to send voting delegates to all meetings of the Association, and such delegates would constitute the constituency interested in discussing the questions of an institutional character, such, for example, as the following:

(*a*) Requirements in mathematics for admission to college and for the Bachelor's degree. These questions are at present undergoing reconsideration and should have the attention of those who are mathematically qualified to speak with some authority.

(*b*) Changes in the curricula as regards undergraduate courses in mathematics. These questions are also undergoing careful consideration, and every institution should be vitally concerned to know what other institutions are doing and why radical modifications are being proposed, such, for example, as the "Combined courses" discussed at this meeting, or the question of introducing the subject of projective geometry as early as the sophomore year, considered by Professor Bussey in the November 1913 issue of the MONTHLY.

(*c*) Questions concerning mathematical libraries, such as the best selection of books for a small library, say of 200, 300, or 500 volumes, the best means of purchasing such lists of books, the best methods of administering such libraries in the interests of the students. Advisory pronouncements by this body, after careful consideration, will be of the utmost advantage in enabling the smaller institutions to secure libraries where none now exist or where the existing facilities are entirely inadequate. In this connection the institutional delegates would naturally work with the committee of the Association on libraries already appointed.

On motion of Professor J. N. Van der Vries, it was unanimously voted that the institutional delegates should meet as a separate department for the discussion of questions of an institutional character, and that a committee of delegates be appointed to prepare such a program for the next meeting, the chairman of this committee to be the presiding officer at this meeting and one of its members to act as secretary of the meeting.

The general impression of those in attendance at this meeting was that there is ample opportunity for wide usefulness through departmental meetings of

institutional delegates, and that it is well worth while for institutions to become members and take part in such meetings.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

The Council held two meetings, nine members being present on each occasion. The following is an outline of the business transacted.

(1) The Council having already voted to hold the annual meeting in New York City during the Christmas holidays, in affiliation with the American Association for the Advancement of Science, it was decided to appoint two committees for this meeting, one on program and one on arrangements. Professor T. S. Fiske has since been asked to act as chairman of the latter committee and has consented to act. The further committee appointments will be announced in the next issue of the MONTHLY. The exact time and place of the meeting was left to be arranged in conference with the committees of the American Association for the Advancement of Science and the American Mathematical Society.

(2) With the approval of the Council, the President has empowered the Committee on Mathematical Requirements to invite certain local associations of teachers of mathematics (See the report of this committee given above) to appoint representatives from their organizations as additional members. This action was taken in response to a suggestion made by the New England Association of Teachers of Mathematics through Professor Tyler. The Committee was also empowered to add others to its membership, if it is so desired, from outside the Association.

(3) The following eight institutions, on applications duly certified, were elected to institutional membership, making the total number now 61:

George Washington University, Washington, D. C.
Shurtleff College, Alton, Ill.
Drake University, Des Moines, Ia.
Wellesley College, Wellesley, Mass.
Dartmouth College, Hanover, N. H.
Elmira College, Elmira, N. Y.
New York University, New York, N. Y.
State Normal School, La Crosse, Wis.

(4) The following thirteen persons, on applications duly certified, were elected to individual membership, making the total number now 1063:

Winona M. Perry, Judson College, Marion, Ala.
W. A. Moore, Montgomery, Ala.
Grace E. Berry, Pomona College, Claremont, Cal.
Frank Langellotti, Nautical Almanac Office, Washington, D. C.
H. M. Roeser, Bureau of Standards, Washington, D. C.
P. W. Hill, Wabash College, Crawfordsville, Ind.
W. J. Fisher, Honorary Fellow in Physics, Clark University, Worcester, Mass.
C. A. Garabedian, New Hampshire State College, Durham, N. H.

W. O. Wiley, with John Wiley & Sons, New York, N. Y.
J. A. Hacker, Sioux Falls College, Sioux Falls, S. D.
W. E. Boren, State Normal School, Milwaukee, Wis.
L. H. Clark, State Normal School, River Falls, Wis.
Mary B. McMillan, State Normal School, River Falls, Wis.

(5) The Council next considered the report of a committee, consisting of Professor E. H. Moore, Chairman, and Professors R. C. Archibald, Oswald Veblen, and Alexander Ziwet, which was appointed last Spring to consider and report upon the advisability of fostering the production and the publication of articles of an expository and historical nature.

In view of the fact that the number of papers of high character at present being produced in this field is small, this committee decided that it would be wise to learn whether an arrangement could be made with the *Annals of Mathematics*, which now publishes such articles, for developing this field. After correspondence and conference the committee learned that the Board of Editors of the *Annals* was willing to enter into an agreement of the nature described below.

The proposed agreement provides that, for a period of three years, beginning in September, 1917, in consideration of an annual subvention of three hundred dollars from the Association, the Board of Editors of the *Annals* will increase the size of the *Annals* from 200 (its present size) to 300 pages. The added pages will be devoted to expository articles of suitable character so far as these can be secured. The Board of Editors will also fix a subscription price for individual members of the Association to be one-half the ordinary price, which latter would probably be three dollars.

Under this agreement the exclusive editorial control of the *Annals* remains with its present Board of Editors, but the Board agrees to make no change in its present policy without conference and agreement with the committee representing the Association.

At the end of the three-year period this agreement may be renewed by the consent of both parties; but if not, it is agreed that the Board of Editors of the *Annals* shall thereafter conduct the *Annals* as a journal devoted primarily to research, yielding the field of historical and expository articles (not necessarily absolutely but principally) to the publications of the Association.

The committee recommended that this proposed agreement be ratified and adopted by the Association. After careful consideration, the Council voted unanimously to submit this agreement to the members of the Association through the official journal, in accordance with the provisions of Article III, Section 3 of the Constitution. This agreement will go into effect only if ratified by the Council after the views of members of the Association have been ascertained in accordance with that provision. (See copy of the Constitution and By-Laws in the supplementary matter in this issue.)

E. R. Hedrick,
President,
W. D. Cairns,
Secretary.

USE OF TRANSCENDENTAL EQUATIONS IN ANALYTIC GEOMETRY.

By W. R. LONGLEY, Sheffield Scientific School.

The changes in the teaching of college mathematics during the last ten or fifteen years have been made with varying degrees of rapidity in different institutions. The more radical members of the teaching profession have abolished separate courses in trigonometry, analytic geometry, and calculus and have substituted a sequence of courses consisting of Mathematics I, Mathematics II, etc., the content of each course being a function of several variables. Among the more conservative, the classic names of courses have been retained but the content and emphasis have been more or less altered. This is particularly noticeable in analytic geometry. Some years ago an elementary course in this subject was synonymous with a thorough study of the properties of the conic sections. The tendency now is toward a very brief treatment of conics and the introduction of other material having a closer connection with problems of physics and engineering.

This movement is in line with the growing utilitarian bias of education in general and its chief impetus has come from the criticisms and demands of engineers and scientists who have use for mathematics as a tool. The teacher, trained as a pure mathematician, has yielded each point reluctantly with the feeling that the beauty and elegance of his subject are being destroyed. The beauty and elegance may possibly be preserved in some academic institutions but receive scant consideration in the technical schools, where it is felt that something has been wrong with the courses in mathematics. The engineering student has not been able to use the mathematics of the class room, and the value of the mental training received has been questioned.

The desire to improve the product of the department of mathematics, as measured by the ability of the student to use mathematics in technical and scientific work, is causing the experiments now being made. In a general way the ends to be attained are twofold. The first is the cultivation of the power of logical reasoning and the second is the ability to use mathematics as a tool. When the first end alone is sought, experience shows that the average student acquires little, if any, power to make practical applications, while a certain amount of emphasis upon the utility of topics and methods studied does not necessarily detract from their value as mental discipline. The problem before the teachers is the establishment of a proper balance.

It is to be expected that the solution of this problem will be accomplished by short steps and after a large number of trials. One step which has been taken by several writers of recent texts and which has been tried with success in the Sheffield Scientific School is the introduction of transcendental equations as a topic in analytic geometry. The problem is to solve an equation involving algebraic, trigonometric, exponential, and logarithmic expressions, for example, $1 - x = 2 \sin x$, $e^{-x} = \tan \pi x$, etc. The method is to determine the number

and approximate value of the roots graphically, for example, as the abscissas of the intersections of the curves $y = 1 - x$ and $y = 2 \sin x$. Certain roots are then calculated to a specified degree of accuracy by the use of numerical tables. The advantages of this work may be grouped under three heads.

In the first place it offers an opportunity to the teacher to test and correct the mathematical knowledge which the pupil should have acquired by previous study. The curves involved in any problem are being used as tools and, using them in this way, the student learns more of their properties than by merely studying the curves for their own sake. For some reason, difficult to explain, there is a wide difference between drawing a curve as an intermediate step in obtaining an ulterior result and in doing the exercise: "Plot the locus of $y = 4 - x^2$." The theory of logarithms, radian measure of angles, trigonometric and exponential expressions have to be *used,* and if any hazy ideas lurk in the mind of the student the teacher has no difficulty in discovering the fact.

A second advantage lies in the preparation for work to follow. For the calculus, it is a distinct help to have acquired familiarity with the behavior of trigonometric functions as *functions* (of a variable expressed in radian measure) and not merely as expressions occurring in formulas for the solution of triangles, and to have used exponential functions and logarithms to the base e. For the technical courses, the methods used here constitute an important introduction to the construction and use of engineering charts, graphical tables, and nomography.

The third and, perhaps, chief benefit lies in the use of numerical tables of different kinds with insistence upon good methods and accuracy in computation. The teacher frequently loathes numerical calculation and thinks of mathematics as the science of avoiding computation. It is of the greatest importance to consider means of avoiding unnecessary computation, but the habit of neglecting numerical work accounts for much of the dissatisfaction which outsiders feel toward the department of mathematics. Scathing criticism of the inability of college graduates to spell and to use the English language is forcing the teachers of English to give more time to fundamentals which should have been mastered in the preparatory school. The situation is much the same so far as arithmetic is concerned. A large percentage of students coming to college (with trigonometry as an entrance requirement) are not able to use numerical tables and do ordinary arithmetical calculations with speed and accuracy. Unless special attention is given to it, a two-year course in analytic geometry and calculus does not correct the deficiency. Inaccuracy in numerical work is, in general, a habit which the teachers have allowed to develop by giving too much credit for the "right method" of solving a problem. The student has come to think that it is sufficient if he "knows how" to do a certain thing and feels a rude shock when told by an employer or by an instructor in a technical course: "I care nothing about what you *know*; the only thing that counts here is what you can *do*." There is a big gap when the teacher of mathematics merely shows that it is theoretically possible to compute a certain quantity, but omits the actual computation. It is the existence of this gap that accounts, to some extent, for the

fact that so few engineers and scientists actually use their college mathematics. The closing of this gap is an important step in the improvement of the teaching of undergraduates.

In view of the importance of the methods and the value of the practice in solving transcendental equations it seems worth while to give more attention to the topic than is usually done. The texts now merely give a few such equations for solution with no indication of how they may arise. Some instances where this can be done occur in the subject matter of analytic geometry. For example the maximum and minimum points on the curve $y = x \cos x$ are given by the roots of the equation $x = \cot x$. The points of intersection of two polar curves of which one is a spiral ($\rho = a\theta$ or $\rho = e^{a\theta}$) while the equation of the other involves trigonometric functions are found by solving a transcendental equation. While this introduces nothing new theoretically it causes difficulty because the method of solution has been developed with reference to rectangular coördinates.

In college mathematics, as in high school work, a greater significance is given to any topic by the introduction of some problems having a physical meaning. The first of the following examples is a real problem which arose in some construction work and was brought to a college instructor after every member of a large force of employees in an engineering office had exhausted his knowledge of algebra in an attempt to solve it. The equation is easily deduced by elementary mathematics. The equation in the second can not be deduced without a knowledge of mechanics, but the meaning of the problem and the significance of the solutions can be explained by a simple model, which may consist merely of a book and pencil.

Problem 1. A circular arch, ACB, 14 feet long, is to be constructed with an altitude $CD = 2$ feet. Required the radius of the circle.

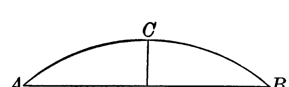


FIG. 1.

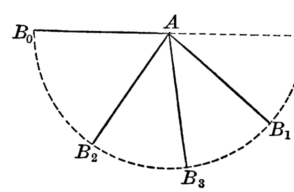


FIG. 2.

If x denotes the reciprocal of the radius, this problem leads to the equation $1 - 2x = \cos 7x$.

Problem 2.[1] A uniform rod is freely movable on a rough inclined plane, whose inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pin fixed through the end A; the bar is held in the horizontal position in the plane and allowed to fall from this position. If θ be the angle through which it falls from rest show that $\sin \theta = m\theta$, where $m = \mu \cot i$.

The problem may be generalized by allowing the rod to fall from a position $\theta = \theta'$ instead of from the horizontal position $\theta = 0$. In this case, if θ increases as the rod falls, the next position of rest is given by the value of θ from the

[1] The statement of this problem is taken from Loney's *Dynamics of a Particle and of Rigid Bodies* (Cambridge, 1909), example 5, p. 217

equation (1) $\sin\theta - m\theta = \sin\theta' - m\theta'$, and, if θ decreases, θ is found from (2) $\sin\theta + m\theta = \sin\theta' + m\theta'$. Suppose the rod is allowed to fall from the horizontal position $AB_0(\theta_0 = 0)$. It swings to the position $AB_1(\theta = \theta_1)$, then back to the position $AB_2(\theta = \theta_2)$, and, after a finite number of swings, comes to rest. Successive applications of equations (1) and (2) show that the angles θ_1, θ_2, etc., are given by the abscissas of the points P_1, P_2, etc., of Fig. 3. Each segment of the zigzag line $OP_1P_2P_3$ which is directed toward the right (OP_1,

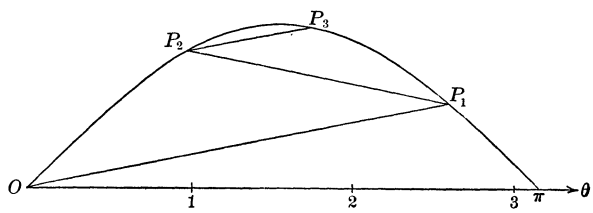


FIG. 3.

P_2P_3) has the slope m, and each segment which is directed toward the left (P_1P_2) has the slope $-m$. The figure is drawn for $i = 45°$, $\mu = 1/5$. By setting $\theta' = 0$ in equation (1) the value of θ_1 is found to be 2.595 radians. Hence the angle B_0AB_1 is equal approximately to 149°. By setting $\theta' = 2.595$ in equation (2) the value of θ_2 is found to be 0.996 radians. Hence the angle B_0AB_2 is equal approximately to 57°. By setting $\theta' = 0.996$ in equation (1) the value of θ_3 is found to be 1.733 radians. Hence the angle B_0AB_3 is equal approximately to 99°. A line drawn from P_3 with slope $-m$ will not intersect the arch of the sine curve. The force equations of the mechanical problem show that the rod will not move from this position.

Problem 3. The diameter of a bicycle wheel is 28 inches and the valve is at the lowest point of the wheel. The wheel is rolled forward until the valve is N inches ahead of its original position. Through what angle has the wheel turned? Assuming that the valve is 12 inches from the center of the wheel the equation to be solved is $N = 14\theta - 12\sin\theta$.

Problem 4. Given a string wrapped around a circle. The locus of the end as it is unwound is the involute of the circle, $x = r\cos\theta + r\theta\sin\theta$, $y = r\sin\theta - r\theta\cos\theta$. Find the length unwound when x or y have given values.

Problem 5. The equation of a damped vibration has the form $x = ae^{-bt}\sin ct$. To find the time when the moving point is at a given distance D from the center, the equation would be put in the form $De^{bt} = a\sin ct$.

A DIFFERENTIATING MACHINE.

By ARMIN ELMENDORF,[1] University of Wisconsin.

A differentiating machine, as its name implies, is a device for drawing the differential or rate curve of any given curve, whether the latter be a curve plotted between two variables connected by an algebraic equation or an empirical curve obtained from experimental data. Its primary interest lies in its use for

[1] Instructor in the Department of Mechanics.

developing rate curves in various technological fields. The problem of rail friction, especially around curves, may be investigated by the use of a machine drawing the rate-of-change-of-velocity curve from velocity time data. From the rate or acceleration curve the friction force on the rails is readily computed by simple relations of mechanics. Many other instances could be cited in which rate curves are of great significance. By the use of the machine, load-deflection curves in impact on beams may be obtained, stresses in beams may be determined when the elastic curve is known, and from statistics such information as mortality and immigration rates become available.

As is well known, the slope of the tangent at any point on a curve is represented by the differential expression dy/dx when the curve is plotted in rectangular coördinates, and if it is desired to plot the rate dy/dx against the variable x it is necessary to determine the magnitude of the slopes by some mechanical means. This involves first the exact location of the tangent. The differentiating machine designed by the author uses for this purpose a small silver mirror which, when set vertically upon the curve, shows the image of the curve. When the image and the curve form a continuous line without a cusp at the junction of the curve and its image the reflector is exactly normal to the curve. Referring now to Fig. 1 in which it is assumed that the tangents have been exactly located,

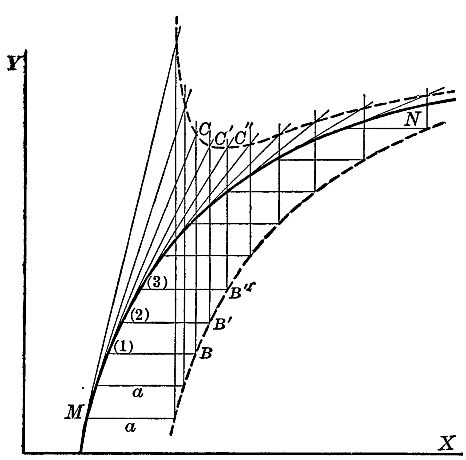


FIG. 1.

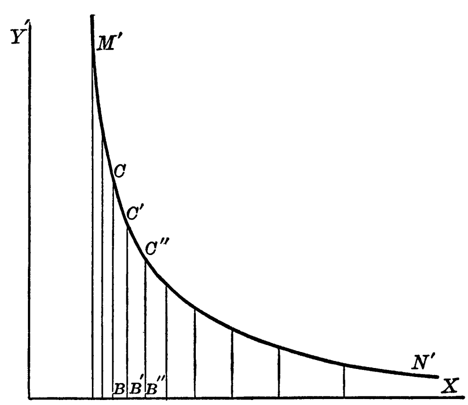


FIG. 2.

it is seen that, if a series of horizontal lines of constant length $a = 1$ be drawn from points along the given curve MN, the vertical distances intercepted by these horizontal lines and the tangents measure the slope at the various points. Thus BC represents the slope at the point (1), $B'C'$ that at point (2), etc. In Fig. 2 these distances have been plotted as ordinates giving the differential curve $M'N'$.

The next consideration is, then, the design of some device that will plot the variable distances between the two dashed lines of Fig. 1, as ordinates to a given base line.

The simple machine illustrated in Fig. 3 fulfills all the kinematic relations

embodied in the motions of the finished machine shown in Fig. 4. The mirror T is set so that the small scratch on the lower edge of the mirror which is vertically under the point O is upon the curve MN, thereby locating the tangent OB. The pin at B is free to slide in the tangent bar and also in the vertical arm EC. Link L is the base line of constant length. It is free to move in the horizontal slot of

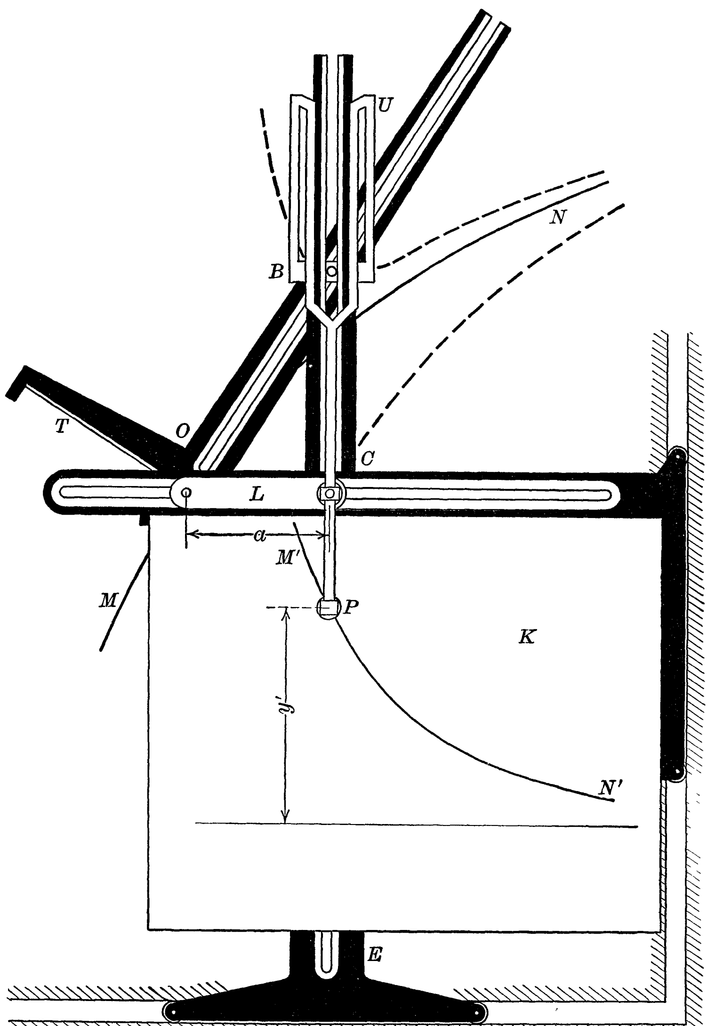


FIG. 3.

the carriage K, so that as the pivot O follows the curve MN, B traces the upper dashed line and C traces the lower line. Motion of the point B is transmitted through a yoke U, Fig. 3, to the tracing point P, enabling the pin B to pass under the point C as happens when the slope changes over from positive to neg-

ative values. When the pin B is under C, the tangent bar is horizontal, indicating a zero slope, and P is down at the base line. As the slope is increased the distance BC is increased and P is drawn up a distance equal to BC; in other words y', the distance drawn up, is in general the slope at some point on MN, or $M'N'$ on the carriage is the differential curve of MN. Fig. 4 clearly shows the

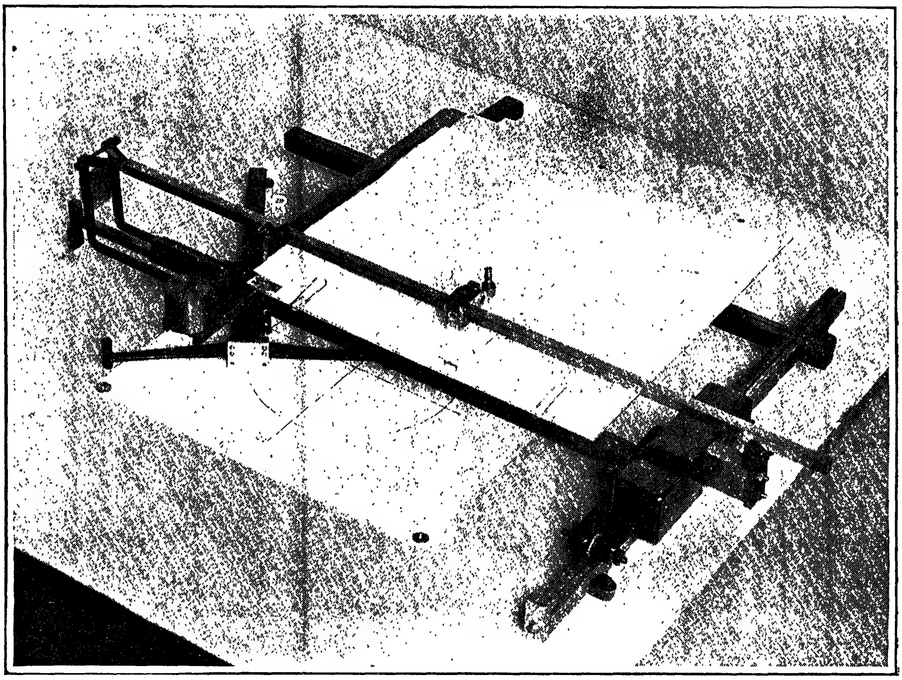


FIG. 4.

two grooves at right angles, one for the carriage and the other for the vertical arm. The mirror is set upon a circle. The tube S is graduated on its upper face so that the operator may obtain the numerical value of the slope at any point by noting the reading under the indicator at R.

In practice the best results are obtained by plotting a series of points on the differential curve and subsequently drawing a smooth curve through these points by hand. The machine has a slope capacity of about 4.5 but this may be changed by varying the length of the link a. For slopes of unity or less fairly smooth continuous curves may be drawn after some experience. The difficulty of obtaining a smooth continuous differential curve will probably remain as an imperfection in all differentiating machines, however perfect mechanically, when the location of the tangent is left to the operator. However, the "point-by-point" method, as a great many tests have verified, gives very accurate results, and only the time lost in filling in the curves by hand would be gained by the continuous process.

RELATIONS BETWEEN THE ASSOCIATION AND THE SOCIETY[1]

Mr. Chairman, Ladies, and Gentlemen: I am very sorry that Professor Ernest William Brown, of Yale University, President of the American Mathematical Society, could not be here to greet and encourage the Mathematical Association of America on this occasion. As an old friend of his I think that I know what his feeling would be on this occasion. As a member of the Council of the American Mathematical Society and as the member of the Society of longest standing present at this meeting (although I say this with some diffidence in the presence of Professor Fine and Professor Woolsey Johnson, who twenty-five years ago helped to bring about the transformation of the small New York Society into a society of national scope), possibly I may take on myself the pleasant duty of expressing to the new Association the felicitations of the older Society. The infant is beginning its activities with much greater resources and with much greater physical vigor than characterized the older Society. After twenty-five years of constant but comparatively slow growth the American Mathematical Society has a membership of less than 800, while the new Association now on the day of its first meeting numbers considerably over 1,000.

These two organizations must have intimate and friendly relations for many reasons.. In the first place, they have a large body of members in common. Men who have played a very important part in the scientific activities and in the administrative conduct of the American Mathematical Society have had a great deal to do with the establishment of the Mathematical Association of America. In the second place, they have, to a large extent, a common aim, or more accurately, closely related and mutually helpful aims. It is, therefore, almost inconceivable that there should be anything but the most friendly rivalry between these bodies. They can be only allies and staunch supporters of each other. The primary aim of the American Mathematical Society is the advancement of mathematical science by the stimulation of research. I take it that the primary aim of the Mathematical Association of America is the advancement of mathematical science by the stimulation of teaching. Those who are interested in contemporary research will naturally wish to attach themselves to the former. Those whose time and strength are largely given to undergraduate teaching may find the latter more helpful. For my part I am inclined to think that every American mathematician should wish to belong to both organizations.

Let us hope that as the Mathematical Association grows in size and influence it may find itself possessed of sufficient funds to enlarge the scope and number of its publications. It may be able to prepare and publish extensive reports on different phases and aspects of mathematical teaching, thus continuing and supplementing for this country the work begun by the International Commission on the Teaching of Mathematics. Then again, it may be able to undertake or assist in the publication of monographs or treatises which would tend

[1] An address given at the dinner of the Association in Cambridge on September 1, 1916, by Professor Thomas S. Fiske, Ex-President of the American Mathematical Society.

to elevate the standard of teaching. Such works sometimes hold forth no prospect of financial return. Abroad they often have the moral support and financial assistance of the government. But in this country some other agency, perhaps the Mathematical Association of America, must supply the need.

In concluding these brief remarks I wish to recommend to all who have not done so recently that they read the address delivered by the late Dr. Emory McClintock on the occasion of his retirement from the presidency of the American Mathematical Society in December, 1894. This address entitled "The Past and Future of the Society" was published in the *Bulletin* for January, 1895. From it I will quote a paragraph which should be of special interest to this newly established Association and of which a few lines have a ring almost prophetical:

"While the Society is not directly concerned in encouraging the study of the higher mathematics among the young, its indirect influence in that direction has undoubtedly been felt, and must be felt increasingly as time goes on. Years ago, when the present century was much younger, the course of study in our colleges was so arranged as to give a large proportion of the time of the undergraduates to the study of mathematics. Subsequently, the tendency in colleges having uniform courses of study was to cut down the number of hours given to this science, as well as to the classics, and to parcel out the time among the modern languages and various sciences. It is believed that even already the organization, the meetings, and the publications of the Society have, by the effect of numbers in association, perceptibly strengthened the tone of the mathematical departments of many institutions of learning and assisted in enabling them, more or less successfully, to stem the hostile tide of sentiment to which I have just referred. I say 'assisted,' for other agencies, especially the journals, have done great good. That the dissemination of knowledge concerning the gigantic strides lately made and still making in mathematical science must in the future have the same favorable effect to an even greater extent is not to be doubted."

NEW BOOKS RECEIVED.

Send all communications to W. H. Bussey, University of Minnesota.

Trigonometric and Logarithmetric Tables. By George Wentworth and David Eugene Smith. Ginn and Company, Boston. v + 104 pages. $0.60.

An Introduction to the Use of Generalized Coördinates in Mechanics and Physics. By William Elwood Byerly. Ginn and Company, Boston, 1916. vii + 118 pages. $1.25.

Engineering Applications of Higher Mathematics. By V. Karapetoff. Part 1, Problems on machine design. Part 2, Problems on hydraulics. Part 3, Problems on thermodynamics. Part 4, Problems on mechanics of materials. Part 5, Problems on electrical engineering. John Wiley and Sons, New York, 1916. xv + 69, v + 103, v + 113, v + 81, v + 64 pages. $0.75 per volume.

MODERN BUSINESS ARITHMETIC. By H. A. Finney and J. C. Brown. Henry Holt and Company, New York, 1916. v + 298 pages.

PLANE AND SOLID GEOMETRY. By William Betz and Harrison E. Webb. With the editorial coöperation of Percy F. Smith. Ginn and Company, Boston, 1916. xii + 507 pages. $1.36.

BOOK REVIEWS.

Subjects for mathematical essays. By CHARLES DAVISON. Macmillan and Co., London, 1915. x + 160 pages. $1.90.

At the outset we may say that the book can be used to advantage as a collection of problems by any teacher of mathematics. This, however, is not the object suggested by the title or by the author in his preface. "The object of what are here called 'mathematical essays' is to coördinate a pupil's knowledge on certain subjects not specially dealt with in textbooks."

Some of the essays certainly give the opportunity for coördination, though the animadversion on textbooks seems not always just. On page 114 we find "State and prove the leading properties in the theory of determinants," and on page 115 "Discuss the principle of proportional parts as applied to mathematical tables." It is not necessary to assume that the 'pupils' are prepared to give presidential addresses to learned societies to explain the presence of these 'subjects.' The trade mark (Trin. 1910) reveals that the pupils are preparing for scholarship examinations where they will be met by a perfectly cynical examiner who throws in just such questions for the purpose of separating the candidates widely.

The essays are not all of this type however; they descend to quite elementary sets of coördinated examples. Among the ways of coördinating the field of elementary mathematics we can scarcely expect to find many of really fundamental interest. A few styles occurring here may be noticed.

Euler's product theorem, page 81, has this set:

1. Prove that

$$\sin \theta = 2^n \cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cos \frac{\theta}{2^3} \cdots \cos \frac{\theta}{2^n} \sin \frac{\theta}{2^n}.$$

2. Prove that

$$\lim_{n \doteq \infty} \left(\cos \frac{\theta}{2} \cos \frac{\theta}{2^2} \cdots \cos \frac{\theta}{2^n} \right) = \sin \frac{\theta}{2}, \text{ etc.}$$

Here coördination apparently consists in taking the purely formal steps in the proof of a theorem and giving them as separate problems in the order required. If any object other than mnemonic is served it is hidden. The worst modern text would presumably say something about the convergence of an infinite product if only something wrong.

In "Euler's Polyhedron Theorem" (page 61) we have:

1. If a cube be constructed face by face, state the number of edges and vertices in the partially constructed figure for each value of F from 1 to 6.

2. Asks the pupil to prove Euler's theorem for any polyhedron.

Now by virtue of that peculiar psychological reaction which enabled the slave in the Platonic dialogue to give a Euclidean demonstration, which enables each bright student to satisfy the teacher unless he be ultra-socratic (and almost the only ultra-socratic teacher is the cold hard world) we may be reasonably certain that it will rarely happen that the cube is ever anything but a *Flächenstück* in the course of its construction, or that any polyhedron so improper as not to obey the theorem will be mentioned. The coördination in this case is of the classic type " the blind leading the blind."

Inevitably some obvious connections must be missed, but the omission in Ptolemy's theorem (page 78) of any reference to the theorem on a straight line seems to require explanation.

Occasionally hypotheses necessary to the conclusion are suppressed as in page 51, "Prove $x^3 + y^3 + z^3 > 3xyz$." These can be supplied by telepathy.

The first part of the work is almost entirely on pure mathematics, only one section bearing a title suggesting applications, and this " On the dip of a stratum " might as well have been " Napier's rule of circular parts." The second part contains in the scholarship papers a good deal of mechanics.

In spite of all that has been specially pleaded the book may well serve a useful purpose, mnemonically, and lay some foundations for broader coördinations. This is probably all the author expected. It should be serviceable to a teacher who desires problems not found in the usual run of American texts.

We note a few unimportant misprints, from which the book appears to be unusually free for a first edition. Page 21, line 15; page 46, line 9; page 81, line 2; page 120, line 15; page 124, line 12. The typography and makeup of the book are very good.

R. P. Baker.

The University of Iowa.

Plane and Spherical Trigonometry with Tables. By George Wentworth and David Eugene Smith. Ginn and Co. 230+104 pages. $1.35.

The total content of this text is practically the same as that of the second revision of Wentworth's *Plane and Spherical Trigonometry,* the principal differences being the addition of a chapter on graphs and the omission of the chapter on "Applications of spherical trigonometry," contained in the older book. Moreover, the general treatment of the subject is the same and the proofs of the individual theorems are, in most cases, identical. In the matter of arrangement, however, the two texts differ materially.

In the present text the authors (as they state in the preface) have followed the rule of " putting the practical before the theoretical." To this end, after defining the trigonometric functions of acute angles, a large amount of space is devoted to problems illustrating the practical uses of each of them, first using natural functions and then logarithms. This covers 76 pages (including a chapter on logarithms) and it is not until page 82 that the student meets the definitions of the functions of angles greater than 90° and begins to get some insight into the

theory of the subject. This will probably appeal to some teachers as carrying a good idea rather far. The so-called "fundamental relations" among the functions play a much less prominent part than in most other modern texts. They are proved on pages 12 and 13, but no special emphasis is laid on them and practically no use is made of them until page 95. Even in the later chapter on identities, the advice to the student as to "how to prove an identity" makes no mention of the use of the fundamental relations. It would seem that the great importance of these relations in later work in mathematics should warrant more effort to impress them upon the memory of the student. Circular measure of angles, inverse functions, graphs and trigonometric identities and equations, as well as the application of trigonometry to algebra are put in later chapters, after the solution of oblique triangles—the logical arrangement both for those who want these subjects and for those who do not.

The book is well printed and bound and presents an attractive appearance. It is quite free from typographical errors, but has two rather surprising mistakes in definition on page 40: In line 1 a logarithm is defined as a *power,* instead of *the exponent of a power,* and in line 19 the statement is made that "any positive, rational number may be taken as the base," thus *including unity* and *excluding e.* Also, the treatment of negative characteristics is likely to be confusing to beginners. But these defects can easily be corrected by any competent teacher, and the book will no doubt receive the same cordial welcome that has been given to others of the same series.

E. P. R. DUVAL.

UNIVERSITY OF OKLAHOMA.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all communications to B. F. Finkel, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

465. Proposed by CYRUS B. HALDEMAN, Ross, Ohio.

Having given $\tan^{-1} 1 = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$, show that

$$\tan^{-1} 1 = 5 \tan^{-1} \tfrac{1}{8} + 2 \tan^{-1} \tfrac{1}{18} + 3 \tan^{-1} \tfrac{1}{57}.$$

466. Proposed by E. B. ESCOTT, Kansas City, Mo.

For what functions, f, are the following relations true:

$$\text{When } \frac{f(x, y, z)}{X} = \frac{f(y, z, x)}{Y} = \frac{f(z, x, y)}{Z}, \text{ then } \frac{f(X, Y, Z)}{x} = \frac{f(Y, Z, X)}{y} = \frac{f(Z, X, Y)}{z}.$$

467. Proposed by IRA M. DE LONG, The University of Colorado.

Determine the function, f, from the functional relation, $f(x + y) = f(x) + f(y) + 2xy$.

GEOMETRY.

497. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two spheres on any line passing through any point common to the two spheres.

498. Proposed by FRANK R. MORRIS, Glendale, California.

To trisect an angle ABC, on BA and BC take D and E equidistant from B. Using DE as a diameter draw the semicircle $DFGE$. With the same radius and D and E as centers draw arcs locating the points F and G on this semicircle. Connect F and G with B. Prove that this method trisects a right angle and a straight angle and that it does not trisect an oblique angle.

CALCULUS.

415. Proposed by GEORGE PAASWELL, New York City, N. Y.

If r is the distance from a fixed point (x, y, z) to a variable point (x', y'), in the plane $z = 0$; determine the values of the integrals $\int\int r\,dx'dy'$ and $\int\int \log(z + r)dx'dy'$ for the two cases

(a) when the integration is extended over the surface of the circle of radius R; and

(b) when the integration is extended over the surface of the rectangle of dimensions a, b.

These integrals are special cases of the direct and logarithmic potentials, the densities of the surface distributions being taken as unity.

416. Proposed by CHARLES N. SCHMALL, New York City, N. Y.

If A be a point on a cycloid and C the corresponding position of the center of the generating circle, show that AC envelops another cycloid half the size of the first.

MECHANICS.

332. Proposed by E. E. MOOTS, University of Arizona.

In any quadrilateral $ABCD$ whose diagonals AB and BD intersect in E, lay off on AC from C, CF equal to AE. Join F to B. Join G, the middle point of BE, to D. On GD lay off GM equal to one-third of GD. Prove that M is the center of gravity of the quadrilateral.

333. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A flywheel 21 feet in diameter makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 pounds. Show that the intensity of stress on a transverse section of rim, assuming that it is unaffected by the arms, is 1,176 lbs. per sq. in. If the safe stress permissible in the material is 6,000 lbs. per sq. in., show that the greatest speed at which the wheel can be run with safety is about 225 revolutions per minute.

NUMBER THEORY.

251. Proposed by HERMAN ROLAND KATNICK, Chicago, Ill.

Determine the character of the positive integer n so that the Diophantine system

$$z + n = x^2, \qquad z - n = y^2$$

shall have an integral solution; and exhibit a method for finding all the values of x, y, z for a given n of such character.

252. Proposed by E. J. MOULTON, Northwestern University.

(A) Show that the number of integers x on the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9\cdot\{[\text{first } p \text{ terms of expansion of } (9 + 1)^r] - {}_rC_{p-1}\cdot 9^{r-p}\}$$

where ${}_rC_{p-1}$ is the coefficient of x^{p-1} in the expansion of $(1 + x)^r$.

(B) Show that the number of integers x on the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 0 at least p times, $p \leqq r$, is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9 + 1)^r].$$

253. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

Prove that $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$ for integral values of n and k.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

452. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Find in the form of a continued fraction the positive root of the equation $x^3 - 2x - 5 = 0$.

SOLUTION BY HORACE OLSON, Chicago, Illinois.

Since the left member of this equation is negative for $x = 2$, and positive for $x = 3$, the positive root of this equation can be written $2 + (1/x_2)$, where $x_2 > 1$. Substituting this expression for x, we obtain

$$x_2^3 - 10x_2^2 - 6x_2 - 1 = 0.$$

By trial, we find that this equation has a root between 10 and 11.

Putting $x_2 = 10 + (1/x_3)$, we obtain $61x_3^3 - 94x_3^2 - 20x_3 - 1 = 0$.

This process can be continued as far as may be desired. A convenient way of making the substitutions is to use Horner's method of solving equations and then to reverse the order of the terms of the equation resulting from each step. From the nature of the process it is evident that each of the successive equations has one, and only one, root greater than 1, since the original equation has one, and only one, root between 2 and 3. The root of the original equation will then be

$$2 + \frac{1}{x_1 +}\frac{1}{x_2 +}\frac{1}{x_3 +}\cdots.$$

This I find to be

$$2 + \frac{1}{10 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{2 +}\frac{1}{1 +}\frac{1}{3 +}\frac{1}{1 +}\cdots.$$

Also solved by H. C. FEEMSTER and J. W. CLAWSON.

453. Proposed by A. J. KEMPER, University of Illinois.

Is the series, whose terms are the reciprocals of all positive integers not containing the figure 0, convergent or divergent?

SOLUTION BY E. J. MOULTON, Northwestern University.

I shall prove the following proposition:

The series S whose terms are the reciprocals of all positive integers not containing the digit 0 at least p times, where p is any positive integer, is convergent.

Let us group the terms of S according to the number of digits required in writing the integers; that is, into groups $g_1, g_2, \cdots$ where the denominators x_{r+1} of the terms in g_{r+1} satisfy the relation

$$10^r \leqq x_{r+1} < 10^{r+1}.$$

The number of terms in g_{r+1} can be shown to be

$$\text{(1)} \qquad N(r+1) = 9\cdot[\text{first } p \text{ terms of the expansion of } (9+1)^r]$$
$$= 9\cdot[9^r + {}_rC_1 9^{r-1} + {}_rC_2 9^{r-2} + \cdots + {}_rC_{p-1}9^{r-p+1}].$$

When $r \geqq 2(p-1)$ we have ${}_rC_1 < {}_rC_2 < {}_rC_3 \cdots < {}_rC_{p-1}$; and hence,

$$N(r+1) \leqq 9\cdot{}_rC_{p-1}\cdot 9^r\cdot p = 9^{r+1}\cdot p\cdot\frac{r(r-1)\cdots(r-p+2)}{(p-1)!} \leqq 9^{r+1}\cdot p\cdot r^{p-1}.$$

Since each term in g_{r+1} is $\leqq 1/10^r$, we have, when $r \geqq 2(p-1)$,

$$g_{r+1} \leqq 9p\cdot\left(\frac{9}{10}\right)^r\cdot r^{p-1}.$$

Designating the right member of this inequality by u_{r+1}, we see that Σu_r is convergent, since

$$\lim_{r=\infty} \frac{u_{r+1}}{u_r} = \lim_{r=\infty} \frac{9}{10}\left(\frac{r+1}{r}\right)^{p-1} = \frac{9}{10} < 1.$$

Hence Σg_r is convergent, and therefore also S is convergent.

NOTE.—One may generalize the proposition thus:

The series S' whose terms are the reciprocals of all positive integers not containing any combination C whatever of the digits $0, 1, 2, \cdots, 9$ (which contains at least one digit) is convergent.

For, suppose C contains p digits. It may be shown that there are not more terms in a group g'_{r+1} in this series than in the corresponding group g_{r+1} previously discussed. When this is established the preceding argument proves the proposition.

When the terms of the series are the reciprocals of all the positive integers the series diverges. The proposition states that if from this divergent series certain terms are stricken out a convergent series is obtained. This is not particularly surprising when one observes that obviously a very large percentage of the integers of a thousand places, for example, contain a zero, and hence most of the terms far out in the series are stricken out.

Note.—This solution was received before the appearance of the article by Dr. Irwin on "A curious convergent series" in the May MONTHLY. Dr. Irwin there proves this generalized theorem by a somewhat different style of argument. He gives on page 50 some details of the reasoning by which equation (1) above may be established.—EDITORS.

Also solved by A. H. HOLMES and G. W. HARTWELL.

454. Proposed by C. N. SCHMALL, New York City.

Prove that a number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

SOLUTION BY FRANK R. MORRIS, Glendale, California.

Let $a, b, c, d, \cdots$ be the digits of any whole number, in order from right to left. Then this number may be written in the form

$$a + 10b + 100c + 1{,}000d + \cdots.$$

Dividing by 9 the quotient is

$$a/9 + (1 + 1/9)b + (11 + 1/9)c + (111 + 1/9)d + \cdots,$$

which equals

$$(b + 11c + 111d \cdots) + \frac{a + b + c + d + \cdots}{9}.$$

The first term of this quotient is integral and the second term is integral only if $a + b + c + d + \cdots$ is divisible by 9. Therefore, the number is divisible by nine if, and only if, the sum of its digits is divisible by nine.

This proof holds for a number containing a decimal fraction, as may be seen by converting the number into a whole number divided by some power of ten.

The above number may also be written

$$a + (9 + 1)b + (99 + 1)c + (999 + 1)d + \cdots$$

or

$$9b + 99c + 999d + \cdots + (a + b + c + d + \cdots),$$

which is a multiple of 9 when $(a + b + c + d + \cdots)$ is a multiple of 9.

Also solved by PAUL CAPRON, ELIJAH SWIFT, J. W. CLAWSON, G. L. WAGAR, H. C. FEEMSTER, W. F. RIGGS, NATHAN ALTSHILLER, E. F. CANADAY, C. A. BARNHART, O. S. ADAMS, G. W. HARTWELL, A. W. SMITH, E. E. WHITFORD, HORACE OLSON, A. H. HOLMES, W. J. THOME, H. S. UHLER, and H. N. CARLETON.

GEOMETRY.

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

SOLUTION (COMPLETED) BY HORACE OLSON, Chicago, Illinois.

Professor Hartwell's solution of this problem in the June issue of the MONTHLY is incomplete; it does not cover the case in which the given line is parallel to the radical axis of the circles.

This case may be considered in two parts, according as the given circles do or do not intersect in real points. If they intersect in real points, the required circle passes through three known real points, and can therefore readily be constructed.

If the given circles do not intersect in real points, we have $x^2 - r^2 = c^2$, where x is the distance from the radical axis to the center of any circle of the system of coaxial circles, r is its radius, and c is a real constant ascertainable from the given circles; c is, in fact, the length of the tangent from the intersection of the line of centers and the radical axis to any circle of the system. The distance d from the radical axis to the given line is either $x + r$ or $x - r$, according as it is $\gtreqless c$, x and r being here the x and r of the required circle. From these two equations, we find

$$x = \frac{d}{2} + \frac{c^2}{2d}, \qquad \text{and} \qquad r = \left|\frac{d}{2} - \frac{c^2}{2d}\right|.$$

Thus, the required circle is unique and can easily be constructed. Professor Hartwell's second solution becomes, in the cases I am considering, $x = \infty$, $r = \infty$; *i. e.*, the radical axis itself.

481. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the locus of the intersection of a pair of perpendicular normals to a parabola $y^2 = 4px$ is the parabola $y^2 = p(x - 3p)$.

SOLUTION BY F. M. MORGAN, Dartmouth College.

The line $y = mx - 2pm - pm^3$ is a normal to the parabola for all values of m. If x and y are considered as constants, then the roots of this equation are the directions of the normals that pass through the point (x, y). Call the roots m_1, m_2, m_3. Now, $m_1m_2m_3 = -(y/p)$. But if two normals are perpendicular, $m_1m_2 = -1$. Therefore, $m_3 = (y/p)$.

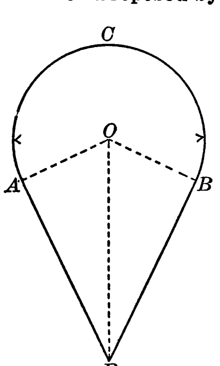
However, $y = m_3x - 2pm_3 - pm_3^3$; *i. e.*,

$$y = \frac{xy}{p} - 2y - \frac{y^3}{p^2},$$

which when simplified takes the form $y^2 = p(x - 3p)$.

Also solved by NORMAN ANNING, H. R. HOWARD, ELIJAH SWIFT, C. K. ROBBINS, J. A. BULLARD, H. L. AGARD, S. E. RASOR, T. O. WALTON, R. W. LORD, HORACE OLSON, C. M. SPARROW, G. W. HARTWELL, R. A. JOHNSON, C. A. EPPERSON, C. E. DIMICK, H. S. UHLER, and the PROPOSER.

482. Proposed by ROBERT G. THOMAS, The Citadel, Charleston, S. C.



In laying out a kite-shaped mile race-track, composed of a circular arc and two intersecting tangents at the ends of the arc, determine the angle at the center of the arc (*a*) when the length of the arc equals the sum of the two tangents, and (*b*) when the arc is equal to the length of each tangent.

SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

Let AP and BP be the two tangents, and ACB the arc composing the race-track; O the center, and OB the radius of the arc. Let $BOP = \varphi$, $OB = r$, and $AP = BP = t$. Then, the length of the arc $ACB = r(2\pi - 2\varphi)$. By the conditions in (*a*), (1) $2r(\pi - \varphi) = 2t$, and (2) $r = t/\tan\varphi$. Eliminating r between (1) and (2), we obtain (3) $\varphi + \tan\varphi = \pi$. By the conditions in (*b*), we have (4) $2r(\pi - \varphi) = t$. Eliminating r between (2) and (4), we obtain (5) $2\varphi + \tan\varphi = 2\pi$. Solving equations (3) and (5) by the Method of Successive Approximations, we find from (3), angle $\varphi = 63° 45' 38.657''$. Whence, angle at center of arc $= 2(\pi - \varphi) = 232° 28' 42.686''$, answer to (*a*). From (5), angle $\varphi = 74° 46' 14.636''$. Whence, angle at center of arc $= 2(\pi - \varphi) = 210° 27' 30.728''$, answer to (*b*).

NOTE.—By the conditions in (*a*) or (*b*) the angle of the arc is independent of the total length of the race-track.

Also solved by C. E. DIMICK, H. L. AGARD, G. PAASWELL, A. H. HOLMES, PAUL CAPRON, W. C. EELLS, C. K. ROBBINS, W. J. THOME, J. A. BULLARD, H. S. UHLER, and the PROPOSER.

CALCULUS.

342. Proposed by CLARIBEL KENDALL, University of Colorado.

Referring to Poincaré's *Science and Hypothesis*, page 65, show that the path between two points in Poincaré's ideal world, requiring the least number of steps of beings such as exist on earth, is the arc of a circle cutting the boundary of the Poincaré world orthogonally.

SOLUTION BY B. F. FINKEL, Drury College.

Poincaré in his *Science and Hypothesis*, page 49, 1905 edition, says: "Suppose a world enclosed in a great sphere and subject to the following laws:

"The temperature is not uniform; it is greatest at the center, and diminishes in proportion to the distance from the center to sink to absolute zero when the sphere is reached in which this world is enclosed.

"To specify still more precisely the law in accordance with which this temperature varies: Let R be the radius of the limiting sphere; let r be the distance of the point considered from the center of this sphere. The absolute temperature shall be proportional to $R^2 - r^2$.

"I shall further suppose that, in this world, all bodies have the same coëfficient of dilatation so that the length of any rule is proportional to its absolute temperature.

"Finally, I shall suppose that a body transported from one point to another of different temperature is put immediately into thermal equilibrium with its new environment."

To make this problem concrete, suppose a man in this ideal world wishes to go from his house to his barn along the shortest path. From the nature of the problem, it is clear that the required path will lie in the plane of the house, barn, and center of the sphere. We shall assume this to be the case. Taking the center of the sphere as origin of rectangular coördinates, let (x_0, y_0) be the coördinates of the house and (x_1, y_1) the coördinates of the barn. Let the length of a step be $k(R^2 - r^2)$. Then if ds is the length of an element of the path, $ds/k(R^2 - r^2)$ is the number of steps taken in traversing this element, and the total number of steps taken in going from the house to the barn is

$$\int_{x_0}^{x_1} \frac{ds}{k(R^2 - r^2)} = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}\,dx}{k(R^2 - x^2 - y^2)}, \qquad \text{where} \qquad y' = \frac{dy}{dx}.$$

This integral is to be made a minimum. The necessary condition for a maximum or minimum is

$$\frac{dF}{dy} - \frac{d^2F}{dxdy'} = 0, \qquad \text{where} \qquad F = \frac{1}{k}\frac{\sqrt{1 + y'^2}}{R^2 - x^2 - y^2}.$$

Hence, we have

$$\frac{2y\sqrt{1 + y'^2}}{(R^2 - x^2 - y^2)^2} - \left(\frac{\dfrac{d^2y}{dx^2}}{(1 + y'^2)^{3/2}(R^2 - x^2 - y^2)} + \frac{2(xy' + yy'^2)}{\sqrt{1 + y'^2}(R^2 - x^2 - y^2)^2} \right) = 0,$$

or

$$(R^2 - x^2 - y^2)\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \frac{2\left(y - x\dfrac{dy}{dx}\right)}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}.$$

Transforming to polar coördinates by the relations $x = \rho\cos\theta$, $y = \rho\sin\theta$, we have

$$R^2\left[\rho^2 - \rho\frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2\right] + \rho^2\left[\rho^2 + \rho\frac{d^2\rho}{d\theta^2}\right] = 0.$$

Let $\rho = R\epsilon^z$. We then have, after some simplification,

$$\frac{\dfrac{d^2z}{d\theta^2}}{1 + \left(\dfrac{dz}{d\theta}\right)^2} = \frac{\epsilon^{-z} + \epsilon^z}{\epsilon^{-z} - \epsilon^z}.$$

Whence, multiplying both sides by $2dz$, and integrating, we have

$$\log\left[1 + \left(\frac{dz}{d\theta}\right)^2\right] = \log\frac{C_1^2}{(\epsilon^{-z} - \epsilon^z)^2}.$$

Hence,

$$1 + \left(\frac{dz}{d\theta}\right)^2 = \frac{C_1^2}{(\epsilon^z - \epsilon^{-z})^2}$$

and

$$\frac{dz}{d\theta} = \pm\frac{\sqrt{C_1^2 - (\epsilon^z - \epsilon^{-z})^2}}{\epsilon^z - \epsilon^{-z}}.$$

Using the $+$ sign, we have

$$d\theta = \frac{(\epsilon^z - \epsilon^{-z})dz}{\sqrt{C_1^2 - (\epsilon^z - \epsilon^{-z})^2}} = \frac{(\epsilon^z - \epsilon^{-z})dz}{\sqrt{C_1^2 + 4 - (\epsilon^z + \epsilon^{-z})}}.$$

Whence, on integration, we have

$$\theta + C_2 = \sin^{-1}\left(\frac{\epsilon^z + \epsilon^{-z}}{\sqrt{C_1^2 + 4}}\right),$$

or

$$\epsilon^z + \epsilon^{-z} = \sqrt{C_1^2 + 4}\sin(\theta + C_2) = 2C\sin(\theta + C_2),$$

where

$$2C = \sqrt{C_1^2 + 4}.$$

Replacing ϵ^z by ρ/R and simplifying, we have

$$\rho^2 - 2CR\rho\sin(\theta + C_2) + R^2 = 0.$$

Transforming to rectangular coördinates, we have

$$x^2 + y^2 - 2CR\sin C_2x - 2CR\cos C_2y + R^2 = 0,$$

the equation of a circle orthogonal to the circle $x^2 + y^2 = R^2$.

380. Proposed by C. N. SCHMALL, New York City.

Show that

$$\int_0^\infty\left[\frac{1}{1^4 + x^2} + \frac{1}{2^4 + x^2} + \frac{1}{3^4 + x^2} + \cdots\right]dx = \frac{\pi^3}{12}$$

where the series in the brackets is infinite.

Criticism on the solution published in the January, 1916, MONTHLY, page 23, by T. H. GRONWALL, New York City.

The solution as published is open to criticism, inasmuch as the integrability term by term of a uniformly convergent series is assumed for an *infinite* interval of integration. Now the familiar argument runs thus:

$$S(x) = \sum_1^\infty s_n(x) = \sum_1^N s_n(x) + R_N(x),$$

N an integer, $|R_N(x)| < \epsilon$ uniformly for $a \leqq x \leqq b$, ϵ arbitrarily small, and N sufficiently large $(N \geqq N_0(\epsilon))$; whence

$$\left|\int_a^b S(x)dx - \sum_1^N\int_a^b s_n(x)dx\right| = \left|\int_a^b R_N(x)dx\right| < \epsilon(b - a),$$

from which the term by term integrability of $S(x)$ follows when a and b are finite. The above inequality has, however, no sense when $b = \infty$, so that the term by term integrability in an infinite interval demands further investigation. If one does not want to invoke general theorems in this direction, such as are given, for instance, in Bromwich's *Infinite Series* or de la Vallee Poussin's *Cour d'Analyse,* one may proceed as follows in this problem.

Let

$$S(x) = \sum_1^\infty \frac{1}{n^4 + x^2}$$

be uniformly convergent for $x \geqq 0$; then the term by term integration in the finite interval $0 \leqq x \leqq a$ is legitimate:

$$\int_0^a S(x)dx = \sum_1^\infty \frac{1}{n^2} \arctan \frac{a}{n^2}.$$

Now $\arctan (a/n^2) < (\pi/2)$ and all the terms of the right-hand series are positive; hence, for any $N \geqq 1$

$$\sum_1^N \frac{1}{n^2} \arctan \frac{a}{n^2} < \int_0^a S(x)dx < \frac{\pi}{2} \sum_1^\infty \frac{1}{n^2}.$$

Letting a tend toward infinity, *while keeping N fixed,* we find

$$\frac{\pi}{2} \sum_1^N \frac{1}{n^2} \leqq \int_0^\infty S(x)dx \leqq \frac{\pi}{2} \sum_1^\infty \frac{1}{n^2},$$

and finally letting N tend to infinity,

$$\int_0^\infty S(x)dx = \frac{\pi}{2} \sum_1^\infty \frac{1}{n^2} = \frac{\pi^3}{12}.$$

This point, that great caution should be exercised when inverting summation and integration in an infinite integration interval, or when the integrands may become infinite between the finite limits of integration, appears to me to be all the more important as the treatment of such questions in the current calculus textbooks is exceedingly inadequate.

393. Proposed by LAENAS G. WELD, Pullman, Ill.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

II. Solution by Frank R. Morris, Glendale, Calif.

In this solution the bricks are considered laid as in practical work.

Consider two of the bricks placed end to end in the same layer, a third in the layer above with its middle at the joint of the first two, and the fourth similarly placed in the layer below. Select as the origin the mid-point of the joint between the two bricks in the same layer and choose the x-axis parallel to the layers. Let the equation of the desired ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since the four quadrants are symmetrical, it will be sufficient to consider the first one only. Call the upper, right-hand corner of the uppermost brick A and the corresponding corner of right-hand brick of the middle layer B. Let m represent the length of a brick and n its thickness. The ellipse will obviously pass through A or B or both. If it passes through A, whose coördinates are $(m/2, 3n/2)$, the equation is

$$\frac{\left(\frac{m}{2}\right)^2}{a^2} + \frac{\left(\frac{3n}{2}\right)^2}{b^2} = 1, \qquad \text{or} \qquad \frac{m^2}{a^2} + \frac{9n^2}{b^2} = 4.$$

Whence,

$$b = \frac{3an}{\sqrt{4a^2 - m^2}}, \qquad \text{and the area } \pi ab \text{ is} \qquad \frac{3\pi a^2 n}{\sqrt{4a^2 - m^2}},$$

which is a function, $f(a)$, of the independent variable a. Differentiating, and setting the derivative equal to 0, we have

$$f'(a) = \frac{6\pi a(2a^2 - m^2)}{(4a^2 - m^2)^{3/2}} = 0.$$

Whence $a = 0$ or $m/\sqrt{2}$.

$f'(a)$ increases continuously as a increases beyond $m/\sqrt{2}$. Hence, the least ellipse which will inclose the four bricks will be the one in which a is as near $m/\sqrt{2}$ as possible. The smallest value of a is given when the ellipse passes through the point B, this value being greater than $m/\sqrt{2}$.

It is perhaps evident that the least ellipse passing through B would pass through A also. It could be proved in the same manner as the first case.

The equation of the ellipse through B is

$$\frac{m^2}{a^2} + \frac{\left(\frac{n}{2}\right)^2}{b^2} = 1 \quad \text{or} \quad \frac{4m^2}{a^2} + \frac{n^2}{b^2} = 4.$$

Solving this simultaneously with the equation derived by substituting the coördinates of A for a and b we have

$$a = \frac{1}{4}\sqrt{\frac{35}{2}}\,m, \qquad b = \frac{1}{2}\sqrt{\frac{35}{3}}\,n.$$

Hence the area πab is

$$\frac{35\pi\sqrt{6}}{48}\,mn.$$

If allowance is made for mortar on all sides of each brick the solution is unchanged except m and n are increased by the thickness of the mortar. If the allowance is made at the joints of the four bricks but not at the points A and B, these points are $((m + t)/2,\ (3n + 2t)/2)$ and $((2m + t/2),\ n/2)$ where t is the thickness of the mortar.

MECHANICS.

317. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the maximum area contained between the path of a projectile and the horizontal line is $v^2\sqrt{3}/8g^2$, where v is the velocity of projection.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

If the particle is projected in a direction making an angle α with the horizontal, the equations of motion are, t being the time,

$$x = v\cos\alpha\cdot t \quad \text{and} \quad y = v\sin\alpha\cdot t - \tfrac{1}{2}gt^2.$$

Equating y to zero, we find the path cuts the horizontal line through the point of projection when $t = 0$ and $t = \dfrac{2v\sin\alpha}{g}$. The area is given by $\displaystyle\int_{t=0}^{t=\frac{2v\sin\alpha}{g}} y\,dx = \frac{2v^4\cos\alpha\sin^3\alpha}{3g^2}$, which is to be a maximum. Setting the first derivative with respect to α equal to 0, we find $\alpha = 60°$. The maximum value of the area is then $= \dfrac{v^4\sqrt{3}}{8g^2}$ (v^2 being incorrect as given in the problem).

Also solved by J. L. RILEY, W. C. EELLS, THEODORE HOWARD, PAUL CAPRON, H. L. AGARD, HORACE OLSON, H. S. UHLER, and ELMER SCHUYLER.

318. Proposed by C. N. SCHMALL, New York City.

Given an inclined plane making an angle φ with the horizontal. A perfectly elastic ball is projected upward at an angle ψ with the inclined plane, so as to ascend it by bounds. Show that as the ball rebounds for the nth time, the angle of inclination of its path to the plane is

$$\tan^{-1}\left(\frac{\tan\psi}{1 - 2n\tan\varphi\tan\psi}\right)$$

and if it rebounds vertically upward, then

$$\cot \psi = (2n + 1) \tan \varphi.$$

SOLUTION BY IMMANUEL KLAUFF, Chicago, Ill.

The equation of the path is, if we let $\varphi + \psi = \alpha$,

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}.$$

The given inclined plane intersects this parabola at P_1 and the slope of the parabola at P_1 is

$$\left(\frac{dy}{dx}\right)_{P_1} = 2 \tan \varphi - \tan \alpha = \tan \omega.$$

The angle of inclination of the first rebound is

$$\omega_1 + \varphi = \gamma_1.$$

where ω_1 is the complement of ω. We have

$$\tan \gamma_1 = \frac{\tan \omega_1 + \tan \varphi}{1 - \tan \omega_1 \tan \varphi} = \frac{\tan \varphi - \tan \omega}{1 + \tan \omega \tan \varphi} = \frac{\tan \alpha - \tan \varphi}{1 - 2 \tan^2 \varphi + \tan \varphi \tan \varphi} = \frac{\tan \psi}{1 - 2 \tan \psi \tan \varphi}.$$

The analysis for the subsequent rebounds is the same; the angle of projection with respect to the inclined plane is for the nth rebound

$$\gamma_n = \tan^{-1} \left[\frac{\dfrac{\tan \psi}{1 - 2(n-1) \tan \psi \tan \varphi}}{1 - \dfrac{2 \tan \psi \tan \varphi}{1 - 2(n-1) \tan \psi \tan \varphi}} \right] = \tan^{-1} \left(\frac{\tan \psi}{1 - 2n \tan \psi \tan \varphi} \right).$$

If the ball rebounds vertically, we have

$$\tan (\gamma_n + \varphi) = \infty, \qquad \text{or} \qquad 1 - \frac{\tan \psi \tan \varphi}{1 - 2n \tan \psi \tan \varphi} = 0.$$

Whence, $1 - (2n + 1) \tan \psi \tan \varphi = 0$; and $\cot \psi = (2n + 1) \tan \varphi$.

Also solved by H. S. UHLER, HORACE OLSON, and the PROPOSER.

NUMBER THEORY.

185. Proposed by R. D. CARMICHAEL, Bloomington, Indiana.

Obtain the complete solution of the equation $\phi(p^\alpha) = \phi(q^\beta)$ where ϕ denotes Euler's ϕ-function, p and q are unknown primes, and α and β are unknown integers.

CRITICISM BY T. H. GRONWALL, New York City.

The solution of this problem on page 227, Volume XX, September, 1913, is open to criticism. From the equation:

$$p^{\alpha-1}(p - 1) = q^{\beta-1}(q - 1),$$

where p and q are primes, and $p > q$, it follows that

$$p - 1 = \lambda q^{\beta-1}, \qquad q - 1 = \lambda p^{\alpha-1}$$

where λ is a positive integer. Since $q < p$, the second equation shows that $\alpha = 1$, and consequently $\lambda = q - 1$,

$$p = 1 + (q - 1)q^{\beta-1},$$

and here the prime q and the exponent β have to be chosen so that p becomes a prime. The error consists in assuming $\lambda = 1$ and hence $q = 2$; that this does not give all the solutions, is evident from the examples:

$$q = 2, \quad \beta = 2, 3, 5 \qquad q = 3, \quad \beta = 2, 3, 5 \qquad q = 5, \quad \beta = 3, 5$$
$$p = 3, 5, 17 \qquad p = 7, 19, 163 \qquad p = 101, 2501.$$

214. (April, 1914.) Proposed by A. J. KEMPNER, University of Illinois.

Let a be a positive integer $\geqq 2$, and let $T(n)$ denote the number of distinct divisors of the positive integer n, including both 1 and n, so that $T(1) = 1$, $T(2) = 2$, $T(3) = 2$, $T(4) = 3, \cdots$. Show that

$$\sum_{n=1}^{n=\infty} T(n)/a^n = \sum_{n=1}^{n=\infty} 1/(a^n - 1).$$

The special case $a = 10$ gives, as is easily seen:

$$9 \sum_{n=1}^{n=\infty} \frac{T(n)}{10^n} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots.$$

SOLUTION BY FRANK IRWIN, University of California.

We prove the proposition first for the special case, $a = 10$. We have,

$$1/9 = .11111111\cdots, \qquad 1/99 = .01010101\cdots,$$
$$1/999 = .001001001\cdots, \qquad 1/9999 = .00010001\cdots,$$

and so on.

Let us determine the sum of all the digits in the nth decimal place. Since the kth row of the above array reads

$$1/99\cdots99 = .00\cdots00100\cdots001\cdots,$$

with k 9's on the left and $(k - 1)$ 0's in each recurring period on the right, it follows that we get a 1 in the nth decimal place whenever k is a divisor of n, and otherwise a zero. We have, then, $T(n)$ units in this place, and since a unit there has the value $1/10^n$, their sum is $T(n)/10^n$; and the sum of our series, $1/9 + 1/99 + 1/999 + \cdots$, is equal to

$$\sum_{n=1}^{n=\infty} T(n)/10^n,$$

as was to be proved.

(It is clear that, regarding the array as a double series, we have a right to put the sum by rows equal to the sum by columns.)

In the general case, where we have any a, we need merely suppose the decimals above written in the scale of a, instead of in that of 10,

$$1/(a - 1) = .1111\cdots, \quad 1/(a^2 - 1) = .0101\cdots, \text{ etc.,}$$

and a like argument holds.

QUESTIONS AND DISCUSSIONS.

[Send all Communications to U. G. MITCHELL, University of Kansas, Lawrence, Kans.]

DISCUSSIONS.

I. RELATING TO NAPIER'S LOGARITHMIC CONCEPT.

BY H. S. CARSLAW, University of Sydney, Australia.

In the March number of the MONTHLY, page 71, Professor Cajori takes exception to the following remark, contained in a paper of mine in the *Mathematical Gazette* (Vol. VIII, page 77):

It is sometimes stated that Napier's Logarithms were obtained from the coördination of two definite series, an arithmetical and a geometrical. For instance, Cajori, in his recent paper on the "History of Logarithms," says:

"Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{cccccccc} v, & v\left(1-\frac{1}{v}\right), & v\left(1-\frac{1}{v}\right)^2, & \cdots & v\left(1-\frac{1}{v}\right)^n, & \cdots, \\ 0, & 1\quad, & 2\quad, & \cdots & n\quad, & \cdots. \end{array}$$

The numbers in the upper series represent successive values of the *sines*; the numbers in the lower series stand for the corresponding logarithms. Thus $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log [10^7(1 - 10^{-7})^n] = n$, where $n = 0, 1, 2, \cdots$."

This statement is incorrect. In Napier's Tables the logarithm of $(10^7 - 1)$ is not 1. It lies between 1 and 1.000,000,1, and he takes it as the mean between these two numbers, namely 1.000,000,05.

I am sorry that Cajori does not also quote the next sentence in my paper: "If Napier had used these two series in the way named above, his work would have more closely resembled that of Bürgi than actually is the case." These words, and the other references to Bürgi's Tables in my paper, were meant to direct attention to the distinction to which I referred.

Bürgi's work was entitled *Arithmetische und Geometrische Progresstabulen,* and the point I was anxious to make clear was that the fundamental conception in Napier's definition of a logarithm involved far more than the coördination of an arithmetical and a geometrical progression. If Napier's logarithms had been the numbers in the series

$$0,\ 1,\ 2,\ \cdots \quad \text{(A.P.)},$$

and his anti-logarithms (in our use of the term), the numbers

$$10^7,\ 10^7\left(1-\frac{1}{10^7}\right),\ 10^7\left(1-\frac{1}{10^7}\right)^2,\ \cdots \quad \text{(G.P.)},$$

then the only difference between his work and that of Bürgi would have been that the latter employed the series

$$\begin{array}{ccccccc} 10\times 0, & 10\times 1, & 10\times 2, & \cdots & 10\times n, & \cdots & \text{(A.P.)}, \\ 10^8, & 10^8\left(1+\frac{1}{10^4}\right), & 10^8\left(1+\frac{1}{10^4}\right)^2, & \cdots & 10^8\left(1+\frac{1}{10^4}\right)^n, & \cdots & \text{(G.P.)}, \end{array}$$

instead of the A.P. and G.P. which Cajori quotes.

It is, of course, true that geometrical series occur in Napier's work. The first 101 terms of the above series are, in fact, the numbers of his "First Table." And such series were employed in his *computation,* because, with his definition of a logarithm, the logarithms of numbers in geometrical progression have a constant difference.

Cajori argues in his last communication that it is in the *computation,* not in the *definition,* or *theory,* that Napier passes from the coördination of the arithmetical and geometrical progressions. In this I cannot agree with him. The kinematical definition of a logarithm (*cf. Constructio,* § 26) indicates, as I have

said in another place,[1] "that he had already in his mind, though it may be very dimly and vaguely, the principles on which Newton, some 50 years later, built up the differential calculus." And it is one of the most amazing features of Napier's work that his kinematical definition of a logarithm should have suggested itself to him before the invention of the calculus.

It seems to me that Cajori reads into our interpretation of the word logarithm —*the number of the ratios*—that the fundamental conception in Napier's work was the coördination of the arithmetical and geometrical series. My view rather is that Napier started with this coördination—the idea was not new—and very probably invented his word logarithm in connection with it. Later, I believe, he advanced to the much more general principle involved in his definition and treatment.[2] However, as Gibson remarks,[3] "Napier alone knew the derivation of the word, and dogmatism in the matter is out of place."

SYDNEY, AUSTRALIA,
April, 1916.

COMMENT BY FLORIAN CAJORI, Colorado College.

In the *Mathematical Gazette* of May, 1915, p. 78, Professor Carslaw criticized me for writing Nap. log $(10^7-1)=1$ instead of Nap. log $(10^7-1)=1.00000005$. In my reply[4] I quoted from Napier's *Constructio* to show that Napier himself considered both values as differing "insensibly" from the true value and that my statement was therefore not "incorrect." I made some other observations but made no reference to Bürgian logarithms, since they are irrevelant to the question.

In the present note Professor Carslaw no longer offers any objection to Nap. log $(10^7-1)=1$, but claims that I ignore the kinematical definition of logarithms given by Napier. While in my former reply I laid no emphasis upon that point (it had not been raised in Professor Carslaw's first criticism), I said enough to show that I had it in mind. What else could my reference to Napier's "theory of moving points" mean?

That Professor Carslaw should claim that I ignored Napier's kinematical definition in my *History of the Exponential and Logarithmic Concepts*, published three years ago, is an indication that he has not read with care my account of Napier. I lay very special emphasis upon that definition, where I say:[5]

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[1] *Proc. R. S. New South Wales*, Vol. XLVIII, p. 55, 1914.

[2] In his study of "Logarithms and Computation" in *The Napier Tercentenary Volume* the great authority on this subject—Dr. J. W. L. Glaisher—refers more than once to the fact that Napier's manner of conceiving a logarithm involved quite a new principle. *E. g.*, on p. 69 he says: "I find no difficulty in perceiving that logarithms might have been introduced at that time in such a manner, as we know that Jobst Bürgi did actually conceive antilogarithms, *i. e.*, as a correspondence between $(1.0001)^r$ and $10r$, for integral values of r, with interpolations; but Napier's conception of a logarithm was of a much more subtle kind, and involved the principle of a mathematical function."

[3] Cf. *The Napier Tercentenary Volume*, p. 115.

[4] AMERICAN MATHEMATICAL MONTHLY, vol. 23, p. 72.

[5] *Ibid.*, vol. xx, p. 6.

"Napier based his explanations upon two considerations: (1) the geometrico-mechanical concept of flowing points, (2) the relations which exist between arithmetic and geometric series. . . . Napier lets the point g move along the definite line TS with a diminishing velocity such that its velocity at T is to that at d, as the distance TS is to the distance dS. At the same time Napier lets a point a move along the line bc (which is of indefinite length) with a uniform velocity which is the same as the initial velocity of the point g. If the two points start to move at the same moment, and if g is at d when a is at c, then the length bc is defined as the logarithm of dS. Napier constructed tables for trigonometric computation. With that end in view he lets TS stand for the *radius*, assigning to it the value 10^7, while dS stands for a given *sine*. At that time trigonometric functions were not thought of strictly as *ratios*."

Then I give Napier's kinematical definition *in his own words;* it is the very definition in the *Constructio*, § 26, which Professor Carslaw says in the above communication that I ignore. What more could I do to emphasize this kinematical definition?

As an easy deduction from this kinematical definition is the one-one correspondence between arithmetic progressions and geometric progressions used by Napier in his computations. The part of Professor Carslaw's criticism which I consider valid is that Nap. log $(10^7 - 1) = 1.00000005$ exhibits "Table I" in Napier's *Constructio* somewhat more closely than does Nap. log $(10^7 - 1) = 1$. This I admitted in my former reply.

COLORADO SPRINGS,
June, 1916.

II. RELATING TO THE QUADRATIC FACTORS OF A POLYNOMIAL.

BY O. E. GLENN, University of Pennsylvania.

The process of finding by trial the rational roots of an equation $f(x) = 0$ with integral coefficients, depending upon the resolution of the last coefficient into its prime factors $s_1, s_2, \cdots$ and the first coefficient into its prime factors $r_1, r_2, \cdots$, and testing by division the possible factors $x - s/r$, is given as an isolated method in books on elementary theory of equations, with no suggestion that it is capable of extensions. As an incident in a published paper[1] on Degenerate Curves I have shown, however, that this process is a particular manifestation of a general one in which, by a finite number of arithmetical trials, one can determine any polynomial $g(x)$ with integral coefficients which is a factor of $f(x)$. As applied to quadratic factors of $f(x)$, and especially when $f(x)$ is a quartic, the extended method is very useful.

By multiplying the roots of $f(x) = 0$ by a properly chosen integer we can assume it in the form

$$f(x) = x^n + p_1x^{n-1} + \cdots + p_n = 0,$$

where the p's are integers. Let a quadratic factor of $f(x)$, with integral coefficients, be $x^2 - \xi x - \eta$. If the roots of $f(x) = 0$ are $-x_1, -x_2, \cdots, -x_n$, we may assume $-\xi = x_1 + x_2$.

[1] *Amer. Journal of Math.*, vol. 32 (1910), p. 79.

We can then prove the following result, under the hypotheses:
The integer ξ must be a prime factor of the integer

$$\begin{aligned} P_n = (x_1 + x_2)(x_1 + x_3) \cdots (x_1 + x_n) \\ (x_2 + x_3) \cdots (x_2 + x_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (x_{n-1} + x_n), \end{aligned}$$

or a product of prime factors of P_n.

The number P_n is an integer because it is a symmetric function of the roots and hence is rational and integral in the integers $p_1, p_2, \cdots, p_n$. The fact that ξ divides P_n *algebraically* does not, however, prove the proposition stated. We prove it as follows: Let the equation whose roots are the $(n) = \frac{1}{2}n(n-1)$ numbers

$$-(x_i + x_j) \qquad (i < j)$$

be

(1) $$\xi^{(n)} + \varphi_1 \xi^{(n)-1} + \varphi_2 \xi^{(n)-2} + \cdots + \varphi_{(n)} = 0.$$

Then all of the functions φ_i, being symmetric in the roots, x_j, are integers, and $\varphi_{(n)} = P_n$. Now ξ, an integer, satisfies this equation, and is a factor of all terms up to the last. Hence ξ is a prime factor of P_n or a product of its prime factors. Likewise η is a factor of p_n.

It is now evident that in order to discover any quadratic factor of the required type of $f(x)$ we have only to resolve P_n and p_n into their prime factors and, with a factor of P_n as ξ, and a factor of p_n as η, test the resulting quadratic for divisibility into $f(x)$.

In the case of a quartic polynomial $f(x) = x^4 + p_1x^3 + p_2x^2 + p_3x + p_4$, we find

$$P_4 = p_1p_2p_3 - p_1^2p_4 - p_3^2.$$

If P_4 or p_4 contain a large number of prime factors the number of trial divisions required to apply the method may be reduced by making use of the formulas for upper limits to the roots. If $L \nless 1$ is a superior limit to the absolute values of the roots, then $2L$ is a superior limit to $|\xi|$ and L^2 a superior limit to $|\eta|$. It is important to note, as well, that in the case of the quartic the ξ's of the two quadratic factors, as ξ_1, ξ_2, must satisfy the condition $\xi_1 + \xi_2 = -p_1$. These conditions often render the number of tests required to resolve a quartic as small as the number required in case of the linear factors.

For illustration consider the quartic

$$f(x) = x^4 - 6x^3 + 3x^2 + 22x - 6 = 0,$$

whose solution, in condensed form, under the theory where the reducing cubic has a commensurable root, is given on page 243 in Volume I of Burnside and Panton's Theory of Equations (sixth edition).

We have

$$P_4 = -2^3.83, \qquad p_4 = -2.3,$$

and 6 is a superior limit to the roots. Hence

$$\xi = \pm 1, \pm 2, \pm 4, \pm 8; \qquad \eta = \pm 1, \pm 2, \pm 3, \pm 6.$$

But $\xi_1 + \xi_2 = 6$, hence the pair (ξ_1, ξ_2) is either $(-2, 8)$ or $(2, 4)$, *i. e.*, $\xi = 2$ or -2. The maximum number of unsuccessful trials is 15. With $\xi = 2$, $\eta = 6$, we have, employing an obvious extension of synthetic division,

$$\begin{array}{cccccc|l}
\underline{1} & -6 & +3 & +22 & -6 & & \underline{2+6} \\
1 & \underline{2} & 6 & & & & \\
 & -\underline{4} & -\underline{8} & -24 & & & \\
 & & 1 & 2 & 6 & & \\
 & & & \overline{0} & \overline{0} & &
\end{array}$$

Thus the remainder $ax + b$ is zero and

$$f(x) = (x^2 - 2x - 6)(x^2 - 4x + 1).$$

Further illustrations of quartics with irrational roots follow.

Let

$$f(x) = x^4 - x^3 - 6x^2 + 5x + 3 = 0.$$

Then

$$P_4 = 2, \; p_4 = 3; \quad \xi = \pm 1, \pm 2; \quad \eta = \pm 1, \pm 3.$$

The only pair (ξ_1, ξ_2) for which $\xi_1 + \xi_2 = 1$ is $(-1, 2)$. Hence the maximum number of unsuccessful trials is three. We find

$$f(x) = (x^2 + x - 3)(x^2 - 2x - 1).$$

Exercise 1.—Show that the equation

$$x^4 + 4x^3 - 4x^2 - 17x + 10 = 0$$

can be resolved by less than nine trial divisions, and that the roots are $\frac{1}{2}(-1 \pm \sqrt{21})$, $\frac{1}{2}(-3 \pm \sqrt{17})$.

Exercise 2.—Solve the equation

$$x^4 + 4x^3 + 4x^2 - 16x + 28 = 0.$$

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

The Secretary of the ASSOCIATION, Professor W. D. CAIRNS, is on leave of absence from Oberlin College during the present academic year. Until further notice he will be in residence at the University of Chicago and his official address will be **5465 Greenwood Avenue, Chicago, Illinois.**

At Dartmouth College, Dr. C. N. HASKINS has been promoted to be Professor of Mathematics on the Chandler Foundation.

Dr. R. W. BURGESS, formerly of Cornell University, has been appointed to an instructorship at Brown University.

Dr. NATHAN ALTSHILLER, who was last year at the University of Colorado, has become instructor in mathematics at the University of Oklahoma.

Dr. A. A. BENNETT, formerly instructor at Princeton University, has been appointed adjunct professor of mathematics at the University of Texas.

Mr. F. E. WOOD and Mr. GLEASON, graduate students at Princeton University, have been appointed to instructorships at Northwestern University.

Mr. F. L. Smith, formerly instructor at Northwestern University, is now instructor in mathematics at Princeton University.

Dr. R. E. Gilman, formerly instructor at Princeton University, has been appointed to an instructorship at Cornell University.

Mr. C. M. Reynolds, Jr., has been appointed to an instructorship in mathematics at Wesleyan University, Middletown, Conn.

Mr. Carl A. GARABEDIAN goes to New Hampshire State College for the coming year as instructor in mathematics.

Professor EVA S. MAGLOTT, for the past thirty-two years professor of mathematics in Ohio Northern University, died during the summer vacation.

Mr. L. L. LOCKE, of the Brooklyn Training School for Teachers, conducted classes in the history and pedagogy of mathematics at the summer session of Grove City (Pa.) College.

Mr. H. W. MYERS, who has recently taken the master's degree at the University of Chicago, has been elected to the professorship of mathematics at Huron College, Huron, South Dakota.

Dr. A. M. HARDING has returned to the University of Arkansas as professor of mathematics and university examiner after a year's leave of absence spent in study at the University of Chicago, where he received the doctorate in June, 1916.

At the University of Nebraska, Mr. A. H. GIST has been transferred from the department of mathematics to that of physics and Dr. ELIZABETH B. GRENNAN has resigned. Mr. Albert BABBITT and Miss MARY COLPITTS have been appointed to instructorships in mathematics.

Dr. J. C. DUNCAN, instructor at Harvard University, has been appointed professor of astronomy and director of the Whitin Observatory at Wellesley College.

On account of the war, many of the European periodicals are weeks and even months late in arriving in this country. It is reported that *L'Education Mathématique* and *Revue de Mathématiques Spéciales* have suspended publication.

AT the University of Kansas, Dr. SOLOMON LEFSCHETZ and Mr. J. J. WHEELER, instructors in mathematics, have been promoted to assistant professorships, and Mr. E. B. MILLER has been appointed instructor in mathematics for the year 1916–17.

Dr. JOHN P. D. JOHN, former president of De Pauw University, Greencastle, Ind., died on August 7, at the age of seventy-three. Dr. John served as professor of mathematics at De Pauw from 1882 to 1889, when he became president. He served as president from 1889 to 1895, resigning to go upon the lecture platform.

Dr. K. P. WILLIAMS, assistant professor of mathematics at Indiana University, served as first lieutenant in the first regiment of Indiana troops now stationed on the Mexican border at Llano Grande, Texas. It was ordered, however, that troops constituting college units should be mustered out of service on September 1, that the men might continue their college work.

The Mathematical Gazette, May, 1916, contains an interesting article by Professor H. S. CARSLAW on "A progressive income-tax," a scheme of taxation introduced in Australia. Schedules are deduced for "Rate of tax upon income derived from personal exertion," and "Rate upon income derived from property." The same number of the *Gazette* also contains a paper by E. H. NEVILLE on the "So-called cases of failure in the solution of linear differential equations."

According to cable reports from London, the Council of Trinity College, Cambridge, has removed Professor BERTRAND RUSSELL from his lectureship in logic and principles of mathematics on account of his having been convicted under the defense of the realm act for publishing a leaflet defending the "Conscientious Objector" to service in the British army. Professor Russell is well known in this country through his mathematical writings.

The *Annals of Mathematics*, June, 1916, contains the following: "On the wronskian test for linear dependence," by M. BÔCHER; "Note on a theorem of envelopes," by W. R. LONGLEY; "Non-essential singularities of functions of several variables," by D. JACKSON; "A congruence of circles," by F. W. BEAL; "A case of iteration in several variables," by A. A. BENNETT; "The arithmetic genus of an algebraic manifold immersed in another," by S. LEFSCHETZ; "A characteristic property of self-projective curves," by L. L. DINES.

Professor E. D. GRANT, of the Michigan College of Mines, Houghton, Mich., was given the degree of Ph.D. in mathematics at the September Convocation of the University of Chicago. Others receiving the doctorate in mathematics at this time were Mr. ARCHIE S. MERRILL, who goes to the University of Montana as assistant professor, and Miss GILLIE LAREW, who returns to her post as professor of mathematics in Randolph-Macon Woman's College.

Hunter College for Women, New York City, maintains a lively Mathematical Club, to membership in which all students specializing in mathematics are eligible. The club has for its aim the study and investigation of mathematical subjects which are crowded out of the ordinary curriculum. Reports are given once a month on such topics as: "Non-Euclidean geometry," "Geometry of motion," "Hyperspace," "The golden age of mathematics," "The nature of mathematics," "Zero," "Early surveying instruments," "The earliest mathematicians," etc. Meetings are held once a month, and a small membership fee is charged.

The attention of members is called to a proposed agreement between the Association and the *Annals of Mathematics*, to be found elsewhere in this issue among the actions of the Council. This agreement will not go into effect unless it is finally ratified by the Council after the views of the members have been secured. Comments should be mailed to the Secretary at an early date. (His present address is 5465 Greenwood Avenue, Chicago, Ill.)

Articles of an expository nature have appeared from time to time in the *Annals*, many of which have been of real importance and of great value to mathematics in this country. It is felt that a greater opportunity for publication of articles of this type is much needed, and that the encouragement of such work is a proper function of the Association. The further opportunity to assist in the promotion of such a worthy project seems very attractive to many of those who have given it consideration.

The advantage secured to the members of the Association whereby the enlarged *Annals* may be obtained at a price actually less than the present price will certainly commend itself to all, and it is hoped that, if the arrangement is finally consummated, many members will avail themselves of this opportunity.

The plan mentioned above will in no wise interfere with the present policy of the MONTHLY. The papers which properly appear in the MONTHLY are necessarily brief, since a variety of articles of different types is needed. In fact, the pressure upon the MONTHLY by the large number of good articles of this sort and by the numerous other necessary features well illustrated in the present issue, has already caused the expansion of the MONTHLY by at least twenty per cent. in the current volume, and still further enlargement will be required in the near future.

The attendance at the first summer meeting of the Association was most gratifying, especially in view of the fact that the threatened strike on the rail-

roads came just at the time when those at a distance were preparing to start. Many were thus deterred from going who had definitely planned to be present. Nevertheless, the 111 members in attendance were widely distributed and represented all parts of the country in about the proportion that might be expected; namely 48 from New England, 27 from the Middle States, 27 from the Middle West, five from the South, and three from the far West.

The twenty-third summer meeting and eighth colloquium of the American Mathematical Society were held at Harvard University during the week of September 4–9, 1916. About ninety-eight members were in attendance and over forty papers were presented. The freshman dormitories of Harvard University were opened for the accommodation of the members, and in this way all were thrown together in social intercourse far more effectively than would have been the case if they had been scattered about in the Boston hotels.

Eighty-four persons attended the dinner on Monday evening, where the after-dinner addresses were devoted chiefly to the early history of the Society, it being the twenty-fifth anniversary of its organization as a national body. Professor THOMAS S. FISKE, who was the moving spirit in organizing the New York Mathematical Society in 1888, was also chiefly instrumental in the movement for expanding this local body into the American Mathematical Society in 1891, though this name was not actually taken until 1894. Professor Fiske, being the member of longest standing present at this meeting, was appropriately selected to speak on the early history of the Society, and he gave many interesting items of a personal character, including the reading of letters written to him as Secretary by Professors A. R. FORSYTH, ARTHUR CAYLEY, and J. W. L. GLAISHER, of Cambridge, England, and by Professors SIMON NEWCOMB, HENRY B. FINE, WOOLSEY JOHNSON, and DR. EMORY MCCLINTOCK, of this country. Professors Johnson and Fine were also present and gave personal reminiscences in response to the call of the toastmaster. It was made clear that the founding of the *Bulletin*, and later of the *Transactions*, had been the two great steps taken by the Society in establishing its place in the mathematical world as a producer of research of a high character. Other speakers were Professor G. D. BIRKHOFF, who acknowledged the great indebtedness to the older men who have made possible the present high standards of the Society and spoke of the great responsibility resting upon the younger men for maintaining and advancing these standards; Professor F. N. COLE, who praised the constitution and harmonious working of the Society and the devotion and willing service of its officers past and present; Professor A. G. WEBSTER, who is one of the Society's representatives on the United States Naval Board; Professor J. L. COOLIDGE, who extolled the ideals of the mathematicians; and Professor E. R. HEDRICK, who spoke of the relations of the Society to its younger sister, the Mathematical Association of America, and pledged the support of the new Association in every good work within its field for the advancement of the interests of mathematics in this country.

On Tuesday evening, Professor D. E. Smith entertained the members in an

informal gathering by reading an exceedingly interesting paper on "The History of Problems."

The colloquium lectures were delivered by Professor G. C. Evans of Rice Institute on "Topics from the Theory and Applications of Functionals, including Integral Equations"; and Professor Oswald Veblen of Princeton University on "Analysis Situs." Each gave five lectures, the two courses extending over the three days, September 6 to 8. This was the eighth colloquium which the Society has held, the last one before this having been at the University of Wisconsin two years ago. These courses of lectures are now published in book form by the Society, and thus a series of volumes quite unique in character in the research field is coming into existence.

This was unanimously regarded as the most successful summer meeting ever held by the Society, the attendance at both this meeting and that of the Association having undoubtedly been stimulated by the juxtaposition of the two.

On the second and third following pages are reprinted the Constitution and By-Laws of the MATHEMATICAL ASSOCIATION OF AMERICA, together with the list of officers and members of the Council for 1916. Special attention is called to Article III, Sections 1 and 2, of the Constitution and Section 2 of the By-Laws (concerning officers, tenure of office, and election of officers) by way of preparation for the first application of these regulations to be made about November first. The provision for nomination of officers through open primaries was intended to emphasize the opportunity presented to every member for active participation in the affairs of the Association.

Further attention is called to Article III, Section 3, of the Constitution which deals with the method of transacting the official business of the Association. While the Council, as a representative body, is vested with full authority in all matters, yet it may not "make or alter any question of policy" without first giving all members of the ASSOCIATION ample opportunity for the expression of opinion for or against such proposed action. The first opportunity for individual participation under this provision of the Constitution is now offered in connection with the proposed agreement with the *Annals of Mathematics* explained on page 288 of this issue.

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

ARTICLE I—NAME AND PURPOSE.

1. This organization shall be known as THE MATHEMATICAL ASSOCIATION OF AMERICA.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

ARTICLE III—OFFICERS.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

ARTICLE IV—MEETINGS.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

ARTICLE V—SECTIONS.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

ARTICLE VI—OFFICIAL JOURNAL.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

ARTICLE VII—DUES.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

ARTICLE VIII—AMENDMENTS.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who were admitted to membership before April 1, 1916, constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

Following are the Officers of the Association for 1916

President, E. R. HEDRICK, University of Missouri;

Vice-Presidents, E. V. HUNTINGTON, Harvard University, and
G. A. MILLER, University of Illinois;

Secretary-Treasurer, W. D. CAIRNS, Oberlin College;

Additional members of the Executive Council:

To serve for one year

D. N. LEHMER, University of California
R. E. MORITZ, University of Washington
K. D. SWARTZEL, Ohio State University
OSWALD VEBLEN, Princeton University

To serve for two years

R. C. ARCHIBALD, Brown University
FLORIAN CAJORI, Colorado College
M. B. PORTER, University of Texas
J. W. YOUNG, Dartmouth College

To serve for three years

B. F. FINKEL, Drury College
E. H. MOORE, University of Chicago
J. N. VAN DER VRIES, University of Kansas
ALEXANDER ZIWET, University of Michigan

The American Mathematical Monthly

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THE AMERICAN MATHEMATICAL MONTHLY

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VOLUME XXIII NOVEMBER, 1916 NUMBER 9

ON THE DIOPHANTINE EQUATION $x^4 + ay^4 = u^2 + bv^2$.

By F. L. CARMICHAEL, University of Alabama.

§ 1. Introduction. Diophantine equations of the general type

$$\alpha x^m + \beta y^m = \gamma z^n \tag{1}$$

have been treated by several authors, notably by Desboves.[1] Here α, β, γ, m, n are given integers and integers x, y, z are to be determined so as to satisfy the equation. Recently, R. D. Carmichael[2] has given a partial treatment and has proposed a general investigation of the Diophantine equation

$$x^4 + ay^4 = u^2 + bv^2, \tag{2}$$

where a and b are given integers and integers x, y, u, v are to be determined so that equation (2) shall be satisfied. In case $a = 0$ or $b = 0$ the latter equation reduces to a special case of the former. The object of the present paper is to develop the theory of equation (2) in general and also for the four particular cases $a = 0$, $b = 0$, $b = a$, $b = \sigma^2 a$, where σ is an integer or the reciprocal of an integer. To a small extent (indicated by references below) there is a duplication of previous results; but the principal formulæ obtained are believed to be new. The methods employed throughout the paper are elementary.

§ 2. Case when a = 0. The equation $u^2 + bv^2 = x^4$. The set of numbers $u^2 + bv^2$ forms a domain with respect to multiplication. This character is put in evidence by the easily verified formulæ:

$$(\alpha_1^2 + b\beta_1^2)(\alpha_2^2 + b\beta_2^2) = (\alpha_1\alpha_2 \pm b\beta_1\beta_2)^2 + b(\alpha_1\beta_2 \mp \alpha_2\beta_1)^2. \tag{3}$$

[1] *Nouvelles Annales de Mathématiques* (2), 18 (1879): 265–279, 398–410, 433–444, 481–499.

[2] R. D. Carmichael, *Diophantine Analysis*, pp. 46–48, 54.

In particular, we have

$$(\alpha^2 + b\beta^2)^2 = (\alpha^2 - b\beta^2)^2 + b(2\alpha\beta)^2. \tag{4}$$

Let $x = \alpha^2 + b\beta^2$. Then, by a repeated use of (4), we obtain

$$x^4 = [(\alpha^2 - b\beta^2)^2 - b(2\alpha\beta)^2]^2 + b[4\alpha\beta(\alpha^2 - b\beta^2)]^2. \tag{5}$$

From the last result we see that the equation

$$u^2 + bv^2 = x^4 \tag{6}$$

has the following two-parameter solution:

$$\begin{aligned} x &= \alpha^2 + b\beta^2, \\ u &= \alpha^4 - 6b\alpha^2\beta^2 + b^2\beta^4, \\ v &= 4\alpha\beta(\alpha^2 - b\beta^2). \end{aligned} \tag{7}$$

These formulæ are given by Desboves (l. c., p. 270).

Again, from (4) we may write

$$\begin{aligned} x^2 &= \alpha_1^2 + b\beta_1^2, \\ u &= \alpha_1^2 - b\beta_1^2, \\ v &= 2\alpha_1\beta_1. \end{aligned} \tag{8}$$

If we now treat similarly the first of equations (8), we will clearly be led to a solution of (6). This, however, will be the same as (7).

In order to obtain more general solutions of (6), let us consider the equation

$$x^2 + by^2 = \alpha_1^2 + b\beta_1^2. \tag{9}$$

Taking account of the double sign in the right-hand member of (3), we have readily the following solution of equation (9):

$$\begin{aligned} x &= s_1s_2 + bt_1t_2, \\ y &= s_1t_2 - s_2t_1, \\ \alpha_1 &= s_1s_2 - bt_1t_2, \\ \beta_1 &= s_1t_2 + s_2t_1. \end{aligned} \tag{10}$$

If we impose the condition that $s_1t_2 - s_2t_1 = 0$ or $s_1t_2 = s_2t_1$, we may obviously obtain a solution of (6) by combining equations (8) and (10). Put $s_1 = m - n$, $t_2 = m + n$, $s_2 = p - q$, $t_1 = p + q$. Then the condition $s_1t_2 = s_2t_1$ reduces to $m^2 - n^2 = p^2 - q^2$; and this is satisfied if

$$\begin{aligned} m &= \lambda_1\lambda_2 + \mu_1\mu_2, \\ n &= \lambda_1\mu_2 + \lambda_2\mu_1, \\ p &= \lambda_1\lambda_2 - \mu_1\mu_2, \\ q &= \lambda_1\mu_2 - \lambda_2\mu_1. \end{aligned} \tag{11}$$

Then we have

$$
\begin{aligned}
s_1 &= \lambda_1\lambda_2 + \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2\mu_1, \\
s_2 &= \lambda_1\lambda_2 - \mu_1\mu_2 - \lambda_1\mu_2 + \lambda_2\mu_1, \\
t_1 &= \lambda_1\lambda_2 - \mu_1\mu_2 + \lambda_1\mu_2 - \lambda_2\mu_1, \\
t_2 &= \lambda_1\lambda_2 + \mu_1\mu_2 + \lambda_1\mu_2 + \lambda_2\mu_1.
\end{aligned} \tag{12}
$$

It is now clear that equations (8), (10) and (12) afford a four-parameter solution of (6).

In the special case when $\mu_1 = 0$ we have $s_1 = s_2$, $t_1 = t_2$. If we write $s_1 = s_2 = \alpha$, $t_1 = t_2 = \beta$, then the solution afforded by (8), (10), (12) becomes identical with solution (7). Hence (8), (10), (12) afford a solution which is a generalization of the previously known solution (7).

As illustrative examples of the solution of (6) afforded by (8), (10), (12), we have the following:

$$
\begin{array}{llllll}
b = 3, & \lambda_1 = 2, & \mu_1 = 1, & \lambda_2 = 1, & \mu_2 = 2: & 142^2 + 3\cdot 78^2 = 14^4, \\
b = 5, & \lambda_1 = 2, & \mu_1 = 1, & \lambda_2 = 1, & \mu_2 = 2: & 439^2 + 5\cdot 132^2 = 23^4, \\
b = 2, & \lambda_1 = 3, & \mu_1 = 1, & \lambda_2 = 1, & \mu_2 = 3: & 17^2 + 2\cdot 56^2 = 9^4.
\end{array}
$$

It should be noticed that the above solutions were not obtained merely by substitution. Factors common to u, v, x^2 have been removed after substitution in the formulæ.

§ 3. Case when $\mathbf{b = 0}$. The equation $\mathbf{x^4 + ay^4 = u^2}$. Consider the equation

$$x^4 + ay^4 = u^2. \tag{13}$$

Let $u = \alpha^2 + a\beta^2$. Then $u^2 = (\alpha^2 - a\beta^2)^2 + a(2\alpha\beta)^2$. Equation (13) will therefore be satisfied if we write

$$
\begin{aligned}
x^2 &= \alpha^2 - a\beta^2, \\
y^2 &= 2\alpha\beta, \\
u &= \alpha^2 + a\beta^2.
\end{aligned} \tag{14}
$$

The first of equations (14) is satisfied if

$$
\begin{aligned}
x &= \alpha_1^2 - a\beta_1^2, \\
\alpha &= \alpha_1^2 + a\beta_1^2, \\
\beta &= 2\alpha_1\beta_1.
\end{aligned} \tag{15}
$$

Now, putting $\alpha_1 = \lambda^2$, $\beta_1 = \mu^2$ in (15) and combining equations (14) and (15), we have

$$
\begin{aligned}
x &= \lambda^4 - a\mu^4, \\
y &= 2\lambda\mu\sqrt{\lambda^4 + a\mu^4}, \\
u &= \lambda^8 + 6a\lambda^4\mu^4 + a^2\mu^8.
\end{aligned} \tag{16}
$$

Hence, if λ and μ are values of x and y, respectively, satisfying (13), it is obvious that (16) will afford a second solution. Therefore, by a repeated use of (16), as many solutions as desired may be found.

Formulæ (16) were first obtained by Desboves (l. c., p. 437) by a method equivalent to that employed above.

From the second of equations (14) it follows that either α or β must contain 2 to an odd power. We shall first suppose it is α and write $\alpha = 2m\rho^2$, where m contains no repeated prime factor. Then, from the second equation (14) we see that we must have $\beta = m\sigma^2$. Now, substituting these values of α and β in the first equation (14), we see that values of ρ and σ satisfying the condition

$$4\rho^4 - a\sigma^4 = \tau^2 \tag{17}$$

will yield solutions of (13). If β contains 2 to an odd power, by reasoning similar to that employed above, we may write $\alpha = m\rho^2$, $\beta = 2m\sigma^2$. Making this substitution in the first equation (14), we see that values of ρ and σ satisfying the condition

$$\rho^4 - 4a\sigma^4 = \tau^2 \tag{18}$$

will yield solutions of (13).

Since equation (13) is known to have no solution when $a = 1$, we can not obtain a general solution free of restricting conditions. However, when $a = 2\lambda\mu(\lambda^2 + \mu^2)$, it can readily be shown that a solution is afforded by

$$\begin{aligned} x &= (\lambda - \mu)^2, \\ y &= 2(\lambda + \mu), \\ u &= \lambda^4 + 12\lambda^3\mu + 6\lambda^2\mu^2 + 12\lambda\mu^3 + \mu^4. \end{aligned} \tag{19}$$

When $a = 8\lambda\mu(\lambda^2 + \mu^2)$, it can readily be shown that a solution is afforded by

$$\begin{aligned} x &= (\lambda - \mu)^4 - 8\lambda\mu(\lambda^2 + \mu^2), \\ y &= 2(\lambda - \mu)(\lambda + \mu)^2, \\ u &= (\lambda - \mu)^8 + 48\lambda\mu(\lambda^2 + \mu^2)(\lambda - \mu)^4 + 64\lambda^2\mu^2(\lambda^2 + \mu^2)^2. \end{aligned} \tag{20}$$

Also, it is obvious that if a is a perfect square diminished by unity, integral solutions are possible. Solutions may be obtained when a has other special forms.

The following taken in order are examples illustrating the use of formulæ (16), (19), (20):

$$\begin{aligned} a &= 3, & \lambda &= 1, & \mu &= 2: & 47^4 + 3\cdot 28^4 &= 2593^2, \\ a &= 20, & \lambda &= 2, & \mu &= 1: & 1^4 + 20\cdot 6^4 &= 161^2, \\ a &= 80, & \lambda &= 2, & \mu &= 1: & 79^4 + 80\cdot 18^4 &= 6881^2. \end{aligned}$$

§ 4. Case when b = a. The equation $\mathbf{x^4 + ay^4 = u^2 + av^2}$. From formulæ (3) we see that the equation

$$x^4 + ay^4 = u^2 + av^2 \tag{21}$$

is satisfied if

$$(22)\qquad \begin{aligned} x^2 &= s_1s_2 - at_1t_2,\\ y^2 &= s_1t_2 + s_2t_1,\\ u &= s_1s_2 + at_1t_2,\\ v &= s_1t_2 - s_2t_1. \end{aligned}$$

Let $s_1 = \lambda - 3a\mu$, $s_2 = \lambda$, $t_1 = \lambda + a\mu$, $t_2 = -\mu$. Then equations (22) become

$$(23)\qquad \begin{aligned} x &= \lambda - a\mu,\\ y^2 &= \lambda^2 + (a-1)\lambda\mu + 3a\mu^2,\\ u &= \lambda^2 - 4a\lambda\mu - a^2\mu^2,\\ v &= 3a\mu^2 - (a+1)\lambda\mu - \lambda^2. \end{aligned}$$

It may be easily verified (cf. R. D. Carmichael, l. c., p. 25) that

$$(24)\qquad (m^2+amn+bn^2)^2 = (m^2-bn^2)^2+a(m^2-bn^2)(2mn+an^2)+b(2mn+an^2)^2.$$

Therefore, from the second of equations (23), we may write

$$(25)\qquad \begin{aligned} y &= m^2 + (a-1)mn + 3an^2,\\ \lambda &= m^2 - 3an^2,\\ \mu &= 2mn + (a-1)n^2. \end{aligned}$$

It is now evident that equations (23) and (25) afford a two-parameter solution of (21).

Again, make the following substitution in (22): $s_1 = \lambda - 3a\mu$, $s_2 = \lambda + a\mu$, $t_1 = a\mu$, $t_2 = \lambda$. We then obtain

$$(26)\qquad \begin{aligned} x^2 &= \lambda^2 - (a^2+2a)\lambda\mu - 3a^2\mu^2,\\ y &= \lambda - a\mu,\\ u &= \lambda^2 + (a^2-2a)\lambda\mu - 3a^2\mu^2,\\ v &= \lambda^2 - 4a\lambda\mu - a^2\mu^2. \end{aligned}$$

Making use of (24), we see that the first of equations (26) is satisfied if

$$(27)\qquad \begin{aligned} x &= m^2 - (a^2+2a)mn - 3a^2n^2,\\ \lambda &= m^2 + 3a^2n^2,\\ \mu &= 2mn - (a^2+2a)n^2. \end{aligned}$$

It is now clear that equations (26) and (27) afford a two-parameter solution of (21).

It should be observed that the above methods are capable of yielding an unlimited number of two-parameter solutions. Thus solutions may be obtained by making the following substitutions in (22):

$$\begin{array}{llll} s_1 = \lambda + 2a\mu, & s_2 = \lambda, & t_1 = -2\mu, & t_2 = \lambda + 2a\mu, \\ s_1 = \lambda + 3a\mu, & s_2 = \lambda, & t_1 = -\mu, & t_2 = \lambda + 4a\mu, \\ s_1 = \lambda - 8a\mu, & s_2 = \lambda, & t_1 = -4\mu, & t_2 = \lambda + a\mu, \\ s_1 = \lambda + 3a\mu, & s_2 = \lambda, & t_1 = -3\mu, & t_2 = \lambda + 3a\mu. \end{array}$$

From the above values of s_1, s_2, t_1, t_2 integers x are obtained by mere substitution. The corresponding expressions for y^2 contain λ^2 with coefficient unity. Similarly, the following substitutions yield y directly, the expressions for x^2 containing λ^2 with coefficient unity:

$$\begin{array}{llll} s_1 = \lambda, & s_2 = \lambda + a\mu, & t_1 = 4a\mu, & t_2 = \lambda, \\ s_1 = \lambda + 2a\mu, & s_2 = \lambda + 2a\mu, & t_1 = 2a\mu, & t_2 = \lambda, \\ s_1 = \lambda + 3a\mu, & s_2 = \lambda + 3a\mu, & t_1 = 3a\mu, & t_2 = \lambda, \\ s_1 = \lambda - 3a\mu, & s_2 = \lambda + a\mu, & t_1 = 9a\mu, & t_2 = \lambda. \end{array}$$

From the manner of formation of the expressions for x^2 and y^2, it is easy to see that as many substitutions as desired may be found such that one will be a perfect square and the other will contain λ^2 with coefficient unity.

The following taken in order are examples illustrative of the above methods:

$$a = 2,\quad m = 1,\quad n = 1,\quad \lambda = -5,\quad \mu = 3:\quad 11^4 + 2\cdot 8^4 = 109^2 + 2\cdot 74^2,$$

$$a = 2,\quad m = 1,\quad n = 1,\quad \lambda = 13,\quad \mu = -6:\quad 19^4 + 2\cdot 25^4 = 263^2 + 2\cdot 649^2.$$

§ 5. **Case when $b = \sigma^2 a$. The equation $x^4 + ay^4 = u^2 + \sigma^2 a v^2$.** In order to develop the theory for this case we shall treat the more general equation

$$x^4 + ay^4 = u^2 + bv^2 \tag{28}$$

by means of more general formulæ than have been used in the preceding sections. This may be done by extending either of the sets of numbers $x^4 + ay^4 - bv^2$ and $u^2 + bv^2 - ay^4$ so as to form a multiplicative domain. Extending the latter, the required set of numbers is $u^2 + bv^2 - az^2 - abw^2$.

Consider the equation

$$u^2 + bv^2 - ay^4 - abw^2 = x^4. \tag{29}$$

If $g(u, v, y, w) = u^2 + \alpha v^2 + \beta y^2 + \alpha\beta w^2$, it can readily be verified (cf. R. D. Carmichael, l. c., p. 37) that

$$\begin{aligned} \{g(u, v, y, w)\}^2 &= g(u^2 - \alpha v^2 - \beta y^2 + \alpha\beta w^2,\ 2uv - 2\beta yw,\ 2yu + 2\alpha wv,\ 0), \\ &= g(u^2 - \alpha v^2 + \beta y^2 + \alpha\beta w^2,\ 2uv,\ 2\alpha wv,\ 2yv). \end{aligned} \tag{30}$$

Hence, using the first of equations (30), we see that (29) will be satisfied if

$$(31) \qquad \begin{aligned} u &= u_1^2 - bv_1^2 + ay_1^2 - abw_1^2, \\ v &= 2u_1v_1 + 2ay_1w_1, \\ y^2 &= 2y_1u_1 + 2bw_1v_1, \\ w &= 0, \\ x^2 &= u_1^2 + bv_1^2 - ay_1^2 - abw_1^2. \end{aligned}$$

Suppose first that $b = m^2a$, where m is an integer. Also, put $u_1 = m(\lambda - a\mu)$, $v_1 = 2\lambda$, $y_1 = 2m(\lambda - a\mu)$, $w_1 = 4\mu$ in (31). Then

$$(32) \qquad \begin{aligned} u &= m^2[\lambda^2 - (8a^2 + 2a)\lambda\mu + (4a^3 - 15a^2)\mu^2], \\ v &= 4m(\lambda^2 + 3a\lambda\mu - 4a^2\mu^2), \\ y &= 2m(\lambda + a\mu), \\ x^2 &= m^2[\lambda^2 + (8a^2 - 2a)\lambda\mu - (4a^3 + 15a^2)\mu^2]. \end{aligned}$$

From (24) we see that the last of equations (32) is satisfied if

$$(33) \qquad \begin{aligned} x &= m[\alpha^2 + (8a^2 - 2a)\alpha\beta - (4a^3 + 15a^2)\beta^2], \\ \lambda &= \alpha^2 + (4a^3 + 15a^2)\beta^2, \\ \mu &= 2\alpha\beta + (8a^2 - 2a)\beta^2. \end{aligned}$$

Equations (32) and (33) afford two-parameter solutions of equations of the general type

$$(34) \qquad x^4 + ay^4 = u^2 + m^2av^2.$$

Now, put $a = m^2b$, $u_1 = m(\lambda - b\mu)$, $v_1 = 2m^2\lambda$, $y_1 = 2m(\lambda - b\mu)$, $w_1 = 4\mu$ in (31). Then, after removing a factor m^2 common to x^2, y^2, u, v, we have

$$(35) \qquad \begin{aligned} u &= \lambda^2 - (8m^2b^2 + 2b)\lambda\mu + (4m^2b^3 - 15b^2)\mu^2, \\ v &= 4m(\lambda^2 + 3b\lambda\mu - 4b^2\mu^2), \\ y &= 2(\lambda + b\mu), \\ x^2 &= \lambda^2 + (8m^2b^2 - 2b)\lambda\mu - (4m^2b^3 + 15b^2)\mu^2. \end{aligned}$$

Making use of (24) we see that the last of equations (35) is satisfied if

$$(36) \qquad \begin{aligned} x &= \alpha^2 + (8m^2b^2 - 2b)\alpha\beta - (4m^2b^3 + 15b^2)\beta^2, \\ \lambda &= \alpha^2 + (4m^2b^3 + 15b^2)\beta^2, \\ \mu &= 2\alpha\beta + (8m^2b^2 - 2b)\beta^2. \end{aligned}$$

Equations (35) and (36) afford two-parameter solutions of equations of the general type

$$x^4 + m^2by^4 = u^2 + bv^2. \tag{37}$$

If we put $m = 1$ in (32), (33) and (35), (36), we are led to solutions of (21). These are identical.

As examples illustrating the use of the above methods, we have the following:

$b = m^2a,\ a = 1,\ m = 2,\ b = 4,\ \alpha = 2,\ \beta = 1$: $1^4 + 1\cdot 22^4 = 319^2 + 4\cdot 182^2$,

$a = m^2b,\ b = 1,\ m = 2,\ a = 4,\ \alpha = 2,\ \beta = 1$: $11^4 + 4\cdot 46^4 = 4231^2 + 1\cdot 152^2$.

§ 6. The general case. The equation $x^4 + ay^4 = u^2 + bv^2$. For the purpose of developing the theory for the general case we may employ equations (31). Put $y_1 = 0$, $v_1 = 2bw_1$ in (31). Then

$$\begin{aligned} u &= u_1^2 - (4b^3 + ab)w_1^2,\\ v &= 4bu_1w_1,\\ y &= 2bw_1,\\ x^2 &= u_1^2 + (4b^3 - ab)w_1^2. \end{aligned} \tag{38}$$

The last equation (38) is satisfied if

$$\begin{aligned} x &= u_2^2 + (4b^3 - ab)w_2^2,\\ u_1 &= u_2^2 - (4b^3 - ab)w_2^2,\\ w_1 &= 2u_2w_2. \end{aligned} \tag{39}$$

Equations (38) and (39) clearly afford a solution of (28).

Now, put $y_1 = 0$, $w_1 = 2bv_1$ in equations (31). They then become

$$\begin{aligned} u &= u_1^2 - (b + 4ab^3)v_1^2,\\ v &= 2u_1v_1,\\ y &= 2bv_1,\\ x^2 &= u_1^2 + (b - 4ab^3)v_1^2. \end{aligned} \tag{40}$$

The last of equations (40) will be satisfied if we write

$$\begin{aligned} x &= u_2^2 + (b - 4ab^3)v_2^2,\\ u_1 &= u_2^2 - (b - 4ab^3)v_2^2,\\ v_1 &= 2u_2v_2. \end{aligned} \tag{41}$$

Equations (40) and (41) afford a solution of (28).

Again, using the second formula (30), the last of equations (31) is satisfied if

$$
(42)\qquad
\begin{aligned}
x &= u_2^2 + bv_2^2 - ay_2^2 - abw_2^2,\\
u_1 &= u_2^2 - bv_2^2 - ay_2^2 - abw_2^2,\\
v_1 &= 2u_2v_2,\\
y_1 &= 2bw_2v_2,\\
w_1 &= 2y_2v_2.
\end{aligned}
$$

Put $y_2 = w_2 = bv_2$. Then, from the third equation in (31), we have the further restriction

$$
(43)\qquad y^2 = 4b^2v_2^2[u_2^2 + 2u_2v_2 - (b + ab^2 + ab^3)v_2^2],
$$

which is satisfied in accordance with (24) if

$$
(44)\qquad
\begin{aligned}
y &= 2bv_2[m^2 + 2mn - (b + ab^2 + ab^3)n^2],\\
u_2 &= m^2 + (b + ab^2 + ab^3)n^2,\\
v_2 &= 2mn + 2n^2.
\end{aligned}
$$

It will now be seen that equations (31), (42), (44) afford a solution of (28).

The following taken in order are examples illustrative of the above methods:

$a=5,\ b=2,\ u_2=1,\ w_2=1,\ u_1=-21,\ w_1=2:\quad 23^4+5\cdot 8^4 = 273^2+2\cdot 336^2,$

$a=3,\ b=2,\ u_2=1,\ v_2=1,\ u_1=95,\ v_1=2:\quad 93^4+3\cdot 8^4 = 8633^2+2\cdot 380^2,$

$a=3,\ b=2,\ n=1,\ u=1,\ u_2=39,\ v_2=4:\quad 977^4+3\cdot 560^4=663457^2+2\cdot 618864^2.$

If we operate on the last of equations (31) by means of the first formula (30) and make certain easy transformations, we will be led at once to the following solution of (28):

$$
(45)\qquad
\begin{aligned}
x &= \alpha^4 - a\beta^4 + b\gamma^2,\\
y &= 2\alpha\beta\sqrt{\alpha^4 + a\beta^4 - b\gamma^2},\\
u &= (\alpha^4 + a\beta^4 - b\gamma^2)^2 - 4b\alpha^4\gamma^2 + 4a\alpha^4\beta^4,\\
v &= 4\alpha^2\gamma(\alpha^4 + a\beta^4 - b\gamma^2).
\end{aligned}
$$

These formulæ have been developed in full by R. D. Carmichael (l. c., pp. 46–47) and will therefore be dismissed with this remark.

GRAPHICAL CONSTRUCTIONS FOR A FUNCTION OF A FUNCTION AND FOR A FUNCTION GIVEN BY A PAIR OF PARAMETRIC EQUATIONS.[1]

By W. H. ROEVER, Washington University.

It is easy to see that if $y = \phi(u)$ and $u = \psi(x)$ are the equations of the orthographic projections of a space curve on the planes y–O–u and u–O–x determined by the system of rectangular cartesian axes O–x, u, y, then $y = \phi[\psi(x)]$ is the equation of the orthographic projection of this space curve on the plane x–O–y (see Fig. 1).

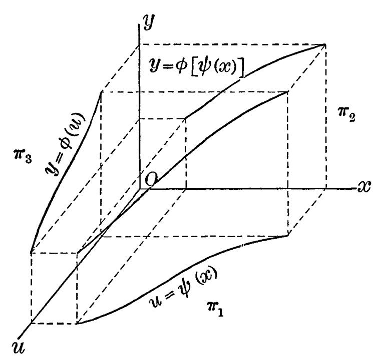

FIG. 1.

Let us now suppose the planes $\pi_1 \equiv xOu$ and $\pi_3 \equiv uOy$ to be revolved around the axes of x and y respectively until they coincide with the plane $\pi_2 \equiv yOx$, and in such a way that the revolved positions of the positive u-axis coincide with the negative portions of the unrevolved axes of x and y (see Fig. 2). If a general point of space be denoted by P, let its orthographic projections on π_1, π_2, and π_3 be denoted by $'P$, P'' and $'''P$ respectively. During the revolutions just made the point P'' remains unchanged, but the points $'P$ and $'''P$ assume the new positions P' and P''' respectively (see Fig. 2). The plane π_2, which now also contains the revolved positions of the planes π_1 and π_3, is the *drawing plane of Mongean Descriptive Geometry* and the unrevolved axes of x and y are called the *first* and *second ground lines* respectively (see Fig. 3). The internal bisector of the right angle formed by the positive portions of these ground lines is called the *line b*. From the method of obtaining the points P', P'', P''' of the drawing plane it follows that:

(1) *P' and P'' lie on the same perpendicular to the first ground line,*

---

[1] Presented to the American Mathematical Society, November 27, 1915.

After this paper was written my attention was called to a construction by Professor E. H. Moore which is practically identical with the one here given. Professor Moore's construction is contained under *Linkage B* in his article "Cross-section paper as a mathematical instrument" published in *The School Review*, May, 1906. Dr. A. J. Kempner has also discovered a construction which includes the one here given. His paper will be published in a later number of this MONTHLY.

(2) *P'' and P''' lie on the same perpendicular to the second ground line,*

(3) *lines through P' and P''' parallel respectively to the first and second ground lines, meet in a point P° of the line b* (see Fig. 3). It is important to observe that this is true regardless of the position of P in space; that is, P need not necessarily lie in the first octant as shown in Fig. 2. Following general usage we shall

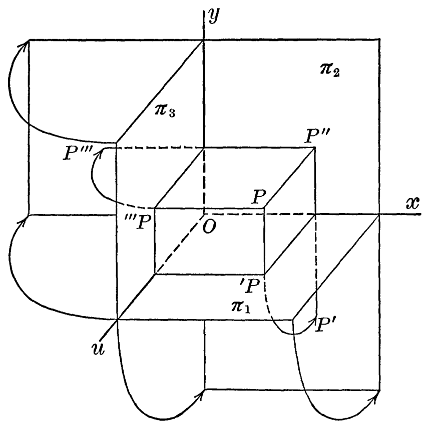

FIG. 2.

call the points P', P'', P''' the *projections in the drawing plane* of the point P of space. We may then say that *no matter what the position of a point in space may be, its projections in the drawing plane are three vertices of a rectangle whose sides are parallel to the ground lines, and whose fourth vertex lies on the line b.*

By combining the facts stated in the two preceding paragraphs we are led to the *Construction* (Fig. 3). Assume a pair of perpendicular lines (one horizontal and the other vertical) intersecting in a point O. Let these represent respectively the axes of x and y, x being positive to the right of O and y positive above O. Let each of these lines also represent the axis of u, in which however the positive and negative directions are the reverse of those in x and y. Then draw through O the line b bisecting internally the angle formed by the positive portions of the axes of x and y. Now plot with respect to the axes of x and u the graph of $u = \psi(x)$ and with respect to u and y the graph of $y = \phi(u)$. All this being done, we find the required graph of $y = \phi[\psi(x)]$ by drawing through each point P° of the line b a horizontal line cutting $u = \psi(x)$ in P' and a vertical line cutting $y = \phi(u)$ in P''', and then through P' a vertical line and through P''' a horizontal line. The last two lines intersect in a point P'' of the required graph.

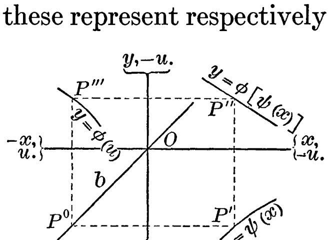

FIG. 3.

In Fig. 4, this construction is used to find the graph of $y = e^{1/x}$ from the graphs of $y = e^u$ and $u = 1/x$.

This method also enables one to construct a curve which is given by a pair of parametric equations

$$x = f(u), \qquad y = \phi(u).$$

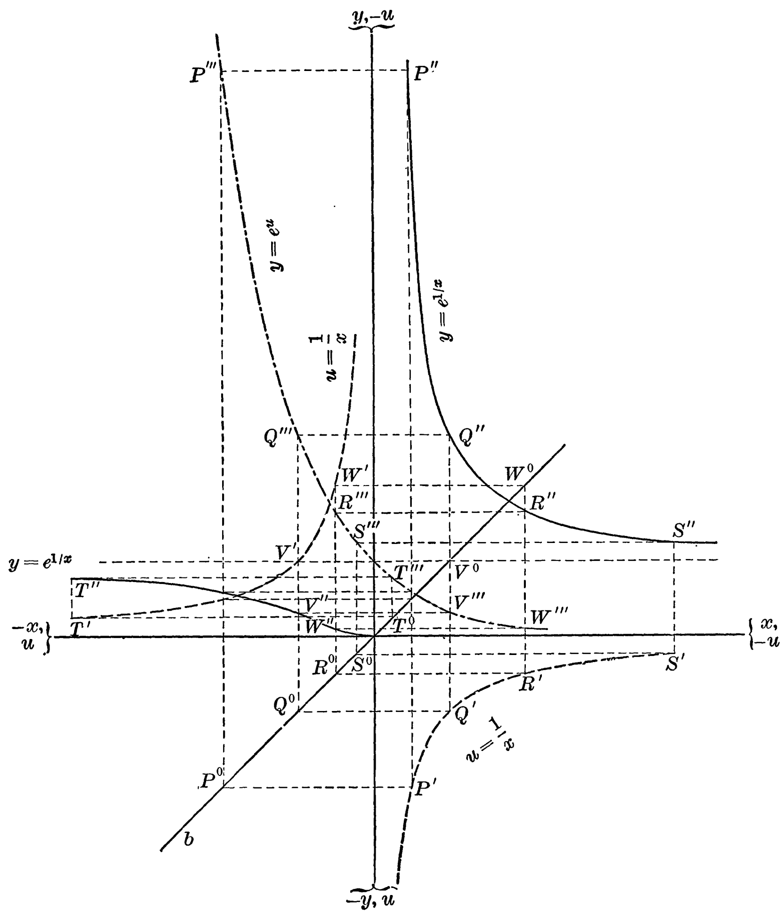

FIG. 4.

For the problem of eliminating u between these equations is the same as that of finding the function $y = \phi(u)$ of the function $u = \psi(x)$, where $u = \psi(x)$ is obtained by solving $x = f(u)$ for u in terms of x. In Fig. 5, the method is used to construct the curve of which the parametric equations are

$$x = u^2, \qquad y = u^3.$$

It is interesting to note that for $u = 0$ both of the functions $f(u) = u^2$, $\phi(u) = u^3$ have vanishing derivatives and hence the curve $y^2 = x^3$ has a singularity—in this case a cusp—at the origin. The curves in Figs. 4 and 5 have been accurately

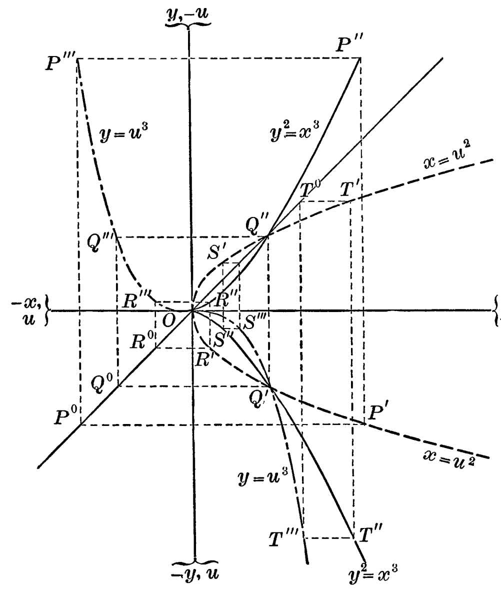

FIG. 5.

constructed. However, the construction under consideration often enables one to determine the salient features of the graph of $y = \phi[\psi(x)]$ from a mere rough plotting of the graphs of $y = \phi(u)$ and $u = \psi(x)$.[1]

A NOTE ON THE SUM OF THE REMAINDERS OF A SERIES.

By GLENN JAMES, Purdue University.

Suppose the series

(1) $$a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

has the sum S, and consider the series formed from its remainders, which is

(2) $$S + [S - a_0] + [S - (a_0 + a_1)] + \cdots + [S - (a_0 + a_1 + \cdots a_{n-1})] + \cdots$$

or

(3) $$R_0 + R_1 + R_2 + \cdots + R_n + \cdots.$$

The assumption that (1) converges is equivalent to the assumption that

$$\lim_{n=\infty} R_n = 0.$$

Moreover, any convergent or divergent series

$$H_0 + H_1 + H_2 + \cdots + H_n + \cdots,$$

[1] It might be remarked that the addition of the 45° line b on ordinary rectangular plotting paper would make of that paper a splendid medium for carrying out the above construction.

for which $\lim\limits_{n=\infty} H_n = 0$, is the remainder series of

$$(H_0 - H_1) + (H_1 - H_2) + \cdots + (H_{n-1} - H_n) + \cdots,$$

which converges since its sum is H_0.

Hence, there is an infinite class of convergent series whose remainder series converge and an infinite class whose remainder series diverge. The following theorems determine, upon the basis of the simpler convergence tests, certain types of series which belong to the former class.

THEOREM I. *Any series whose terms alternate in sign, never increase in absolute value, and tend to zero, possesses an absolutely convergent remainder series if the original series converges absolutely.*

Given an absolutely convergent series,

$$(4) \qquad a_0 - a_1 + a_2 - a_3 + \cdots,$$

where

$$a_0 \geqq a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq 0 \quad \text{and} \quad \lim_{n=\infty} a_n = 0.$$

The sum of such a series lies between zero and the first term, and the nature of such a series is not altered by dropping any number of consecutive terms from the beginning of the series. Also $S - S_{n-1}$, $(= R_n)$, is the sum of (4) after the $(n-1)$th term. Hence,

$$|R_n| < |a_n|, \qquad n = 0, 1, 2, 3, \cdots.$$

By hypothesis $\sum\limits_0^\infty a_n$ is absolutely convergent. Therefore, $\sum\limits_0^\infty R_n$ is absolutely convergent.

The following example shows that a series whose terms alternate in sign, continually decrease numerically, and tend to zero, does not necessarily have a convergent remainder series:

$$(5) \quad S = (\tfrac{1}{3} + \tfrac{3}{4}) - (\tfrac{1}{3} + \tfrac{3}{8}) + (\tfrac{1}{4} + \tfrac{3}{16}) - (\tfrac{1}{4} + \tfrac{3}{32}) + (\tfrac{1}{5} + \tfrac{3}{64}) - (\tfrac{1}{5} + \tfrac{3}{128}) \cdots = \tfrac{1}{2}.$$

This series does not converge absolutely, for each term is greater in absolute value than the corresponding term of the divergent series $\sum\limits_3^\infty (1/n)$. Noting that $R_1 = R_0 - a_0$, $R_2 = R_1 - a_1$, $\cdots$, the remainder series is evidently

$$\tfrac{1}{2} - (\tfrac{1}{3} + \tfrac{1}{4}) + \tfrac{1}{8} - (\tfrac{1}{4} + \tfrac{1}{16}) + \tfrac{1}{32} - (\tfrac{1}{5} + \tfrac{1}{64}) + \tfrac{1}{128} - \cdots$$

or

$$\tfrac{1}{2} - (\tfrac{1}{3} + \tfrac{1}{8}) - (\tfrac{1}{4} + \tfrac{1}{32}) - (\tfrac{1}{5} + \tfrac{1}{128}) - \cdots,$$

which plainly diverges.

THEOREM II. *If a series converges by the ratio test, its remainder series converges.*

Consider first a series $\sum\limits_0^\infty a_n$ for which $\lim\limits_{n=\infty} |a_n/a_{n-1}| = \alpha < 1$. Then for any

positive ϵ there exists an n' such that

$$\alpha + \epsilon > \left|\frac{a_n}{a_{n-1}}\right| > \alpha - \epsilon, \qquad n \geqq n'.$$

Hence to every $n' + i$ there corresponds an ϵ_i, $|\epsilon_i| < 2\epsilon$, $(i = 1, 2, 3, \cdots)$, such that

$$\left|\frac{a_{n'}}{a_{n'-1}}\right| = \left|\frac{a_{n'+1}}{a_{n'}}\right| + \epsilon_1 = \left|\frac{a_{n'+2}}{a_{n'+1}}\right| + \epsilon_2 = \cdots.$$

Let the ratio $|a_{n'}/a_{n'-1}|$ be denoted by $K_{n'}$, and collect terms. Then

$$(6) \qquad K_{n'} = \left|\frac{a_{n'}}{a_{n'-1}}\right| = \frac{|a_{n'+1}| + \epsilon_1 |a_{n'}|}{|a_{n'}|} = \frac{|a_{n'+2}| + \epsilon_2 |a_{n'+1}|}{|a_{n'+1}|} = \cdots.$$

Whence

$$K_{n'}|a_{n'-1}| = |a_{n'}|,$$
$$K_{n'}|a_{n'}| = |a_{n'+1}| + \epsilon_1 |a_{n'}|,$$
$$K_{n'}|a_{n'+1}| = |a_{n'+2}| + \epsilon_2 |a_{n'+1}|,$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

Adding these identities, we have

$$K_{n'}[|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots] = [|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots] + [\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots].$$

Whence, making use of (6),

$$(7) \quad K_{n'} = \frac{|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} + \frac{\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} = \left|\frac{a_{n'}}{a_{n'-1}}\right|.$$

Now

$$|[\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots |] < 2\epsilon[|a_{n'}| + |a_{n'+1}| + \cdots] < 2\epsilon[|a_{n'-1}| + |a_{n'}| + \cdots].$$

Hence

$$\left|\frac{\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots}\right| < 2\epsilon.$$

But $\lim\limits_{n'=\infty} 2\epsilon = 0$, which gives

$$(8) \qquad \underset{n'=\infty}{L} K_{n'} = \underset{n'=\infty}{L} \frac{|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} = \underset{n'=\infty}{L} \left|\frac{a_{n'}}{a_{n'-1}}\right| = \alpha.$$

Let $R_n' = |a_n| + |a_{n+1}| + \cdots$, and $R_n = a_n + a_{n+1} + \cdots$. Then $\sum\limits_0^\infty R_n'$ converges by (8). But $|R_n| \leqq R_n'$. Therefore $\sum\limits_0^\infty R_n$ converges.

Consider now a series $\sum\limits_0^\infty a_n$ for which the ratio a_n/a_{n-1} approaches no limit but remains less than α which is less than unity for all values of n greater than n'.

Then the series $\sum\limits_0^\infty a_n$ and its remainder series $\sum\limits_0^\infty R_n$ are dominated respectively by the series

$$(9) \qquad \sum_0^\infty A_n = \sum_0^{n'-1} |a_n| + \sum_{m=0}^\infty |a_{n'}|\alpha^m$$

and its remainder series $\sum_0^\infty \bar{R}_n$. But (9) is a series to which the above proof applies. Hence $\sum_0^\infty \bar{R}_n$ and consequently $\sum_0^\infty R_n$ converge.

COROLLARY. *If the ratio of the nth to the* $(n-1)$*th term of a convergent series of positive terms has a limit, the corresponding ratio for the remainder series has the same limit.*

If $\sum_0^\infty a_n$ converges, the first part of the proof of the above theorem does not require that α, $(= \lim_{n=\infty} | a_n/a_{n-1} |)$, be less than unity. The corollary follows, then, from (8).

Examples (*a*), (*b*) and (*c*), below, illustrate the preceding theorem and corollary.

(*a*)
$$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2.$$
The remainder series is
$$S' = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 3.$$
(*b*) The geometric series

(i)
$$\sum_0^\infty ar^n = \frac{a}{1-r}, \qquad |r| < 1,$$
has the remainder series

(ii)
$$\sum_{n=0}^\infty [ar^n + ar^{n+1} + \cdots] = \sum_0^\infty \frac{ar^n}{1-r} = \frac{a}{(1-r)^2}.$$
The remainder series of (ii) is

(iii)
$$\sum_0^\infty \frac{ar^n}{(1-r)^2} = \frac{a}{(1-r)^3}.$$
Thus, the *m*th remainder series of (i) is
$$\sum_{n=0}^\infty \frac{ar^n}{(1-r)^m} = \frac{a}{(1-r)^{m+1}}, \qquad |r| < 1.$$
Hence the sums of the successive remainder series increase indefinitely when $1 > r > 0$ and approach zero when $0 > r > -1$.

(*c*)
$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots.$$
The remainder series is
$$S' = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$
whence
$$\lim_{n=\infty} \frac{a_n}{a_{n-1}} = L\frac{n-1}{n+1} = L\frac{R_n}{R_{n-1}} = L\frac{n-1}{n} = 1.$$

A further classification of series, as to the convergence or divergence of their remainder series, might be made upon the basis of other convergence tests. And in the case of series with variable terms it would be of interest to study such properties as uniform convergence with reference to their remainder series.

BOOK REVIEWS.

Analytic Mechanics. By JOHN ANTHONY MILLER, Ph.D., Professor of Mathematics, Swarthmore College, and SCOTT BARRETT LILLY, C.E., Assistant Professor of Engineering, Swarthmore College. D. C. Heath and Company, 1915.

The motive of the authors of this book is described in the preface in the following words: "We have attempted to write a rigorous, teachable introduction to the study of mechanics. We believe that certain fundamental principles of mechanics, used in common by students in the various branches of engineering, in theoretical physics, and in celestial mechanics are essential to the satisfactory progress of a student in any of these fields. To this end, we have chosen as our subject matter only such fundamental theorems." The result is a book of 292 pages, consisting of an introduction and sixteen chapters treating of the following topics: Composition and resolution of forces acting on a particle; statics of a particle; forces acting on a rigid body; vectors; statics of a rigid body; center of gravity; friction; flexible cords; kinetics of a particle; motion of a particle in a plane curve; work and energy; dynamics of a rigid body; kinetic friction. The difficult problem of choice of material has been solved with sufficient success to produce a book without important omissions and of usable dimensions. A number of excellent treatises on mechanics have been published in the English language during the past generation, but the best of these have been unadapted for use as introductory textbooks because of their unmanageable dimensions. Of the numerous attempts which have been made to meet the demand for a satisfactory introductory course, the book of Miller and Lilly is among the most successful in the important matter of selection of material.

The authors have also attained a considerable degree of success in their attempt "always to be rigorous" while producing "a book that is distinctly teachable." Teachable it undoubtedly is; explanations are stated in simple, direct language; too much is not presupposed in the way of mathematical knowledge and acumen on the part of the student. The exercises for the student are for the most part sufficiently simple and designed to illustrate the principles of mechanics rather than the elegance of a mathematical method. It is mainly in the explanation of principles, rather than in the application to particular cases, that there occur what the reviewer regards as lapses of rigor.

The treatment shows the sympathy for practical applications which would be expected from the fact that one of the authors is a teacher of engineering. Engineering applications are however brought in incidentally rather than by formal separation in the treatment; thus the determination of stresses in jointed frames is explained in the chapters on statics of a particle and statics of a rigid body.

The laws of motion are stated in the language of Newton (the English rendering being nearly identical with that of Thomson and Tait), and very little is said by way of explaining the meaning of the laws. The equation of motion

for a particle is habitually employed in the form $F = (W/g)a$; but this is deduced from the equation $F = kma$ which is given as the algebraic expression of the second law and is thus implicitly taken as the "fundamental" equation. The explanation of basic principles may perhaps fairly be characterized as traditional; that is, it is in the main the method which has predominated in English and American books since the publication of Thomson and Tait's *Natural Philosophy*. To a considerable extent the lapses of rigor referred to above are characteristic of this traditional method. It seems worth while to refer in some detail to certain of the defects which, according to the view of the present writer, have been perpetuated in this traditional treatment. These defects are associated with the following subjects: (1) The concept of force, (2) the composition of velocities and accelerations, (3) the theory of work and energy.

1. The vagueness of the definition of force is perhaps the most serious defect in the traditional treatment. A proper explanation should wholly remove this vagueness. The practical concept of force involves the following points: (*a*) A force is a push or a pull exerted upon a body or portion of matter; (*b*) every force acting on one body is exerted by some other body (*i. e.*, there is some body which does the pushing or pulling); (*c*) when a body A is pulling or pushing a body B, the body B is at the same time pulling or pushing A equally in the opposite direction and along the same line; (*d*) a force tends to change the velocity of the body upon which it acts. The traditional definition and explanation usually give no explicit statement of either (*b*) or (*c*)—often scarcely a hint of these important facts—and in many cases there is no appeal even to the notion of push or pull in explaining force.[1]

The book under review shares the traditional vagueness in the explanation of force. At the beginning of Chapter I we read: "Force is that which changes, or tends to change, the motion of matter. It is either a push or a pull." There is no hint here or in the context that two bodies or portions of matter are concerned, nor any intimation that the real meaning of the law of action and reaction is the above statement (*c*).[2] Moreover the explanation is marred by statements which are not merely vague but incorrect: "That which moves matter is *force*" (p. 4); "In a general way we have known that if a body is moving it is acted upon by some force" (p. 6).

The vagueness which results from a failure to keep definitely in mind the points above designated as (*b*) and (*c*) appears in the treatment of internal and external forces in the dynamics of a rigid body (Art. 249). There is no clear

[1] An examination of six books taken at random among textbooks on mechanics published during the past ten years failed to find in any one the use of the words push or pull to explain force; and it is fair to say that the majority of books either give no hint at all that two bodies are concerned in the action of every force or refer to it merely incidentally. Undoubtedly many students have completed a course in mechanics without having their attention called either to this fact or to the real meaning of the law of action and reaction.

[2] Some textbook writers appear to take it for granted that the explicit statement (*c*) is unnecessary because sufficiently implied by the statement "to every action there is an equal and contrary reaction." Nothing is more certain, however, than that able writers have repeatedly been led into mistakes by the failure to recognize that (*c*) is the real meaning of the third law.

statement of the essential distinction between these two classes of forces (that an internal force is exerted upon some part of the body whose motion is being studied *by another part of the same body*, while an external force is exerted *by some other body*). When this is clearly stated, and when the law of action and reaction is understood to have the meaning (*c*), it is at once seen that the internal forces cancel out of the equations obtained by summing the equations of motion written for all individual particles. This cancellation depends merely upon the fact that, for any pair of forces constituting action and reaction, the vector sum is zero and the sum of the moments about any axis is zero; a serious confusion of language, if not of thought, is involved in the statement that "the internal forces are in equilibrium among themselves."[1]

2. The vagueness of the traditional treatment of the composition of velocities is well illustrated by the words "if a particle is subject to two simultaneous velocities" (p. 158 of the book under review). It is true that such language is very common and that there is a stock explanation of its meaning. This explanation refers to a particle moving with a certain velocity with respect to a body which is itself in motion. Thus if a particle is supposed to "move along a rod with uniform speed" while the rod is "carried parallel to itself" with uniform velocity, it is easily seen that the actual velocity of the particle is the vector sum of the velocity of the rod and the velocity of the particle with respect to the rod. There is no vagueness here; but the explanation throws no light whatever upon the meaning of simultaneous velocities *with respect to a single base or body of reference.* Yet this is precisely what needs explanation, since the language commonly employed implies that a body may have at the same time two different velocities *with respect to the same base.* The fact is that, so long as motion is referred to one definite base, a particle has at any instant one definite velocity. This actual velocity, being a directed or vector quantity,[2] may *for the purposes of mathematical analysis or computation* be resolved into components by the rules of vector addition; but this does not at all warrant the statement that a particle has (or is "subject to") two or more different velocities at the same time.[3]

What has just been said of velocity applies also to acceleration, so far as the meaning of vector composition and resolution is concerned. An acceleration is

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[1] The statement that an action and reaction constitute a pair of counterbalancing forces is one of the most unfortunate errors which are perpetuated by the carelessness of able writers. Action and reaction never counterbalance each other; they act on different portions of matter.

[2] The fact that "velocity is a vector" is not a proposition requiring proof (as might be inferred from the language used on p. 159); the very conception of the instantaneous motion of a particle makes its velocity a directed magnitude.

[3] The logical point involved may be illustrated by an analogy. The position of a particle may be specified by a vector OA, drawn from a fixed origin O to the instantaneous position A, and this position-vector may be expressed as the sum of two vectors OB, OC so taken that $OBAC$ is a parallelogram. If we were to interpret this as meaning that the particle at A may be regarded as occupying simultaneously the positions B and C, we should be making the same kind of statement that is made when it is said that a particle has at the same instant two different velocities.

It may be remarked also that the use of the words "subject to a velocity" seems to indicate a confusion of thought, since they imply that the continuance of a velocity must be due to some external cause—a notion explicitly negatived by the first law of motion.

a vector quantity and may, for the purposes of analysis or computation, be resolved into components; but this does not warrant the statement that a particle has at the same time two different accelerations with respect to one given base. *Apart from the idea of force* the resolution of the actual acceleration into vector components is an arbitrary geometric process. When we come to interpret the second law of motion as applied to a particle acted upon by more than one force, there is of course a special reason for treating the acceleration as made up of vector components, each associated with a particular force; the explanation of this fact is however not a proof that " acceleration is a vector " but is an explanation of the law of composition of forces. The idea that acceleration is a directed or vector quantity is purely kinematical—quite independent of any law involving force; in fact the full understanding of Newton's second law must presuppose the notion of acceleration defined as the rate of change of the velocity-vector.

3. In the traditional treatment there is a logical confusion associated with the definitions of work and energy. Energy is defined as *the ability of a body to do work,* while work is defined as being done not *by a body* but *by a force.*

In the book under review the chapter on Work and Energy begins with an admirably simple and clear elementary treatment of work done by a force. Kinetic energy is then defined by an algebraic formula, and a proof is given of the theorem that " the change in the kinetic energy of a particle equals the work done by the resultant force acting on it." Then follows a brief discussion of the condition for the existence of a force function U, leading to the definition of potential energy as " the negative of the function U." Thus far there is no lack of logical rigor. There follows however an Article headed " The Energy of a Particle is its Ability to do Work " in which we read: " It seems reasonable from the Third Law of Motion that the equations[1] . . . may be read the other way, viz., that in changing from the velocity v to the velocity v_0 the particle will do work on another body equal to w. This comes to saying that a particle will do as much work in giving up its velocity as has been done on the particle in order that it might acquire it." But what is meant by a particle " doing work upon a body "? This question is nowhere answered. The passage quoted is in fact fairly representative of the logical vagueness of the traditional treatment. To remove this vagueness it is necessary to adopt a clear *definition* of " work done by a body," from which it may be *proved* that the " ability of a particle to do work " (*i. e.*, the quantity of work it will do in coming to rest) is equal to $\frac{1}{2}mv^2$. With reference to the above quotation it is also to be said that the citation of the third law of motion is quite irrelevant and encourages confusion as to the meaning of that law.

The space occupied by the foregoing criticisms is out of proportion to their applicability to the book under review; the criticisms are in fact directed rather at the traditional treatment than at the example of it presented by this book. The usefulness of a book as a text depends far more upon the treatment of concrete

[1] The equations referred to express the fact that " the change of the kinetic energy of a particle equals the work done by the forces acting on it."

applications than upon the logical rigor with which fundamental principles are established. The book of Miller and Lilly seems to be decidedly usable as a class textbook, and is likely to find favor among teachers.

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PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

468. Proposed by H. C. FEEMSTER, York College, Nebraska.

In each of the following series find the nth term and sum:

(a) $$2 + 5 + 9 + 15 + 24 + \cdots$$
(b) $$1 + 6 + 10 + 20 + 35 + \cdots$$
(c) $$1 + 5 + 15 + 35 + 70 + \cdots$$

469. Proposed by T. H. GRONWALL, New York City.

Show that the equation

$$f(x) = 2ax^4 + (1 - b)x^3 + b(1 - b)x - 2ab = 0,$$

where $0 < b < 1$, $a > 0$ and $a^2 > b$ has only one positive root and that this root lies between the roots of $g(x) = x^2 - 2ax + b = 0$.

470. Proposed by ERNEST W. BROWN, Yale University.

There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, where s is an integral multiple of $\frac{1}{2}$?

GEOMETRY.

499. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

500. Proposed by R. T. McGREGOR, Bangor, California.

$OABC$, $OA'B'C'$ are two straight lines such that AA', BB', CC' are parallel. AB', $A'C$ meet in P; $A'B$ and AC' meet in Q. Show by synthetic projective geometry that PQ is parallel to AA'. MILNE's *Projective Geometry*, Chap. I, Ex. 20.

501. Proposed by R. P. BAKER, University of Iowa.

Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box.

502. Proposed by R. P. BAKER, University of Iowa.

A designer of machinery requires a curve having the following properties:

(1) A closed curve touching a given circle at two diametral points and enclosing it.
(2) The sum of the three radii from the center of this circle to the curve which make with each other angles of 120° is constant.

(3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

CALCULUS.

417. Proposed by H. S. UHLER, Yale University.

To the degree of approximation indicated, show that

$$(\sqrt{-1})^{\sqrt{-1}} = 0.207,879,576,351.$$

418. Proposed by B. F. FINKEL, Drury College.

A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grandstands, training quarters, etc. If the land costs $200 an acre, what will be the least possible cost of the land required?

GRANVILLE'S *Differential and Integral Calculus*, p. 116.

Is there anything wrong with this problem? Explain the contradiction involved in the solution.

MECHANICS.

334. Proposed by HORACE OLSON, Chicago, Illinois.

A particle of elasticity e is projected with a velocity v at an angle ϕ with a plane inclined to the horizontal at an angle ψ; its plane of motion is perpendicular to the inclined plane. Show that after $2v \sin \phi/g(1 - e) \cos \psi$ seconds it will cease to rebound and will move along the plane with an initial velocity $v \cos \phi - 2v \sin \phi \tan \psi/(1 - e)$ and a uniform acceleration, $g \sin \psi$, down the plane.

(This problem is a generalization of problem 289 in Mechanics, which appeared in the March, 1914, issue of the MONTHLY.)

335. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the elevation of the particle when the velocity downwards is V is equal to LT where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity V^2/L.

NUMBER THEORY.

254. Proposed by HORACE OLSON, Chicago, Illinois.

Find three integers, x, y, and z, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all perfect squares.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

456. Proposed by PAUL CAPRON, U. S. Naval Academy.

If

$$S_{i,n} = \sum_{k=1}^{k=n-i+1} \frac{(i + k - 1)!}{(k - 1)!},$$

show that $S_{i,n}$ is equal to $1/(i + 1)$ times the last term of $S_{i+1,n+1}$; as, for instance, that

$$S_{1,n} = 1 + 2 + \cdots + n = \frac{n}{2}(n + 1),$$

that

$$S_{2,n} = 1\cdot 2 + 2\cdot 3 + \cdots + n(n + 1) = \tfrac{1}{3}(n - 1)n(n + 1),$$

etc.

I. Solution by Howard C. Feemster, York College, Nebraska.

$$S_{1,n}=\frac{\underline{|1}}{\underline{|0}}+\frac{\underline{|2}}{\underline{|1}}+\cdots+\frac{\underline{|n}}{\underline{|n-1}}=1+2+\cdots+n=\frac{n}{2}(n+1);$$

$$S_{2,n}=\frac{\underline{|2}}{\underline{|0}}+\frac{\underline{|3}}{\underline{|1}}+\cdots+\frac{\underline{|n}}{\underline{|n-2}}=1\cdot2+2\cdot3+\cdots+(n-1)n=\tfrac{1}{3}n(n-1)(n+1),$$

for $n-1$ terms; and

$$S_{3,n}=\frac{\underline{|3}}{\underline{|0}}+\frac{\underline{|4}}{\underline{|1}}+\cdots+\frac{\underline{|n}}{\underline{|n-3}}=1\cdot2\cdot3+2\cdot3\cdot4+\cdots+\frac{(n-2)(n-1)n(n+1)}{4},$$

for $n-2$ terms.

Now assume

$$\begin{aligned}S_{i,n}=\frac{\underline{|i}}{\underline{|0}}+\frac{\underline{|i+1}}{\underline{|1}}+\frac{\underline{|1+3}}{\underline{|2}}+\cdots+\frac{\underline{|n}}{\underline{|n-i}}&=1\cdot2\cdot3\cdots i+2\cdot3\cdots(i+1)\\ &\quad+3\cdot4\cdots(i+2)+\cdots+(n-i+1)(n-i+2)\cdots n\\ &=\frac{(n-i+1)(n-i+2)\cdots(n)(n+1)}{i+1},\end{aligned}\tag{1}$$

for $n-i+1$ terms.

But the formula is true for $n-i+1=1$ and $n-i+1=2$; for

$$\frac{\underline{|i}}{\underline{|0}}=\frac{1}{i+1}\cdot\frac{\underline{|i+1}}{\underline{|1}}\qquad\text{and}\qquad\frac{\underline{|i}}{\underline{|0}}+\frac{\underline{|i+1}}{\underline{|1}}=\frac{\underline{|i+2}}{i+1}.$$

If (1) is true

$$\begin{aligned}S_{i,n+1}&=\frac{(n-i+1)(n-i+2)\cdots(n)(n+1)}{i+1}+(n-i+2)(n-i+3)\cdots n(n+1)\\ &=\frac{(n-i+1)(n-i+2)\cdots n(n+1)+(i+1)(n-i+2)(n-i+3)\cdots(n)(n+1)}{i+1}\\ &=\frac{(n-i+2)(n-i+3)\cdots(n)(n+1)(n+2)}{i+1},\end{aligned}$$

which is of the same form as (1) when the number of terms is increased by unity, and hence the proposition is true in general.

II. Solution by the Proposer.

Let $\phi(x)\equiv a^n+a^{n-1}x+a^{n-2}x^2+\cdots+a^{n-i}x^i+\cdots+x^n$;
then

$$\frac{d^i\phi(x)}{dx^i}=\phi^{(i)}(x)\equiv i!a^{n-i}+\frac{(i+1)!}{1!}a^{n-i-1}x+\frac{(i+2)!}{2!}a^{n-i-2}x^2+\cdots+\frac{n!}{(n-1)!}x^{n-i};$$

and

$$\phi(a)=na^n,\qquad\phi^{(i)}(a)=S_{i,n}a^{n-i},\qquad\phi^{(n)}(a)=S_{n,n}a^\circ=n!.$$

Let $f(x)\equiv(x-a)\phi(x)\equiv x^{n+1}-a^{n+1}$; then $f(a)=0$,

$$\begin{array}{ll}f'(x)\equiv(x-a)\phi'(x)+\phi(x); & f'(a)=\phi(a)=na^n.\\ f''(x)\equiv(x-a)\phi''(x)+2\phi'(x); & f''(a)=2\phi'(a)=2S_{1,n}\cdot a^{n-1}.\\ \cdots\cdots & \cdots\cdots\\ f^{(i+1)}(x)\equiv(x-a)\phi^{(i+1)}(x)+(i+1)\phi^{(i)}(x); & f^{(i+1)}(a)=(i+1)\phi^{(i)}(a)=(i+1)S_{i,n}\cdot a^{n-i}.\\ \cdots\cdots & \cdots\cdots\\ & f^{(n+1)}(a)=(n+1)\phi^{(n)}(a)=(n+1)!\end{array}$$

Since $f(a+x) \equiv (a+x)^{n+1} - a^{n+1} \equiv \binom{n+1}{1}a^n x + \binom{n+1}{2}a^{n-1}x^2 + \cdots + \binom{n+1}{i+1}a^{n-i}x^{i+1} + \cdots + x^{n+1}$, and again,

$$f(a+x) \equiv f(a) + xf'(a) + \frac{x^2}{\underline{|2}}f''(a) + \cdots + \frac{x^{i+1}}{(i+1)!}f^{(i+1)}(a) + \cdots + \frac{x^{n+1}}{(n+1)!}(n+1)!,$$

we have, on comparison of coefficients,

$$\frac{1}{(i+1)!}f^{(i+1)}(a) = \binom{n+1}{i+1}a^{n-1} = \frac{(n+1)!}{(i+1)!(n-i)!}a^{n-1},$$

$$f^{(i+1)}(a) = (i+1)\phi^{(i)}(a) = (i+1)S_{i,\,n} \cdot a^{n-1} = \frac{(n+1)!}{(n-i)!}a^{n-1},$$

so that

$$S_{i,\,n} = \frac{1}{i+1} \cdot \frac{(n+1)!}{(n-i)!},$$

and $(n+1)!/(n-i)!$ is the last term of $S_{i+1,\,n+1} = \sum_{k=1}^{k=n-i+1} (i+k)!/(k-1)!$.

Also solved by HORACE OLSON.

457. Proposed by FRANK IRWIN, University of California.

If a be any number prime to m and m/a be developed as a continued fraction,

$$\frac{m}{a} = a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +} \cdots + \frac{1}{a_{k-1} +}\frac{1}{a_k},$$

with $a_1 \neq 0$, then there will exist a number b such that

$$\frac{m}{b} = a_k + \frac{1}{a_{k-1} +} \cdots + \frac{1}{a_2 +}\frac{1}{a_1}.$$

Show that $ab \equiv \pm 1$ (mod. m) and determine the sign.

SOLUTION BY THE PROPOSER.

Write m and a as functions of the a's: $m = [a_1, a_2, \cdots a_k]$, $a = [a_2, a_3, \cdots a_k]$, using the *Gaussische Klammer* notation, see BACHMANN, *Niedere Zahlentheorie*, vol. 1, p. 104; or, with a different symbol, $K(a_1, a_2, \cdots a_k)$, CHRYSTAL'S *Algebra*, part II, 2d ed., p. 495.

Similarly, $b = [a_{k-1}, \cdots a_2, a_1]$, and if p_{k-1}/q_{k-1} be the next to last convergent to m/a, $p_{k-1} = [a_1, a_2, \cdots a_{k-1}]$. But, by an elementary property of these expressions, $[a_1, a_2, \cdots a_{k-1}] = [a_{k-1}, \cdots a_2, a_1)$; that is, $p_{k-1} = b$.

Again, since p_{k-1}/q_{k-1}, and m/a are successive convergents, $mq_{k-1} - ap_{k-1} = (-1)^k$; that is, $mq_{k-1} - ab = (-1)^k$, or $ab \equiv (-1)^{k-1}$ (mod. m).

GEOMETRY.

483. Proposed by LAENAS G. WELD, Pullman, Illinois.

A circle is inscribed in a triangle. In each of the three spandrils exterior to the circle another circle is inscribed; in the remaining spandrils three other circles; and so on ad infinitum. Show that the sum of the areas of these circles is given by the formula:

$$\Sigma = \frac{\pi}{4} \cdot \frac{\Delta^2}{S^2}\left[\frac{1}{\sin(A/2)} + \frac{1}{\sin(B/2)} + \frac{1}{\sin(C/2)} - 2 + \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right].$$

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let A, B, C and a, b, c denote the angles and sides opposite these angles of the given triangle. Since the center of the inscribed circle is at the intersection of the bisectors of the angles of the triangle, the centers of the circles inscribed in the spandrils are on these bisectors. Let R be the radius of the inscribed circle and R_1, R_2, $\cdots$ R_n the radius of the circles whose centers are A_1, A_2, $\cdots$ A_n, respectively.

From the triangle AOB we can easily show that

$$AO = \frac{c \sin B/2}{\sin (A + B)/2} \quad \text{and} \quad R = \frac{c \sin A/2 \sin B/2}{\sin (A + B)/2};$$

and since

$$AA_1 = AO - A_1O = AO - R - R_1, \qquad AA_1 = \frac{R_1}{\sin A/2}.$$

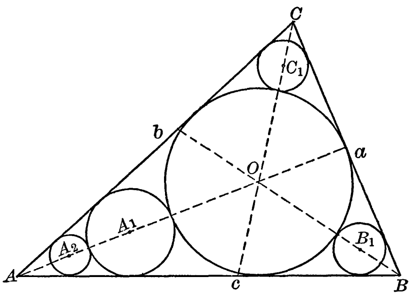
Hence, we have

$$\frac{R_1}{\sin A/2} = \frac{c \sin B/2}{\sin (A + B)/2} - \frac{c \sin A/2 \sin B/2}{\sin (A + B)/2} - R_1.$$

Solving for R_1 we get

$$R_1 = \frac{c \sin A/2 \sin B/2}{\sin (A + B)/2} \cdot \frac{1 - \sin A/2}{1 + \sin A/2}.$$

In the same way we can prove from $AA_2 = AO - A_2O$ that



$$R_2 = \frac{c \sin A/2 \sin B/2}{\sin (A + B)/2} \cdot \left(\frac{1 - \sin A/2}{1 + \sin A/2}\right)^2,$$

and finally, for circles on OA, $R_n = R(F_a)^n$; and similarly for circles on OB, $R_n = R(F_b)^n$; for circles on OC, $R_n = R(F_c)^n$, where

$$F_a = \frac{1 - \sin A/2}{1 + \sin A/2}, \qquad F_b = \frac{1 - \sin B/2}{1 + \sin B/2}, \qquad F_c = \frac{1 - \sin C/2}{1 + \sin C/2}.$$

The sum Σ of the areas of all these circles will then be

$$\Sigma = \pi R^2 + \pi R^2(F_a)^2 + \pi R^2(F_a)^4 + \pi R^2(F_a)^6 + \cdots \pi R^2(F_a)^{2n} + \pi R^2(F_b)^2 + \pi R^2(F_b)^4 + \cdots \pi R^2(F_b)^{2n} + \pi R^2(F_c)^2 + \pi R^2(F_c)^4 + \cdots \pi R^2(F_c)^{2n};$$

or,

$$\Sigma = \pi R^2[1 + (F_a)^2 + (F_a)^4 + \cdots (F_a)^{2n} + 1 + (F_b)^2 + (F_b)^4 + \cdots (F_b)^{2n} + 1 + (F_c)^2 + (F_c)^4 + \cdots (F_c)^{2n} - 2].$$

Adding the terms of each of these geometric series by the usual formula, we have

$$\Sigma = \pi R^2 \left[\frac{(F_a)^{2n} - 1}{(F_a)^2 - 1} + \frac{(F_b)^{2n} - 1}{(F_b)^2 - 1} + \frac{(F_c)^{2n} - 1}{(F_c)^2 - 1} - 2 \right].$$

If n approaches infinity, we have

$$\Sigma = \pi R^2 \left[\frac{1}{1 - (F_a)^2} + \frac{1}{1 - (F_b)^2} + \frac{1}{1 - (F_c)^2} - 2 \right].$$

Replacing the values of F_a, F_b, and F_c we get

$$\Sigma = \frac{\pi R^2}{4} \left[\frac{(1 + \sin A/2)^2}{\sin A/2} + \frac{(1 + \sin B/2)^2}{\sin B/2} + \frac{(1 + \sin C/2)^2}{\sin C/2} - 2 \right].$$

Let Δ be the area of the given triangle, then $\Delta = \frac{1}{2}(a + b + c)R$ or $R = \Delta S$ where $S = \frac{1}{2}(a + b + c)$.

Substituting the value of R in Σ, expanding the binomials, and reducing we finally get,

$$\Sigma = \frac{\pi}{4}\cdot\frac{\Delta^2}{S^2}\left[\frac{1}{\sin A/2}+\frac{1}{\sin B/2}+\frac{1}{\sin C/2}-2+\sin A/2+\sin B/2+\sin C/2\right].$$

Also solved by F. R. Morris, Horace Olson, G. W. Hartwell, J. W. Clawson, Paul Capron, and J. W. Cromwell.

484. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that when spheres of uniform size are packed in the closest possible manner there is, in the interior of the mass, about 26 per cent. of voids.

Solution by Laenas G. Weld, Pullman, Ills.

The given space may be divided into (equal) rhombic dodecahedrons. Spheres inscribed to these will be packed in the closest possible manner, since each has contact with twelve equal spheres. Now, the rhombic dodecahedron may be conceived as follows: Place a cube and a regular octahedron in such relation that each of the twelve edges of either figure is bisected at right angles by an edge of the other; then join the extremities of the edges so related, thus forming twelve rhombs, which are the faces of the figure in question. Each rhomb has for one of its diagonals an edge of the cube and for the other the corresponding edge of the octahedron. Taking the edge of the octahedron as 1, that of the cube related to it as above is equal to $\frac{1}{2}\sqrt{2}$. Moreover the radius of the inscribed sphere is equal to $\frac{1}{2}$, which is also the altitude of each of the twelve rhombic pyramids into which the dodecahedron may be resolved. The area of the base of each of these pyramids is one half the product of its diagonals, which is equal to $\frac{1}{4}\sqrt{2}$; whence the volume of the dodecahedron is

$$D = 12(\tfrac{1}{3}\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}\sqrt{2}) = \tfrac{1}{2}\sqrt{2}.$$

The volume of the sphere is $S = \pi/6$.

The ratio S/D is equal to the ratio of the aggregate volume of the spheres to that of the space in which they are packed, since this space is completely filled by the circumscribing rhombic dodecahedrons. This ratio is $0.7405-$. The voids, therefore, occupy $0.2595+$ of the space, or about 26 per cent.

Also solved with slightly different results by Paul Capron, G. Paaswell, J. W. Clawson, and Herbert N. Carleton.

CALCULUS.

400. Proposed by H. S. UHLER, Yale University.

The axis of a prism whose right section is a regular polygon of apothem a and n sides passes through the center of a sphere of radius R. Show that, in general, the volume may be expressed by the formula:

$$V = \tfrac{4}{3}\pi R^3 + \tfrac{2}{3}a^2 n\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}$$
$$+\tfrac{1}{3}an(3R^2-a^2)\sin^{-1}\left[\frac{2a\left(R^2-a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}}{R^2-a^2}\right]-\tfrac{4}{3}nR^3\sin^{-1}\left[\frac{R\sin\frac{\pi}{n}}{(R^2-a^2)^{1/2}}\right].$$

Also discuss the special cases when $a = R\cos(\pi/n)$ and when $n = \infty$.

Solution by R. K. Morley, Worcester, Mass.

Using cylindrical coördinates

$$V = 4n\int_0^{\pi/n}\int_0^{a\sec\theta}\int_0^{\sqrt{R^2-}}\rho dz d\rho d\theta = -\tfrac{4}{3}n\int_0^{\pi/n}(R^2-a^2\sec^2\theta)^{3/2}d\theta + \tfrac{4}{3}n\int_0^{\pi/n}R^3 d\theta$$
$$= -\tfrac{4}{3}n\int_0^{\pi/n}(R^2-a^2\sec^2\theta)(R^2-a^2\sec^2\theta)^{1/2}d\theta + \tfrac{4}{3}\pi R^3.$$

For brevity, set $c^2 = R^2 - a^2$. Then

$$V = -\tfrac{4}{3}n\int_0^{\pi/n} \frac{R^2(R^2 - a^2\sec^2\theta)d\theta}{(R^2 - a^2\sec^2\theta)^{1/2}} + \tfrac{4}{3}na\int_0^{\pi/n}(c^2 - a^2\tan^2\theta)^{1/2}\cdot a\sec^2\theta d\theta + \tfrac{4}{3}\pi R^3,$$

$$= -\tfrac{4}{3}nR^3\int_0^{\pi/n}\frac{R\cos\theta\, d\theta}{(c^2 - R^2\sin^2\theta)^{1/2}} + \tfrac{4}{3}nR^2a\int_0^{\pi/n}\frac{a\sec^2\theta d\theta}{(c^2 - a^2\tan^2\theta)^{1/2}}$$

$$+ \tfrac{4}{3}na\left[\frac{a\tan\frac{\pi}{n}}{2}\sqrt{c^2 - a^2\tan^2\frac{\pi}{n}} + \frac{c^2}{2}\sin^{-1}\left(\frac{a\tan\frac{\pi}{n}}{c}\right)\right] + \tfrac{4}{3}\pi R^3,$$

$$= -\tfrac{4}{3}nR^3\sin^{-1}\left(\frac{R\sin\frac{\pi}{n}}{c}\right) + \tfrac{4}{3}nR^2a\sin^{-1}\left(\frac{a\tan\frac{\pi}{n}}{c}\right)$$

$$+ \tfrac{2}{3}na^2\tan\frac{\pi}{n}\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}} + \tfrac{2}{3}na(R^2 - a^2)\sin^{-1}\left(\frac{a\tan\frac{\pi}{n}}{c}\right) + \tfrac{4}{3}\pi R^3,$$

$$= \tfrac{4}{3}\pi R^3 + \tfrac{2}{3}a^2n\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}}\tan\frac{\pi}{n} - \tfrac{4}{3}nR^3\sin^{-1}\left[\frac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right]$$

$$+ \tfrac{2}{3}an(3R^2 - a^2)\sin^{-1}\left[\frac{a\tan\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right]. \quad (1)$$

This form for the result is more convenient than that required, but if desired this may be transformed into the required form by means of the relation $2A = \sin^{-1}(2\sin A\cos A)$ applied to the last term.

Special Case I. $a = R\cos(\pi/n)$. Then the polygonal cross-section of the prism is inscribed in a great circle of the sphere, and

$$V_{\text{I}} = \text{volume of sphere} - n \text{ spherical segments of altitude } R - a$$

$$= \tfrac{4}{3}\pi R^3 - n\pi\left[R(R - a)^2 - \frac{(R - a)^3}{3}\right],$$

$$= \tfrac{4}{3}\pi R^3 - n\pi\left(\frac{2}{3}R^3 - R^2a + \frac{a^3}{3}\right).$$

Putting $a = R\cos(\pi/n)$ in (1) throughout except in the coefficient of the last term and reducing

$$V_{\text{I}} = \tfrac{4}{3}\pi R^3 - \tfrac{2}{3}\pi nR^3 + \pi nR^2a - \tfrac{1}{3}\pi na^3,$$

$$= \tfrac{4}{3}\pi R^3 - n\pi\left(\frac{2}{3}R^3 - R^2a + \frac{a^3}{3}\right),$$

as before.

Special Case II. $n = \infty$. In this case there are some indeterminates to be evaluated. From the second term of (1)

$$\lim_{n=\infty} n\tan\frac{\pi}{n}\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}} = \lim_{n=\infty}\frac{\tan\frac{\pi}{n}}{\frac{1}{n}}\cdot\lim_{n=\infty}\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}}$$

$$= \lim_{n=\infty}\frac{\sec^2\frac{\pi}{n}\cdot\left(-\frac{\pi}{n^2}\right)}{-\frac{1}{n^2}}\cdot\sqrt{R^2 - a^2} = \pi\sqrt{R^2 - a^2}.$$

From the third term of (1),

$$\lim_{n=\infty}\left[\frac{\sin^{-1}\dfrac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}}{\frac{1}{n}}\right] = \lim_{n=\infty}\left[\frac{\dfrac{R\cos\frac{\pi}{n}\cdot\left(-\frac{\pi}{n^2}\right)}{(R^2 - a^2)^{1/2}\sqrt{1 - \frac{R^2}{R^2 - a^2}\sin^2\frac{\pi}{n}}}}{-\frac{1}{n^2}}\right] = \frac{R\pi}{(R^2 - a^2)^{1/2}}.$$

From the last term of (1) in a similar manner

$$\lim_{n=\infty} n \sin^{-1}\left[\frac{a \tan\dfrac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right] = \frac{\pi a}{(R^2 - a^2)^{1/2}}$$

Substituting these values in (1) and reducing

$$V_{\text{II}} = \lim_{n=\infty} V = \tfrac{4}{3}\pi[R^3 - (R^2 - a^2)^{3/2}]$$

which is the result for the familiar problem of the volume cut from a sphere of radius R by circular cylinder of radius a when the center of the sphere is on the axis of the cylinder.

Also solved by Paul Capron, A. W. Smith and the Proposer.

401. Proposed by LAENAS G. WELD, Pullman, Ill.

Given a continuum of triangles whose sides are in arithmetical progression, the common difference being h: (*a*) The ratio of the mean value of all the triangles, the mean of whose three sides is not greater than μ, to the area of the triangle, the mean of whose three sides is equal to μ, is $(\mu + 2h)/3\mu$. Indicate the limiting values of this ratio and show that, when it is equal to 1/2, the triangle whose mean side is μ is right angled. (*b*) The ratio of the mean value of the areas of the circles inscribed in all these triangles, the mean of whose three sides is not greater than μ, to that of the circle inscribed in the triangle, the mean of whose three sides is equal to μ, has the limiting values 1/2 and 1/3. When the triangle whose mean side is μ is right angled, the ratio in question is 4/9. (*c*) Of the circles circumscribed about these triangles the minimum has the radius $2h$.

Solution by A. H. Wilson, Haverford College.

Let $x - h$, x, and $x + h$ be the three sides. Then the area is

$$[s(s - a)(s - b)(s - c)]^{1/2} = \tfrac{1}{4}[3x^2(x^2 - 4h^2)]^{1/2}.$$

The mean area of all triangles, the mean of whose three sides is not greater than μ, is

$$\frac{\sqrt{3}}{4(\mu - 2h)}\int_{2h}^{\mu} x\sqrt{x^2 - 4h^2}\,dx = \frac{\sqrt{3}}{12}\sqrt{\mu^2 - 4h^2}(\mu + 2h),$$

the least value of x being $2h$.

The area of the triangle for which $x = \mu$ is

$$\tfrac{1}{4}[3\mu^2(\mu^2 - 4h^2)]^{1/2} = \frac{\sqrt{3}}{4}\mu(\mu^2 - 4h^2)^{1/2}.$$

(*a*) The ratio of these two areas is $(\mu + 2h)/3\mu$. The limits of this ratio occur for the values $\mu = 2h$ and $\mu = \infty$, and are, respectively, 2/3 and 1/3. If the ratio is equal to 1/2, then $\mu = 4h$; and the sides $3h$, $4h$, and $5h$ are those of a right triangle.

(*b*) The area of the inscribed circle is

$$\pi r^2 = \pi(x^2 - 4h^2)/12. \quad (r = [(s - a)(s - b)(s - c)/s]^{1/2}).$$

The mean of such areas for triangles of the first class is

$$\frac{\pi}{12(\mu - 2h)}\int_{2h}^{\mu}(x^2 - 4h^2)\,dx = \frac{\pi}{36}(\mu^2 + 2h\mu - 8h^2).$$

The area of the circle of the second class is $\pi(\mu^2 - 4h^2)/12$; and the ratio of the two areas is $(\mu + 4h)/3(\mu + 2h)$. The limiting values of this ratio are 1/2 and 1/3, and for the value 4/9 the mean side is $4h$, and the triangle is right-angled.

(*c*) The area of the circumscribed triangle is $\pi R^2 = \pi(abc)^2/16s(s - a)(s - b)(s - c) = \pi(x^2 - h^2)^2/3(x^2 - 4h^2)$. For a minimum, equate the derivative to 0; and there results for x the value $\sqrt{7}h$, giving a minimum. For this value of x the radius of the circumscribed circle is $2h$.

Also solved by Elijah Swift.

402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2), if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x \partial y} = 0$$

where (x', y') are the current coördinates of points on the tangents.

SOLUTION BY C. E. DIMICK, New London, Connecticut.

If (x, y) be a double point of the curve $u \equiv f(x, y) = 0$, $\partial u/\partial x = \partial u/\partial y = 0$ and the two values of dy/dx at the double point must satisfy the quadratic

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

(Todhunter's Diff. Calc., pages 319, 320.)

If the two tangents are perpendicular, the product of their slopes is -1, and since the product of the roots of the quadratic $a + bx + cx^2 = 0$ is a/c we have

$$\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = -1 \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The equations of the two tangents will be $y - y_1 = l_1(x - x_1)$, and $y - y_1 = l_2(x - x_1)$, where l_1 and l_2 are the roots of the quadratic given above. Transposing and multiplying, the equation of the two tangents taken together will be

$$(y - y_1)^2 - (x - x_1)(y - y_1)(l_1 + l_2) + l_1 l_2 (x - x_1)^2 = 0,$$

which becomes on substituting the values of the sum and product of the roots of the quadratic

$$(y - y_1)^2 + 2(x - x_1)(y - y_1)\frac{\partial^2 u}{\partial x \partial y}\Big/\frac{\partial^2 u}{\partial y^2} + (x - x_1)^2\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = 0,$$

which, upon transforming to (x_1, y_1) as origin, becomes

$$y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial x^2} = 0.$$

But $\partial^2 u/\partial x^2 = -\partial^2 u/\partial y^2$ as the tangents are perpendicular.

Hence the equation reduces to

$$(y^2 - x^2)\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0,$$

the minus sign as given in the problem being incorrect.

Also solved by I. A. BARNETT, C. K. ROBBINS, GRACE M. BAREIS, J. A. BULLARD, F. M. MORGAN, and H. L. AGARD.

MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

SOLUTION BY H. S. UHLER, Yale University.

Let f denote the force exerted by the solid sphere of mass m_1 on the shell of radius of gyration k and of mass m. Also, let (x_1, y_1) and (x, R) be the coordinates of the centers (C_1 and C) of the solid and hollow bodies, respectively. f, x, x_1, and y_1 will be functions of the time t. By using the symbols k, m, and m_1 the equations given below apply to two cases in addition to the one proposed, namely, (i) a sphere within a cylindrical shell, and (ii) a cylinder inside a hollow cylinder. Moreover, in all three cases, the bodies are only required to have their centers of mass coincide with their geometric centers, so that the inner body may be hollow and all the bodies may be non-homogeneous radially, that is, the bodies may be built up of similar concentric or coaxial shells of different densities for the several layers. Neither does the density have to be a continuous function of the radius. In the given case $k^2 = 2(R^5 - r^5)/[5(R^3 - r^3)]$, $m = \frac{4}{3}\pi\delta(R^3 - r^3)$, and $m_1 = \frac{4}{3}\pi\delta r_1^3$, where δ symbolizes the constant density of the material involved.

Consider the outer body and take moments around the instantaneous axis through A. Then

$$m(k^2 + R^2)\frac{d^2\phi}{dt^2} = mgR \sin \alpha + fR \sin \theta,$$

where θ is the angle which the line $\overline{C_1C}$ makes at any time t with the fixed direction $\overline{AC}$ or $\overline{OY}$. The geometric meaning of ϕ is obvious from its defining equation $R\phi = x$ (no slipping of the body on the inclined plane). Then

$$\frac{d^2\phi}{dt^2} = \frac{1}{R}\cdot\frac{d^2x}{dt^2}$$

so that, writing s for $1 + k^2/R^2$,

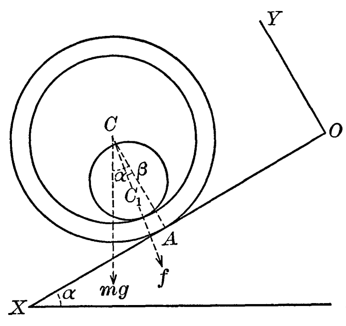
$$ms\frac{d^2x}{dt^2} = mg \sin \alpha + f \sin \theta. \tag{1}$$

The forces acting parallel to $\overline{OX}$ on the inner body fulfil the equation

$$m_1\frac{d^2x_1}{dt^2} = m_1g \sin \alpha - f \sin \theta. \tag{2}$$

The forces parallel to $\overline{OY}$ on this body satisfy the relation

$$m_1\frac{d^2y_1}{dt^2} = - m_1g \cos \alpha + f \cos \theta. \tag{3}$$



From the diagram, we see that

$$x_1 - x = \rho \sin \theta, \tag{4}$$

and

$$R - y_1 = \rho \cos \theta, \tag{5}$$

where $\rho \equiv r - r_1$, $(r > r_1)$.

Of the several differential equations that can be derived from the preceding five relations, the one connecting θ and t is the simplest.

Differentiating (4) and (5) twice each with respect to t we obtain, respectively,

$$\frac{d^2x_1}{dt^2} - \frac{d^2x}{dt^2} = \rho \cos \theta\frac{d^2\theta}{dt^2} - \rho \sin \theta\left(\frac{d\theta}{dt}\right)^2 \tag{4'}$$

and

$$\frac{d^2y_1}{dt^2} = \rho \sin\theta \frac{\theta d^2\theta}{dt^2} + \rho \cos\theta \left(\frac{d\theta}{dt}\right)^2. \tag{5'}$$

Multiplying equations (1), (2), (3), (4′), and (5′) by $m_1 \cos\theta$, $-ms\cos\theta$, $-(ms + m_1)\sin\theta$, $mm_1s\cos\theta$, and $m_1(ms + m_1)\sin\theta$, in the order named, and then adding, we get

$$\rho(ms + m_1\sin^2\theta)\frac{d^2\theta}{dt^2} + m_1\rho\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2 + g(ms + m_1)\cos\alpha\sin\theta - \frac{mgk^2}{R^2}\sin\alpha\cos\theta = 0. \tag{6}$$

Equation (6) may be reduced to the first order by observing that

$$2m_1\sin\theta\cos\theta = \frac{d}{d\theta}(ms + m_1\sin^2\theta), \qquad \text{and} \qquad \frac{d^2\theta}{dt^2} = \frac{d\theta}{dt}\cdot\frac{d(d\theta/dt)}{d\theta}.$$

Accordingly

$$\rho(ms + m_1\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2 - 2g(ms + m_1)\cos\alpha\cos\theta - \frac{2mgk^2}{R^2}\sin\alpha\sin\theta + c' = 0. \tag{7}$$

When $t = 0$, let $\theta = \theta_0$ and $d\theta/dt = \omega_0$, giving

$$c' = 2g(ms + m_1)\cos\alpha\cos\theta_0 + (2mgk^2/R^2)\sin\alpha\sin\theta_0 - \rho(ms + m_1\sin^2\theta_0)\omega_0^2.$$

Consequently,

$$t + f(\theta_0) = \int \frac{d\theta\sqrt{\rho(ms + m_1\sin^2\theta)}}{\sqrt{\begin{aligned}&\rho(ms + m_1\sin^2\theta_0)\omega_0^2 + 2g(ms + m_1)\cos\alpha(\cos\theta - \cos\theta_0)\\ &\qquad - \frac{2mgk^2}{R^2}\sin\alpha(\sin\theta_0 - \sin\theta)\end{aligned}}}. \tag{8}$$

Since the analysis has been explicitly reduced to the evaluation of an indefinite integral the problem may be considered as formally solved. For, equation (8) theoretically gives θ as a function of t so that (5) then expresses the dependence of y_1 (and so also of dy_1/dt and d^2y_1/dt^2) upon t. Or, we may substitute directly in (5′) to get $d^2y_1/dt^2 = F(\theta) = \Phi(t)$. By suitably combining (1), (2), and (4′) it is ideally possible to exhibit x, x_1, dx/dt, dx_1/dt, d^2x/dt^2, and d^2x_1/dt^2 as explicit functions of t. However, since the writer does not know how to evaluate the integral in (8), save as a series development, he cannot profitably continue the general solution. Of course, the periodic nature and other properties of the function can be readily demonstrated but nothing is thereby added to what we know in advance from purely dynamical considerations.

Nevertheless, in the course of the investigation, we have noticed certain general and special facts which seem to be interesting. In the first place, elimination of f from (2) and (3) gives

$$\cos\theta\frac{d^2x_1}{dt^2} + \sin\theta\frac{d^2y_1}{dt^2} = g\sin(\alpha - \theta). \tag{9}$$

This equation shows that the sum of the components normal to $\overline{CC_1}$ of d^2x_1/dt^2 and d^2y_1/dt^2 is equal to the component in the same direction of the acceleration due to gravity, that is, the total linear acceleration of m_1 parallel to the common tangent of the inner and outer bodies is equal to the component of g in the same direction. This is a natural consequence of the perfect smoothness of the surfaces of contact. Relation (9) also indicates explicitly how d^2x_1/dt^2 can be obtained as a function of t as soon as θ and y_1 have been evaluated in terms of the time. Multiplying equations (4′) and (5′) in order by $\cos\theta$ and $\sin\theta$, and then subtracting the sum from (9) we find

$$\rho\frac{d^2\theta}{dt^2} + \cos\theta\frac{d^2x}{dt^2} = g\sin(\alpha - \theta). \tag{10}$$

This equation admits of the same physical interpretation as (9), relative motion of the bodies now being involved, and shows how to get d^2x/dt^2 directly as a function of the time.

As a second case, the result of adding equations (1) and (2) is

$$ms\frac{d^2x}{dt^2} + m_1\frac{d^2x_1}{dt^2} = g(m + m_1)\sin\alpha. \tag{11}$$

If we take a point (ξ, η) which divides the line joining $C_1(x_1, y_1)$ to $C(x, R)$ internally in the ratio $ms : m_1$, then $\xi = (msx + m_1x_1)/(ms + m_1)$ so that (11) becomes

$$\frac{d^2\xi}{dt^2} = \frac{(m + m_1)g\sin\alpha}{ms + m_1}. \tag{12}$$

Therefore, this point, which is definitely associated with the moving system, has the property of constant acceleration parallel to the inclined plane. Equation (12) can be integrated at once giving $d\xi/dt$ and ξ as explicit functions of the time, quite independently of θ. On the other hand, $\eta = (msR + m_1y_1)/(ms + m_1)$; hence,

$$\frac{d^2\eta}{dt^2} = \frac{m_1}{ms + m_1} \cdot \frac{d^2y_1}{dt^2},$$

which is, in general, a function of θ.

An important special case of the motion of the system of bodies is brought out by (6). This equation is satisfied for all time if θ maintains the constant value θ' given by

$$\tan \theta' = \frac{mk^2}{mk^2 + (m + m_1)R^2} \cdot \tan \alpha; \tag{13}$$

for then $d^2\theta/dt^2 = d\theta/dt = 0$. Relation (13) shows that $\alpha > \theta' > 0$, as in the diagram. Under these conditions (4) and (5) give, respectively, $dx_1/dt = dx/dt$ and $dy_1/dt = 0$. Consequently, if we start with the centers of the inner and outer bodies at the respective points $(x_0 + \rho \sin \theta', R - \rho \cos \theta')$ and (x_0, R), and impart equal linear velocities parallel to the incline, the inner body will not oscillate relative to its constraining wall but will maintain the constant angular position θ' while the shell slips under it. The equations of motion of the bodies are now easily shown to be

$$(ms + m_1)\frac{d^2x}{dt^2} = (ms + m_1)\frac{d^2x_1}{dt^2} = g(m + m_1) \sin \alpha,$$

$$(ms + m_1)\frac{dx}{dt} = (ms + m_1)\frac{dx_1}{dt} = g(m + m_1) \sin \alpha \cdot t + (ms + m_1)v_0,$$

$$(ms + m_1)x = \tfrac{1}{2}g(m + m_1) \sin \alpha \cdot t^2 + (ms + m_1)(v_0t + x_0),$$

$$(ms + m_1)x_1 = \tfrac{1}{2}g(m + m_1) \sin \alpha \cdot t^2 + (ms + m_1)(v_0t + x_0 + \rho \sin \theta').$$

Finally, if we introduce a new variable angle λ, defined by the equation $\lambda = \theta - \theta'$, and make use of (13) we can change the expression under the radical in the denominator of the integral in (8) to the form

$$\rho(ms + m_1 \sin^2 \theta_0)\omega_0^2 + 4g(ms + m_1) \cos \alpha \sec \theta' \sin \tfrac{1}{2}[(\theta' - \theta_0) + \lambda] \sin \tfrac{1}{2}[(\theta' - \theta_0) - \lambda],$$

which assumes the same value when λ is assigned values which are numerically equal but of opposite sign. Unfortunately for the analysis, the numerator of the integrand does not possess this kind of symmetry.

No other correct solution of this problem was received. Should any one integrate equation (8) we shall be glad to publish the result.—EDITORS.

NUMBER THEORY.

223. (October, 1914) Proposed by T. E. MASON, Purdue University.

Show that

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}}$$

is an integer, r, s, and t being positive integers. Generalize to the case of n integers, $r, s, t, u, \cdots$. [Carmichael's *Theory of Numbers*, page 28.]

SOLUTION BY FRANK IRWIN, University of California.

Suppose we have rst objects, and let us divide them into t classes of rs objects each, then each class into s sub-classes of r objects each, and let us call each such classification, without any reference to order, a "classification" *par excellence.* We assert that the total number of such classifications is

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}},$$

which expression is, consequently, an integer.

For, let us set up an order among the classes, among the sub-classes in each class, and among the objects in each sub-class. Then from any given classification we may, by permutations of the objects which make changes in the order merely, but do not change the constituents of the classes and sub-classes—which, then, leave the classification as such the same—we may, by such operations, obtain various permutations of the totality of the rst objects. First, without changing the order of the classes or sub-classes, we may permute among themselves the objects in each sub-class. For any one sub-class this may be done in $r!$ ways, and, therefore, for all the st sub-classes in $(r!)^{st}$ ways. Next, we may permute among themselves the sub-classes in each class. For any one class this may be done in $s!$ ways; for all the t classes, then, in $(s!)^t$ ways. Finally, we may permute the classes in $t!$ ways.

Combining these various kinds of permutations, we see that each classification gives rise to $t!(s!)^t(r!)^{st}$ permutations of the rst objects. Since each of these latter permutations may be derived, by the process explained, from some classification or other, and two different classifications cannot give rise to the same permutation, it follows that the number of permutations of the rst objects is $t!(s!)^t(r!)^{st}$ times the number of classifications; that is, since there are $(rst)!$ permutations there are, as was to be shown, $(rst)!/t!(s!)^t(r!)^{st}$ classifications.

The argument may evidently be generalized to show (with a change in the notation) that

$$\frac{(r_1r_2\cdots r_n)!}{r_1!(r_2!)^{r_1}(r_3!)^{r_1r_2}\cdots(r_n!)^{r_1r_2\cdots r_{n-1}}}$$

is an integer.

228. Proposed by HERMON C. KATANIK,* Indianapolis, Ind.

Deduce a formula for the difference between any two squares, and thus show that (1) The difference between any two consecutive squares is of the form $2n+1$; (2) The difference between any two squares is even or odd according to whether they are separated by an odd or even number of squares; (3) The differences of the squares of the consecutive terms of any arithmetic progression form another arithmetic progression.

SOLUTION BY WALTER C. EELLS, Whitman College.

Let T_i be the ith term. Then $T_n = n^2$, $T_{n+k} = (n+k)^2$.

$$T_{n+k} - T_n = (n+k)^2 - n^2 = (n+k-n)(n+k+n) = k(2n+k).$$

(1) If $k = 1$, $T_{n+1} - T_n = (2n+1)$.

(2) When separated by an odd number of squares $T_{n+2k'} - T_n = 2k'(2n+2k')$, which is even. When separated by an even number of squares $T_{n+2k'+1} - T_n = (2k'+1)(2n+2k'+1)$, which is odd.

(3) Since k is constant for any given arithmetic series, $T_{n+mk} - T_{n+(m-1)k}$ is constant and is the constant difference for another arithmetic series.

Also solved by HORACE OLSON and HERBERT N. CARLETON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kans.

A number of questions published in this department have been standing for some time without having been answered. We are re-publishing these in full this month in the hope that some of our readers may thereby be stimulated to send in suitable replies.

12. In view of the notation used by Professor Slobin in his "Note on Certain Algebraic Equations" published in the MONTHLY for April, 1914, pages 113–115, a discussion would be desirable as to the best notation for complex roots in general, and in particular for eliminating the conspicuous ambiguities introduced by the notation above cited.

15. We are in receipt of the following communication from Mr. W. E. Heal, of Washington, D. C.: "In the Proceedings of the Royal Society of Edinburgh, Vol. VII, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

* Deceased since proposing this problem.

"'If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"'This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube.'

"The writer has failed to see how this 'easy proof' follows and has been unable to find the question discussed or even mentioned in Tait's collected works. Can some reader of the MONTHLY supply the missing link or links?"

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

21. For the diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$x = 3, \quad 4, \; 5, \; 9, \; 23, \; 282, \; 375, \; 378661,$$

$$y = -2, \; -1, \; 2, \; 4, \; 5, \; 43, \; 52, \; 5234.$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

26. Why should not the nomenclature of mathematics be made uniform? For example, why call a circle a *portion of a plane* in elementary geometry and a *curve* in analytic geometry? Why call a sphere a *ball* at one time and a *surface* at another time? And so on through all the configurations of two- and three-dimensional geometry.

28. Is it possible to obtain $\int \cos \theta^2 d\theta$ without expanding $\cos \theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

30. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

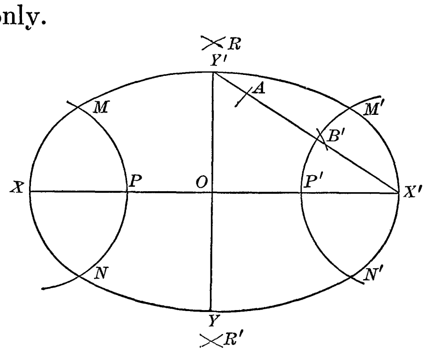
NEW QUESTION.

33. Under what conditions or to what extent is Mr. Iwerson's construction, given below, a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

AN APPROXIMATE CONSTRUCTION FOR AN ELLIPSE.

By RICHARD IWERSON, Everett, Washington.

Given the axes of an ellipse, to construct approximately the curve by use of ruler and compasses only.



Draw XX', the major axis, and YY', the minor axis, perpendicular to and bisecting each other at O. Draw the line $X'Y'$. With O as center and Ox as radius describe an arc cutting $X'Y'$ at B. With Oy as radius and X' as center

describe an arc cutting $X'Y'$ at A. With AB as radius and X as center describe an arc cutting XO at P and extending on each side of XO toward Y and Y'. With the same radius (AB) and P as center draw an arc through X cutting the the arc just drawn in the points M and N. Construct in similar manner the arc $M'N'$ through X'. With NN' as radius and N and N' as centers describe arcs toward Y' cutting each other, obviously, in some point R on OY' (produced, if necessary). With R as center and RN as radius draw the arc NN'. In similar manner draw the arc MM'. The curve $XMY'M'X'N'YNX$ is the required approximation.

Note.—The above construction obviously can not be a true ellipse, except for $a = b$, since the curvature on the arc NYN' is constant while the curvature of the ellipse is variable $(= a^4b^4/(b^4x^2 + a^4y^2)^{3/2}$. This suggests that the construction is a close approximation only in case the difference $a - b$ is small. There are, however, important applications of ellipses, such, for example, as the paths of the planets, in which the difference $a - b$ is small. In such cases Mr. Iwerson's construction might give valuable results if there were some definite criterion (possibly some function of the difference $a - b$ or of the eccentricity of the ellipse) which would readily and accurately measure the degree of approximation. Question 33 above is asked in the hope that some of our readers may be able to furnish such a criterion or suggest useful applications.—U. G. M.

DISCUSSIONS.

Relating to the Number of Terms between Two Given Terms of an Ordered Polynomial.

By O. E. Glenn, University of Pennsylvania.

A complete homogeneous polynomial P of order m in n letters $x_1, x_2, \cdots, x_n$ may be said to have its terms arranged in normal order when the exponents in any two terms of P, as

$$t = C_{k_1k_2\cdots k_n}x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n},\ t' = C_{l_1l_2\cdots l_n}x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n},$$

where t comes before t', satisfy the condition that the first non-vanishing difference of the set

$$k_n - l_n,\ k_{n-1} - l_{n-1},\ \cdots,\ k_p - l_p,\ \cdots,\ (k_1 - l_1),$$

is negative. We wish to prove that the number of terms of P between t and t', including t' but not including t, is given by the formula

$$(2)\qquad N\begin{pmatrix}k_1, \cdots, k_n\\ l_1, \cdots, l_n\end{pmatrix} = \left\{\begin{array}{l} \binom{l_1}{0} + \binom{l_1+1}{0} + \cdots + \binom{k_1-1}{0} \\ + \binom{l_1+l_2+1}{1} + \binom{l_1+l_2+2}{1} + \cdots + \binom{k_1+k_2}{1} \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \binom{l_1+\cdots+l_{n-2}+n-3}{n-3} + \binom{l_1+\cdots+l_{n-2}+n-2}{n-3} \\ \qquad\qquad + \cdots + \binom{k_1+\cdots+k_{n-2}+n-4}{n-3} \\ + \binom{l_1+\cdots+l_{n-1}+n-2}{n-2} + \binom{l_1+\cdots+l_{n-1}+n-1}{n-2} \\ \qquad\qquad + \cdots + \binom{k_1+\cdots+k_{n-1}+n-3}{n-2} \end{array}\right\},$$

where the numbers $\binom{a}{b}$ are ordinary binomial numbers, and are defined to be zero when b is negative.

All terms in this formula have plus signs provided

$$l_1 + \cdots + l_s < k_1 + \cdots + k_s (s = 1, \cdots, n-1).$$

If, for any of these values of s, $l_1 + \cdots + l_s = k_1 + \cdots + k_s$, the line number s, viz.,

$$\binom{l_1 + \cdots + l_s + s - 1}{s-1} + \binom{l_1 + \cdots + l_s + s}{s-1} + \cdots + \binom{k_1 + \cdots + k_s + s - 2}{s-1}$$

disappears from the formula. But if $l_1 + \cdots + l_s > k_1 + \cdots + k_s$ for a value of s, then line number s is to be replaced by, or is to be interpreted as signifying

$$-\binom{k_1 + \cdots + k_s + s - 1}{s-1} - \binom{k_1 + \cdots + k_s + s}{s-1} - \cdots - \binom{l_1 + \cdots + l_s + s - 2}{s-1}.$$

We have $k_1 + k_2 + \cdots + k_n = l_1 + l_2 + \cdots + l_n = m$, and P is ordered according to ascending powers of x_n, the coefficients of the various powers of x_n being polynomials in $n-1$ letters $x_1, \cdots, x_{n-1}$, *i. e.*, in a common terminology, are $(n-1)$-ary forms. Beginning to enumerate terms from the lower (right-hand) end of P we observe that $N\begin{pmatrix} k_1, & k_2, & \cdots, & k_n \\ 0, & 0, & \cdots, & m \end{pmatrix}$ equals 1 plus the number of terms in an $(n-1)$-ary linear form, plus the number of terms in an $(n-1)$-ary quadratic form, plus the number in a cubic, plus and so on to and including the number of terms in an $(n-1)$-ary $(m - k_n - 1)$-ic form; plus 1 plus the number of terms in an $(n-2)$-ary linear form, plus the number in an $(n-2)$-ary quadratic and so on to and including the number in an $(n-2)$-ary $(m - k_n - k_{n-1} - 1)$-ic form, and so forth. This double entry summation extends to include the (last) row 1 plus the number of terms in a linear unary form, plus the number in a quadratic unary form, plus and so on to and including the number in a unary $(m - k_n - k_{n-1} - \cdots - k_2 - 1)$-ic form. Hence

$$(3) \quad N\begin{pmatrix} k_1, & \cdots, & k_n \\ 0, & \cdots, & m \end{pmatrix} = \left\{ \begin{array}{l} \binom{0}{0} + \binom{1}{0} + \cdots + \binom{k_1 - 1}{0} \\ + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{k_1 + k_2}{1} \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \binom{n-3}{n-3} + \binom{n-2}{n-3} + \cdots + \binom{k_1 + \cdots + k_{n-2} + n - 4}{n-3} \\ + \binom{n-2}{n-2} + \binom{n-1}{n-2} + \cdots + \binom{k_1 + \cdots + k_{n-1} + n - 3}{n-2} \end{array} \right\}.$$

This serving as a general formula we have immediately

$$N\begin{pmatrix} k_1, & \cdots, & k_n \\ 0, & \cdots, & m \end{pmatrix} - N\begin{pmatrix} l_1, & \cdots, & l_n \\ 0, & \cdots, & m \end{pmatrix} = N\begin{pmatrix} k_1, & \cdots, & k_n \\ l_1, & \cdots, & l_n \end{pmatrix},$$

equal to formula (2) above.

For illustration, a complete quaternary cubic in normal order is the following:

$$x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$$
$$+ x_1x_3^2 + x_2x_3^2 + x_3^3 + x_1^2x_4 + x_1x_2x_4 + x_2^2x_4 + x_1x_3x_4$$
$$+ x_2x_3x_4 + x_3^2x_4 + x_1x_4^2 + x_2x_4^2 + x_3x_4^2 + x_4^3.$$

The number of terms between $x_1^2x_3$ and $x_3x_4^2$ is

$$N\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} = \left\{ \begin{array}{c} \binom{0}{0} + \binom{1}{0} \\ + \binom{1}{1} + \binom{2}{1} \\ + \binom{3}{2} + \binom{4}{2} \end{array} \right\} = 14.$$

The total number of terms in a complete ternary form of order m is

$$1 + N\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & m \end{pmatrix} = \left\{ \begin{array}{c} \binom{0}{0} + \binom{1}{0} + \cdots + \binom{m-1}{0} \\ + \binom{1}{1} + \binom{2}{1} + \cdots + \binom{m}{1} \end{array} \right\} + 1 = \tfrac{1}{2}(m+1)(m+2).$$

EXERCISE. Show that the number of terms between $x_1^3x_2^2x_3^2x_4^{m-7}$ and $x_1^2x_2x_3^2x_4^{m-5}$ in a complete ordered quaternary polynomial of order m ($m > 6$) is 59.

EXERCISE. Show that

$$N\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \end{pmatrix} = 31.$$

NOTES AND NEWS.

SEND COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

Dr. OTTO DUNKEL, formerly instructor at the University of Missouri, has been appointed assistant professor of mathematics at Washington University, St. Louis, Mo.

"Differential equations and implicit functions in infinitely many variables" is the title of a paper by Dr. W. L. HART of Harvard University, abstract of which appeared in the *Proceedings* of the National Academy of Sciences for June, 1916.

Mr. EDWARD B. ESCOTT is now connected with the auditing department of the Kansas City Life Insurance Company, Kansas City, Mo.

Mr. H. C. CLEVENGER, of Urbana, Ill., has accepted an instructorship in mathematics at the University of Minnesota.

Dr. R. L. BÖRGER, associate in mathematics at the University of Illinois, goes to Ohio University at Athens as professor of mathematics.

Dean THOMAS F. HOLGATE, professor of mathematics at Northwestern University, has been chosen by the trustees *ad interim* president of the University.

The *Washington University Studies,* January, 1916, contains a paper by Professor W. H. ROEVER on "Some recent work on the problem of the deviations of freely falling bodies." This paper is a sixteen-page résumé of the publications of ROEVER, CAJORI, HALL, WOODWARD, MOULTON, and others, which have appeared in *Science, Bulletin and Transactions of the American Mathematical Society,* the *Astronomical Journal, Physical Review* and other periodicals.

An expository volume on integral equations appeared in July, 1916, with the title *Elementi della Teoria delle Equazioni Integrali Lineari* by G. VIVANTI. The book appears in the extensive collection called, *Manuali Hoepli,* and contains about four hundred pages. In the preface the author refers to the present European war and to the need of scientific emancipation of Italy. The present volume is a contribution in that direction.

The Philosophical Transactions of the Royal Society of London, Vol. 216, 1916, has a paper by K. PEARSON, the noted biometrician, on "Mathematical contributions to the theory of evolution."

The July, 1916, number of *The Mathematical Gazette* contains the concluding article by E. H. NEVILLE, on "So-called cases of failure in the solution of linear differential equations" referred to in the October issue of the MONTHLY. The introductory part of this discussion appeared in the May issue of the *Gazette.*

The first number of Vol. 18 of *The Annals of Mathematics* has appeared containing the following papers: "On a surface of lowest degree passing through a given curve in space," by T. HYASHI; "A practical method of determining elementary divisors," by H. T. BURGESS; "Conjugate systems with equal point invariants," by L. P. EISENHART; "On the derivative of a function at a point," by J. F. RITT; "An existence theorem for the solution of a type of real mixed difference equations," by A. A. BENNETT; "Double elliptic geometry in terms of point and order alone," by J. R. KLINE; "An application of a group of order sixteen to a configuration of an elliptic cubic," by A. EMICH.

Parts II and III, of Vol. 40, *Rendiconti del Circolo Matematico di Palermo,* contain fourteen research papers among which are the following in English: "A new theorem in analytic conics," by A. HAWKESWORTH; "The meaning of Plücker's equations for a real curve," by J. L. COOLIDGE; "On Stieltjes's integral," by T. HAYASHI; "Plane nets periodic of period three under the Laplacian transformation," by J. O. HASSLER.

Miss ELEANORA HARRIS, head of the department of mathematics at the Hutchinson, Kan., high school, and secretary of the Kansas Association of Mathematics Teachers, has been markedly successful in conducting a mathematics club for the advanced students in the high school. They have discussed

the practical uses of graphs, of the slide rule, and of determinants of the second and third orders. They have studied the history of our numerals, of logarithms, and of the Pythagorean theorem. And they have debated the question whether one year of algebra and one year of geometry should not be required of all students for graduation from the high school.

The fourth annual meeting of the Mathematics Section of the California High School Teachers' Association was held in July at the University of California under the chairmanship of Professor HENRY W. STAGER, Fresno Junior College. The following program was presented: "Non-technical discussion of some questions of general interest in geometry," by Professor E. J. WILCZYNSKI, University of Chicago; "Mathematics as an applied subject," by Dr. E. R. SNYDER, California Commissioner of industrial and vocational education; "Minimum requirements in mathematics,'' by Professor HENRY W. STAGER, Fresno Junior College; "Some of the difficulties a mathematics teacher of to-day must face," by FRANK R. MORRIS, Glendale High School. The discussions centered largely upon the present attacks upon the teaching of mathematics in the secondary schools by the leaders of the vocational movement. It was urged that united effort be made to meet these attacks and that all teachers of mathematics as individuals and in organizations should continue to defend the cultural value of mathematics.

In order to make the work of the Section more continuous and more effective, a standing committee on policy, selected from all parts of the state, was authorized. An educational survey of the teaching of mathematics in California was also authorized, the survey to be undertaken by the committee on policy in coöperation with the State Commissioner for secondary schools. The official reading course for the past year as printed in the MONTHLY for March, 1916, pp. 77–78, was adopted for the year 1916–1917 without change. Professor C. A. NOBLE, University of California, was elected chairman of the Section, and Mr. G. E. MERCER, Palo Alto High School, secretary for the ensuing year.

The Association of Mathematics Teachers of New Jersey is the second, so far as we know, the California association being the first, to organize with the distinct purpose of giving to high school teachers the "upward look"; or, to use the words of one of the organizers of the New Jersey society, to choose as the field of activity what may be called "graduate work in high school mathematics." To this end, each program has papers emphasizing the "graduate," or in other words, the "collegiate" point of view. For instance, at one of their earlier meetings, Professor OSWALD VEBLEN, of Princeton University, gave a paper on "The affine geometry"; Professor RICHARD MORRIS, of Rutgers College, spoke on "The auxiliary angle"; Professor C. O. GUNTHER, of Stevens Institute of Technology, on "Trigonometry for the college student"; and Mr. HARRISON E. WEBB, of the Central High School of Newark, on "Geometric definitions of the trigonometric functions." Their subsequent programs have continued to include papers of this character.

Professor H. W. Tyler contributes the following biographical note:

"Professor WEBSTER WELLS, long a member of the Faculty of the Massachusetts Institute of Technology, died in a private hospital near Boston, May 23, 1916. Born in 1851, Professor Wells prepared for the Institute at the Allen School, West Newton, Massachusetts. He was graduated in civil engineering in 1873 and was then appointed instructor in mathematics. In 1880 he went to Germany for further study, spending two years in Leipsic, mainly in private mathematical work. Returning to the Institute in 1882, he continued teaching until his retirement on account of serious impairment of health in 1911. At the time of his return from Germany conditions were not yet favorable for the introduction of advanced or graduate courses, and Professor Wells accordingly devoted his unusual powers of lucid exposition to the preparation and publication of an extended series of text books in the whole field of elementary mathematics. These texts represent an enormous amount of careful labor in a field already so long cultivated. They are in general notable for the clear and skilful arrangement of traditional materials rather than for any attempt at innovations. Professor Wells was a member of the American Mathematical Society since 1895.

Outside of the field of mathematics, Professor Wells was especially interested in music and foreign travel; also early in life in mountaineering. He was for a short time Secretary of the Faculty, and Bursar of the Institute.

NOTES ON THE ASSOCIATION.

The first annual meeting of the Association will take place on Friday and Saturday, December 29, 30, 1916, at Columbia University, New York City. The selection of the place of meeting was determined by the fact that the American Association for the Advancement of Science meets in New York during holiday week, thus bringing together large numbers representing most of the national scientific societies of the country. In particular, the American Mathematical Society will hold its annual meeting at Columbia University on Wednesday and Thursday, December 27, 28, so that the juxtaposition of dates for the Association and the Society will make it convenient for mathematicians to attend both meetings.

The program of the annual meeting of the Association will be in charge of a committee consisting of Professor D. E. Smith of Columbia University, Chairman, Professor E. H. Moore of the University of Chicago, and Professor G. D. Olds of Amherst College. A special sub-committee consisting of Professor J. N. Van der Vries of the University of Kansas, Chairman, Professor J. N. Hart of the University of Maine, and Professor Helen Merrill of Wellesley College, will arrange the program for the meeting of institutional delegates.

The committee on arrangements for the New York meeting consists of Professor T. S. Fiske of Columbia University, Chairman, Professor T. W. Edmondson of New York University, Professor Paul Saurel of the College of the City of New York, Professor Emma Requa of Hunter College, and Professor D. E. Smith of Teachers College. All these committees are authorized to extend

their membership as circumstances may dictate. They will issue preliminary reports as early as seems feasible. It is hoped that the December issue of the MONTHLY may contain a full outline of the programs and arrangements.

Another important committee of the Association has just been organized, namely, the Library Committee which was authorized last spring and consists of the following persons: Professor W. B. Ford (Chairman), University of Michigan; Professor S. Lefschetz, University of Kansas; Professor W. R. Longley, Yale University, and Professor R. E. Root, U. S. Naval Academy. This committee is already considering plans for its operation and will make a public statement through the columns of the MONTHLY within a short time. In general, it may be said that the Association proposes to take a lively interest in the development of mathematical libraries in the colleges and will wield its influence as far as possible toward this end.

At about the time when this issue of the MONTHLY reaches the members of the Association, the nomination ballots will be received from the Secretary, and the first opportunity will be presented for selecting candidates for the officers of the Association by open primaries. This provision of the Constitution is in the nature of an experiment and its successful operation depends entirely upon the thoughtful interest shown by each member. The nominations will close on November 20 and hence prompt action is needed in returning the ballots.

Four sections of the Association have thus far been organized, namely, in Kansas, Ohio, Missouri, and Iowa. Aside from the organization meetings, the Ohio and Kansas sections have each held meetings in which successful programs were carried out. The Missouri Section will conduct its first program on Saturday, November 16, 1916, at St. Louis, Missouri, and the Iowa Section is likewise planning to hold a meeting soon. It seems clear that the gatherings in which the great body of the membership can take part must be in the smaller geographical units rather than in the national meetings. Gratifying as was the attendance at the Cambridge meeting, and gratifying as the still larger attendance at New York is almost certain to be, nevertheless, these national meetings can probably never include more than one tenth to one fifth of the membership. Hence, whatever benefit is to come from attendance upon meetings and mingling with other members must, for the large majority, be found in the sectional meetings.

It should be possible for almost any member to reach some central meeting place within his own state at least once each year. Moreover, the opportunities afforded in the smaller groups for more extended discussions, especially upon those questions which involve local conditions as, for instance, legislation by states on educational matters, should be such as to attract all members of a section and to command their active participation in shaping the development of mathematical interests in their communities.

It is, therefore, confidently expected that, as time goes on, the Association will develop this phase of its activities very strongly and that numerous other sections will be organized in the near future. Already it is reported that steps

are being taken in this direction in at least three other centers, and doubtless the matter is under informal consideration in still other parts of the country. There is no need for haste, but there is need for most careful deliberation in order that any action that is taken may be permanent and effective. A watch-word that may well be considered as indicative of the spirit of this Association is "active participation on the part of all members." It is a fact of more than ordinary significance that already well toward one hundred persons are directly engaged in active service either as members of the Council, as members of the Editorial Board, or as members of standing or special committees. Indeed, if we count the officers and committees of sections already formed and in process of formation, and the contributors to the programs of meetings and to the various departments of the MONTHLY, the number of active workers must be considerably over one hundred. While this may be considered a most favorable beginning for the first year, nevertheless, the ultimate goal should be to enlist at least one thousand active participants in the affairs of the Association. This can be done as soon as the sections are numerous enough to enable every member to attend at least one meeting a year, and as soon as the opportunities for cooperation in connection with the publications of the Association and in connection with the investigations of its standing committees are fully realized. The great work of this Association will doubtless be done through its sectional meetings, where personal contact and individual responsibility will have full play, through its standing committees which in time are sure to influence every phase of mathematical interests in the country, and through its official publications which should become the clearing-house for all these activities and the medium of intercommunication which will keep every member, however remote from the natural geographical centers, in close touch with all that is taking place in the mathematical world.

One most effective service which any member may render at once is to make known to those who are as yet out of touch with the MATHEMATICAL ASSOCIATION OF AMERICA the important events of the past ten months and to see to it that the Secretary has the name and address of any such persons. The Secretary until further notice will be at *5465 Greenwood Avenue, Chicago, Illinois.*

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VOLUME XXIII DECEMBER, 1916 NUMBER 10

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the ASSOCIATION, W. D. CAIRNS, 5465 Greenwood Ave., Chicago, Ill.

THE AMERICAN MATHEMATICAL MONTHLY

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PREFERENTIAL VOTING.

By W. V. LOVITT, Purdue University.

Let there be given three candidates A, B, and C. Let there be S voters and suppose each voter expresses his first, second, and third choice. Suppose the voting to have been done. It is the object of this paper to determine the conditions under which it is possible to assign weights to the votes for first, second, and third choice so that any preassigned candidate may win.

Let the votes which each candidate receives be exhibited by the following table:

	Choice		
	1st	2d	3d
A	A_1	A_2	A_3
B	B_1	B_2	B_3
C	C_1	C_2	C_3

In this table it is to be noted that

$$A_i + B_i + C_i = S \qquad (i = 1, 2, 3),$$

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 = C_1 + C_2 + C_3 = S.$$

Let x, y, and z be the weights assigned to first, second, and third choice with the condition

$$x > y > z.$$

Then the number of points received by A in this contest is given by $A_1x+A_2y+A_3z$, with similar expressions for the number of points received by B and C.

Consider now the three planes

$$(A) \qquad A_1x + A_2y + A_3z = 0,$$

$$(B) \qquad B_1x + B_2y + B_3z = 0,$$

$$(C) \qquad C_1x + C_2y + C_3z = 0.$$

These three planes form a trihedral, unless two or more coincide, with vertex at the origin. Every point in the first octant is within the same one of the eight compartments of space marked off by these three intersecting planes. The three planes

$$(AB) \qquad (A_1 - B_1)x + (A_2 - B_2)y + (A_3 - B_3)z = 0,$$

$$(AC) \qquad (A_1 - C_1)x + (A_2 - C_2)y + (A_3 - C_3)z = 0,$$

$$(BC) \qquad (B_1 - C_1)x + (B_2 - C_2)y + (B_3 - C_3)z = 0,$$

are the loci of points for which respectively A and B, A and C, or B and C are tied. These three planes are co-axial, the axis of the pencil being the line

$$x = y = z.$$

Cut now the system of six planes which we have by the plane

$$(1) \qquad x + y + z = \text{any real positive constant.}$$

Denote the traces of the planes A, B, C, AB, AC, BC on the plane (1) by the symbols a_1, b_1, c_1, a_1b_1, a_1c_1, b_1c_1 respectively (see Fig. 1).

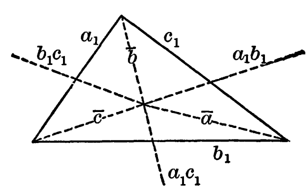


Fig. 1.

The line a_1b_1 and the origin determine a plane which we shall call the plane a_1b_1. In like manner the planes a_1c_1, b_1c_1, and a_1, b_1, c_1 are determined.

Let us denote the region inclosed by the four planes a_1b_1, a_1c_1, b_1, c_1 by $\bar{a}$; likewise, b_1c_1, b_1a_1, a_1, c_1 by $\bar{b}$; and c_1a_1, c_1b_1, a_1, b_1 by $\bar{c}$. It is clear that in each of the regions $\bar{a}$, $\bar{b}$, $\bar{c}$ there are points for which x, y, z are all positive. If now we choose as weights the coördinates of any point in the region $\bar{a}$ ($\bar{b}$ or $\bar{c}$) then A (B or C) is the successful candidate. It remains to show under what conditions it is impossible to find points in $\bar{a}$ ($\bar{b}$ or $\bar{c}$) for which

$$x > y > z.$$

If two candidates A and C receive an equal number of votes for first and second choice they are tie under any system of weights. It becomes then a question as to whether B can win over A. Let *A and B receive the same number of votes for*

first choice while A receives the most votes for second choice. Then

$$B_3 = A_2 \qquad \text{and} \qquad B_2 = A_3.$$

It is now impossible for B to win. For if B is to win we must have

$$xA_1 + yA_2 + zA_3 < xA_1 + yA_3 + zA_2,$$

which reduces to

$$y < z$$

and this is contrary to hypothesis. In like manner we can show that *if A and B have an equal number of votes for second or third choice and A has the greater number of votes for first choice, then it is impossible for B to win.*

There remains the case where any two candidates A and B receive an unequal number of votes for first, second, and third choice. We will now suppose that A receives the most votes for first choice. Under this hypothesis it is clear that the x-axis lies within the region $\bar{a}$, for the weights 1, 0, 0 are such as to make A win.

Designate by ab, ac, bc the lines of intersection of the plane

(2) $$x = \text{any real positive constant}$$

with the planes AB, AC, BC. Designate by O' and P its intersections with the x-axis and with the line $x = y = z$. (See figures 2 and 3.) Let $O'z'$ and $O'y'$

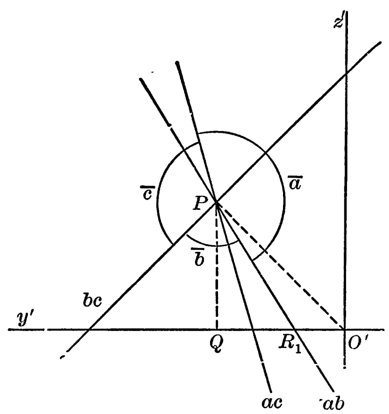


Fig. 2.

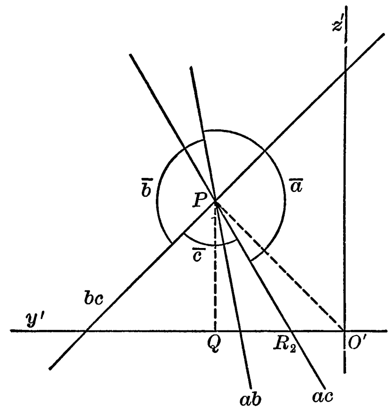


Fig. 3.

be the intersections of the plane (2) with the planes $y = 0$ and $z = 0$. Draw QP perpendicular to $O'y'$. In order that $x > y > z$, we must choose as weights the coördinates of some point within the triangle $O'PQ$. Since the plane AB contains the line $x = y = z$, the line ab must contain the point P. The region $\bar{b}$ and the x-axis are on opposite sides of the plane AB. Hence in order that B may win over A it is necessary that the line ab have a positive slope greater than unity but not infinite. In like manner in order that C may win over A

it is necessary that the line ac have a positive slope greater than unity but not infinite.

Suppose the slopes of ab and ac both positive and greater than unity. If the slope of ab [or ac] is less than that of ac [or ab] then B [or C] may win over both A and C [or B] by choosing as weights the coördinates of any point within the region PQR_1 [or PQR_2]. From Fig. 2 (or Fig. 3) in order that C [or B] may win over B [or C] and A it is necessary that the line bc have a positive slope greater than unity but not infinite and also greater than the slope of ac [or ab].

Thus, in order that either one of B or C at will may win over the other two candidates, A having the greatest number of votes for first choice, it is necessary that each of the three lines ab, ac, bc have a positive slope greater than unity but not infinite and that of these three lines the slope of bc be the greatest.

These conditions are also sufficient. *Illustration:*

	Choice			
	1st	2d	3d	
A	9	2	9	
B	4	11	5	
C	7	7	6	Winner
Weights	4	2	1	A
	6	5	1	B
	3	2	1	C

A DIRECT PROOF OF DE MOIVRE'S FORMULA.

By S. LEFSCHETZ, University of Kansas.

The proposition which will be proved here, and which is practically equivalent to De Moivre's noted theorem, may be stated thus:

If X, Y, Z are three complex numbers of modulus unity, and such that their arguments x, y, z have a zero sum, then $XYZ = 1$.

Let $X = a + ia'$, $Y = b + ib'$, $Z = c + ic'$, and denote their conjugates by $\bar{X}$, $\bar{Y}$, $\bar{Z}$. The moduli being unity, we have $\bar{X} = 1/X$, $\bar{Y} = 1/Y$, $\bar{Z} = 1/Z$. Let A and B be the representative points of X and $\bar{Y}$, using Argand's diagram, O the origin, E the foot of the perpendicular AE from A to OB. As angle $BOA = (x + y)$, it follows that $OE = -c$, $EA = -c'$. The coefficients of the equations of the lines OB and AE are rational in the coördinates of A, B, that is, in a, b, a', b', and therefore the same holds for the coördinates of their intersection E. When these are known, the ratio $OE/OB = -c/1$ is determined rationally. Hence c, and similarly c', can be determined rationally with respect to a, b, a', b'. Therefore.

$$Z = c + ic' = f(a, b, a', b'), \tag{1}$$

where f is a rational function. But

$$a = \frac{1}{2}(X + \bar{X}) = \frac{1}{2}\left(X + \frac{1}{X}\right), \qquad a' = \frac{1}{2i}\left(X - \frac{1}{X}\right), \qquad b = \frac{1}{2}\left(Y + \frac{1}{Y}\right),$$

$$b' = \frac{1}{2i}\left(Y - \frac{1}{Y}\right).$$

Substituting in (1), it can be replaced by a relation such as

$$F(X, Y, Z) = 0, \tag{2}$$

where F is a polynomial in X, Y, Z, of the first degree in Z. When the three variables are arbitrarily permuted, five other relations of the same type are obtained which, for reasons of symmetry, must be equally satisfied. But between X, Y, Z, only one relation can exist, since two of them are arbitrary. Hence the polynomials at the left of the relations analogous to (2), which have just been considered, have a common divisor $\phi(X, Y, Z)$ symmetric with respect to X, Y, Z, and we have $\phi = 0$. As ϕ, like F itself, is necessarily of the first degree in Z, it is linear in the three variables and we have a relation such as

$$A \cdot XYZ + B(XY + YZ + ZX) + C(X + Y + Z) + D = 0, \tag{3}$$

A, B, C being coefficients still to be determined.

If the arguments are changed in sign, X, Y, Z are changed into their conjugates, and (3) is still satisfied. Hence A, B, C, D are real. As the conjugates are also the reciprocals in this case, coefficients in (3) equidistant from the extremes can differ only in sign.

Hence $D = kA$, $C = kB$, $k = \pm 1$. But for $x = y = \pi/2$, we have $z = -\pi$, $X = Y = i$, $Z = -1$. Therefore,

$$A(1 + k) + B[-1 - 2i + k(2i - 1)] = 0.$$

Equating to zero real and complex parts respectively, we obtain

$$(1 + k)(A - B) = 0, \qquad (1 - k)B = 0.$$

Hence, either: (a) $k = +1$, $A = B$; or (b) $k = -1$, $B = 0$, when we can take $A = 1$. But if $x = y = z = 0$, then $X = Y = Z = 1$. Substituting in (3), we have

$$(A + 3B)(1 + k) = 0,$$

which is incompatible with assumption (a).

Hence $k = -1$, $B = 0$, and therefore $XYZ = 1$, as was to be proved.

Since

$$X = \cos x + i \sin x, \qquad Y = \cos y + i \sin y, \qquad z = -(x + y),$$

we have

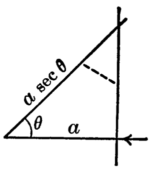
$$(\cos x + i \sin x)(\cos y + i \sin y) = 1/Z = \bar{Z} = \cos(-x-y) - i \sin(-x-y)$$
$$= \cos(x+y) + i \sin(x+y),$$

which is De Moivre's formula. From this the trigonometric addition or subtraction formulas are easily derived in all their generality.

SAILING TO WINDWARD.

BY W. E. BYERLY, Harvard University.

The problem of sailing to a windward goal is usually the simple problem of *getting to windward.* For, if we find the straight path of least time from the boat to a perpendicular to the wind's direction, by tacking at the proper point we shall reach any goal in that perpendicular in the time it would take to reach the perpendicular had we continued on the original course.



The velocity of the boat on any course is some function of θ depending on the model and sailing qualities of the boat, where θ is the angle which the course makes with the direction of the wind, and we shall suppose this function given, and shall represent it by $F(\theta)$.

The time required to get the distance a to windward is the distance sailed, $a \sec \theta$, divided by the velocity, $F(\theta)$, and we wish to make this time a minimum.

Let $u = a \sec \theta / F(\theta)$. Then

$$\frac{du}{d\theta} = \frac{F(\theta)\, a \sec \theta \tan \theta - a \sec \theta F'(\theta)}{[F(\theta)]^2} = 0, \qquad \text{where} \qquad F'(\theta) = \frac{d}{d\theta} F(\theta),$$

or

$$F(\theta) \tan \theta - F'(\theta) = 0. \tag{1}$$

The solution of this equation will give us the angle which our course on each tack should make with the direction of the wind, whether we are using sails alone or sails and auxiliary motor.

In any concrete case equation (1) can be solved by " trial and error " with the aid of a three-place trigonometric table (preferably giving the angles in radians as well as in degrees) with sufficient accuracy for all practical purposes.

The function $F(\theta)$ ought to be determined by experiment for every boat, but as a first approximation, in the case of a sail boat, it may be taken as equal to $k(\theta - \alpha)$, where α, " the angle of repose," is the angle with the direction of the wind within which the boat will not sail; an angle, by the way, that is very easily discovered by experimenting with the boat.

Equation (1) becomes $k(\theta - \alpha) \tan \theta - k = 0$ or

$$\theta - \alpha = \operatorname{ctn} \theta. \tag{2}$$

The θ found from this equation is the angular distance off the wind which we should sail in going to windward, and is evidently independent of the wind's velocity.

For a rather sluggish boat, for which α is 30°, θ is about 60°.

Let us now suppose that the sail boat has a gasolene engine which can give her a velocity v. Then with sail and engine, $F(\theta) = v + k(\theta - \alpha)$, and equation (1) becomes $[v + k(\theta - \alpha)] \tan \theta - k = 0$ or

$$\frac{v}{k} + (\theta - \alpha) = \operatorname{ctn} \theta. \tag{3}$$

Here θ is no longer the same for all breezes but k is easily determined by shutting off the motor and noting the speed on any convenient course, remembering that this speed must be $k(\theta - \alpha)$.

If $\alpha = 30°$, $v = 6$, and $k = 8$, equation (3) gives about 45° for θ.

We shall get to windward, then, under sail and gas combined, most rapidly if we keep the boat about four points off the wind. Can we perhaps do better by dropping sail and going dead to windward under gas alone? To get a mile to windward under sail and motor will take us $(\sec \theta)/[v + k(\theta - \alpha)]$ hours. Computing this value for $\theta = 45°$, $\alpha = 30°$, $v = 6$, and $k = 8$, we get 0.175 hour. To get a mile to windward under gas takes 0.167 hour.

The time required to get a mile to windward under gas on a course making the angle β with the wind is $(\sec \beta)/v$. This is less than 0.175 if β is less than 18°.

Summing up, we see that if short-handed our "sailing directions" should read as follows:

(*a*) For any goal lying in the angle whose vertex is at the boat and whose sides are inclined 18° to the wind's direction, drop sail and head for the goal under gas alone.

(*b*) For any goal between these lines and lines through the boat and inclined 45° to the wind's direction, tack under sail and gas, keeping four points off the wind on each tack.

(*c*) For all other goals, go straight under sail and gas.

For a second and much closer approximation, let us assume that the velocity under sail is $a(\theta - \alpha) + b(\theta - \alpha)^2$. In practice a and b can be determined from the observed speed on two courses, preferably chosen so that the angle $\theta - \alpha$ is small.

For sail alone $F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2$, and equation (1) becomes $[a(\theta - \alpha) + b(\theta - \alpha)^2] \tan \theta = a + 2b(\theta - \alpha)$ or

$$\left(\tan \theta - \frac{2b}{a}\right)(\theta - \alpha) + \frac{b}{a}(\theta - \alpha)^2 \tan \theta = 1. \tag{4}$$

For an example, suppose as before that $\alpha = 30°$, and suppose that the boat it found to make 2 knots when $\theta - \alpha$ is $\frac{1}{4}$ and 3 knots when $\theta - \alpha$ is $\frac{1}{2}$.

We have

$$2 = \frac{a}{4} + \frac{b}{16}; \qquad 3 = \frac{a}{2} + \frac{b}{4}.$$

Whence $a = 10, b = -8, b/a = -0.8$. Equation (4) gives $(\tan\theta + 1.6)(\theta - \alpha) - 0.8(\theta - \alpha)^2 \tan\theta = 1$, and

$$\theta = 55° \text{ approximately.}$$

If we use sail and motor, $F(\theta) = v + a(\theta - \alpha) + b(\theta - \alpha)^2$. Equation (3) becomes

$$[v + a(\theta - \alpha) + b(\theta - \alpha)^2] \tan\theta = a + 2b(\theta - \alpha)$$

or

$$(5) \qquad \left(\tan\theta - \frac{2b}{a}\right)(\theta - \alpha) + \left[\frac{b}{a}(\theta - \alpha)^2 + \frac{v}{a}\right] \tan\theta = 1.$$

For our example, equation (5) becomes

$$(\tan\theta + 1.6)(\theta - \alpha) + [0.6 - 0.8(\theta - \alpha)^2] \tan\theta = 1,$$

$$\theta = 41°.5 \text{ approximately.}$$

The best time for a mile to windward under sail and motor is

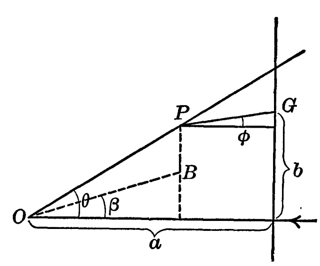
$$(\sec\theta)/[v + a(\theta - \alpha) + b(\theta - \alpha)^2],$$

or about 0.1738 hour. Under motor alone, in this time, the boat would go 1.043 miles, and $\sec^{-1} 1.043$ is about 16°.5.

Our sailing directions would be the same as in our first example, except that in (*a*) 18° would be changed to 16°.5, and in (*b*) 45° would be changed to 41°.5.

Hitherto, we have supposed the crew short-handed, so that sail was neither raised nor lowered in transit.

In a light wind where β is an angle of appreciable magnitude and we are aiming at a goal in the sector $\theta - \beta$, we can do best by using sail and motor



part way and motor only the remainder of the way. Let us consider this new problem. Let θ and β have their old signification and let φ be the angle which

the course makes with the wind after we lower sail. Let P be the turning point and G be the goal. Then, since the whole time of transit is the time that it would take to go the distance $OB + PG$ under motor, we must choose φ so that this distance shall be a minimum.

$$PG = \frac{(a \tan \theta - b) \cos \theta}{\sin (\theta - \varphi)}, \qquad OB = (a - PG \cos \varphi) \sec \beta.$$

$$OB + PG = a \sec \beta - PG(\cos \varphi \sec \beta - 1).$$

This will be a minimum when $PG \cos \varphi(\sec \beta - \sec \varphi)$ is a maximum; and that will be the case when u is a maximum if

$$u = \frac{\cos \theta \cos \varphi}{\sin (\theta - \varphi)} [\sec \beta - \sec \varphi] = \frac{\sec \beta - \sec \varphi}{\tan \theta - \tan \varphi}.$$

$$\frac{du}{d\varphi} = \frac{(\tan \varphi - \tan \theta) \sec \varphi \tan \varphi + (\sec \beta - \sec \varphi) \sec^2 \varphi}{(\tan \theta - \tan \varphi)^2} = 0.$$

$$(\tan \theta - \tan \varphi) \tan \varphi = (\sec \beta - \sec \varphi) \sec \varphi.$$

$$\tan \theta \tan \varphi + 1 = \sec \beta \sec \varphi.$$

$$\cos \varphi + \tan \theta \sin \varphi = \sec \beta. \tag{6}$$

φ determined from equation (6) gives the bearing of the goal when the proper turning point is reached.

If $\theta = 45°$ and $\beta = 18°$, $\varphi = 3°$. If $\theta = 41°.5$ and $\beta = 16°.5$, $\varphi = 3°-$.

If the yachtsman has patience and a soul that longs for accuracy he can amuse himself by trying a third approximation for $F(\theta)$, namely,

$$F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2 + c(\theta - \alpha)^3.$$

THE ACCELERATIONS OF THE POINTS OF A RIGID BODY.

By PETER FIELD AND ALEXANDER ZIWET, University of Michigan.

INTRODUCTION.

It has long been known that, in plane motion, the acceleration field is completely determined by the accelerations of any two points; and the point of zero acceleration, or "acceleration center," for such a motion is discussed in most works on mechanics.[1] The corresponding problems for three dimensions, that is, the determination of the acceleration field from the accelerations of three points and the construction of the acceleration center (or centers), are less widely known. Vector methods are particularly appropriate for their solution. They

[1] Compare AMERICAN MATHEMATICAL MONTHLY, Vol. XXI, 1914, pp. 105–113.

bring out very clearly to what extent the instantaneous motion remains indeterminate when the accelerations of three points are given, and lead to the enumeration of the various cases that may arise concerning the existence of points of zero acceleration. The dynamical interpretation of the result of Art. 20 would also appear to be noteworthy.

For the determination of the central axis from the accelerations of three points compare J. PETERSEN, *Kinematik,* 1884, Art. 36, pp. 47–49, and R. MEHMKE, in *Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt,* p. 77.

I. VELOCITY.

1. Let O, P, Q be any three non-collinear points of a rigid body and put

$$P - O = \boldsymbol{p}, \quad Q - O = \boldsymbol{q}$$

The assumption that O, P, Q are different and not collinear gives

$$\boldsymbol{p} \wedge \boldsymbol{q} \neq 0.$$

The assumption of rigidity gives

$$\boldsymbol{p}^2 = \text{const.}, \quad \boldsymbol{q}^2 = \text{const.}, \quad (\boldsymbol{q} - \boldsymbol{p})^2 = \text{const.}$$

Differentiating with respect to the time we find:

$$\dot{\boldsymbol{p}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \quad \dot{\boldsymbol{p}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{p} = 0, \tag{1}$$

where $\dot{\boldsymbol{p}}$, $\dot{\boldsymbol{q}}$ are the velocities of P, Q, *relative to* O.

If R be any fourth point of the body and we put $R - O = \boldsymbol{r}$ we have the additional rigidity conditions

$$\boldsymbol{r}^2 = \text{const.}, \quad (\boldsymbol{r} - \boldsymbol{p})^2 = \text{const.}, \quad (\boldsymbol{r} - \boldsymbol{q})^2 = \text{const.},$$

whence

$$\dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} + \dot{\boldsymbol{p}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{r} = 0. \tag{2}$$

We proceed to prove that *the velocities* $\dot{O}$, $\dot{P}$, $\dot{Q}$ *of any three non-collinear points* O, P, Q *of a rigid body completely determine the velocities of all points of the body.*

2. Suppose first that $\dot{\boldsymbol{p}} = 0$ and $\dot{\boldsymbol{q}} = 0$ so that O, P, Q have equal velocities. Then by (2)

$$\dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} = 0.$$

Now if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} \neq 0$ so that $\boldsymbol{r}$, $\boldsymbol{p}$, $\boldsymbol{q}$ can be taken as reference vectors for $\dot{\boldsymbol{r}}$ the equations show that $\dot{\boldsymbol{r}} = 0$. If, however, $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} = 0$ we can put $\boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q}$, whence $\dot{\boldsymbol{r}} = a\dot{\boldsymbol{p}} + b\dot{\boldsymbol{q}} = 0$, as before. Hence *if any three non-collinear points have equal velocities the velocities of all points are equal.* The instantaneous mot'on in this case is called a **translation.**

3. Suppose next that $\dot{\boldsymbol{p}} = 0$, $\dot{\boldsymbol{q}} \neq 0$. The conditions (1) and (2) reduce to

$$\dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \quad \dot{\boldsymbol{q}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{r} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{p} = 0, \quad \dot{\boldsymbol{r}} \times \boldsymbol{q} + \dot{\boldsymbol{q}} \times \boldsymbol{r} = 0.$$

As by the first two conditions $\dot{\boldsymbol{q}}$ is normal to both $\boldsymbol{q}$ and $\boldsymbol{p}$ we can put $\dot{\boldsymbol{q}} = k\boldsymbol{p} \wedge \boldsymbol{q}$. The last equation then becomes $\dot{\boldsymbol{r}} \times \boldsymbol{q} + k\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{r} = 0$, *i. e.*, $(\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{q} = 0$; the last three equations can therefore be written in the form

$$(\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{r} = 0, \qquad (\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{p} = 0, \qquad (\dot{\boldsymbol{r}} + k\boldsymbol{r} \wedge \boldsymbol{p}) \times \boldsymbol{q} = 0;$$

hence, if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} \neq 0$:

$$\dot{\boldsymbol{r}} = k\boldsymbol{p} \wedge \boldsymbol{r};$$

if $\boldsymbol{r} \times \boldsymbol{p} \wedge \boldsymbol{q} = 0$ we can again put $\boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q}$, whence $\boldsymbol{r} = b\dot{\boldsymbol{q}} = bk\boldsymbol{p} \wedge \boldsymbol{q} = k\boldsymbol{p} \wedge (\boldsymbol{r} - a\boldsymbol{p}) = k\boldsymbol{p} \wedge \boldsymbol{r}$. Thus, if $\dot{O} = \dot{P} \neq \dot{Q}$, there exists a vector $\boldsymbol{u} = k\boldsymbol{p}$, parallel to OP, such that the velocity of every point R can be derived from it by the formula

$$\dot{\boldsymbol{r}} = \boldsymbol{u} \wedge \boldsymbol{r}.$$

The scalar k is determined by observing that $\dot{\boldsymbol{q}} = k\boldsymbol{p} \wedge \boldsymbol{q}$ whence

$$k = \frac{\operatorname{mod} \dot{\boldsymbol{q}}}{\operatorname{mod} \boldsymbol{p} \cdot \operatorname{mod} \boldsymbol{q} \cdot \sin (\boldsymbol{p}, \boldsymbol{q})}.$$

4. Finally, suppose that $\dot{\boldsymbol{p}} \neq 0$ and $\dot{\boldsymbol{q}} \neq 0$. It can be shown that in this case, too, there exists a vector $\boldsymbol{w}$ such that

$$\dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r},$$

whatever the point R. For, as the vector $\boldsymbol{w}$ is to satisfy the conditions

$$\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}, \qquad \dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q},$$

it must be normal to both $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$; it must therefore be of the form

$$\boldsymbol{w} = k\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}.$$

To determine k we have only to substitute this value of $\boldsymbol{w}$ in the two preceding equations. Owing to the first two of (1) we find

$$\dot{\boldsymbol{p}} = k(\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}) \wedge \boldsymbol{p} = -k\dot{\boldsymbol{q}} \times \boldsymbol{p} \cdot \dot{\boldsymbol{p}},$$

$$\dot{\boldsymbol{q}} = k(\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}) \wedge \boldsymbol{q} = \quad k\dot{\boldsymbol{p}} \times \boldsymbol{q} \cdot \dot{\boldsymbol{q}};$$

and owing to the third of (1) these equations give the same value for k, viz.,

$$k = -\frac{1}{\dot{\boldsymbol{q}} \times \boldsymbol{p}} = \frac{1}{\dot{\boldsymbol{p}} \times \boldsymbol{q}}.$$

The vector

(3) $$\boldsymbol{w} = -\frac{\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}}{\dot{\boldsymbol{q}} \times \boldsymbol{p}} = \frac{\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}}{\dot{\boldsymbol{p}} \times \boldsymbol{q}}$$

gives therefore $\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q}$. That it also gives

(4) $$\dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r},$$

whatever R, appears by putting $R - O = \boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q}$, whence

$$\begin{aligned}\dot{\boldsymbol{r}} &= a\dot{\boldsymbol{p}} + b\dot{\boldsymbol{q}} + c(\dot{\boldsymbol{p}} \wedge \boldsymbol{q} + \boldsymbol{p} \wedge \dot{\boldsymbol{q}}) \\ &= a\boldsymbol{w} \wedge \boldsymbol{p} + b\boldsymbol{w} \wedge \boldsymbol{q} + c[(\boldsymbol{w} \wedge \boldsymbol{p}) \wedge \boldsymbol{q} + \boldsymbol{p} \wedge (\boldsymbol{w} \wedge \boldsymbol{q})] \\ &= a\boldsymbol{w} \wedge \boldsymbol{p} + b\boldsymbol{w} \wedge \boldsymbol{q} + c\boldsymbol{w} \wedge (\boldsymbol{p} \wedge \boldsymbol{q}) \\ &= \boldsymbol{w} \wedge (a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q}) = \boldsymbol{w} \wedge \boldsymbol{r}.\end{aligned}$$

It is thus proved that *the velocities of all points of the body can be found from those of any three non-collinear points.*

5. The velocities $\dot{R}$, $\dot{S}$ of two points R, S are equal if $\dot{\boldsymbol{r}} = \dot{\boldsymbol{s}}$, *i. e.*, if $\boldsymbol{w} \wedge \boldsymbol{r} = \boldsymbol{w} \wedge \boldsymbol{s}$, whence $\boldsymbol{w} \wedge (\boldsymbol{r} - \boldsymbol{s}) = 0$. It follows that all points of any line parallel to $\boldsymbol{w}$ have equal velocities. This shows that the case of Art. 3, where two of the given velocities were assumed equal, is no less general than that of Art. 4; but the case of Art. 3 suggests more directly the introduction of the auxiliary vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

In general, there is no point of zero velocity. But if such a point exists, say O, then all points of the line l through O, parallel to $\boldsymbol{w}$, have zero velocity, and the velocity $\dot{R} = \dot{\boldsymbol{r}} = \boldsymbol{w} \wedge \boldsymbol{r}$ of every other point R is normal to $\boldsymbol{w}$ and $\boldsymbol{r}$ and in magnitude proportional to the distance of R from l. The instantaneous motion in this case is called a **rotation**; the line l is called the **axis**, the vector $\boldsymbol{w}$ the **angular velocity**, of the rotation.

6. Even in the general case the vector $\boldsymbol{w}$ may be called the angular velocity. The instantaneous motion is completely determined by the angular velocity $\boldsymbol{w}$ and the linear velocity $\dot{O}$ of any one point. For then the velocity of every point R is, by (4):

$$\dot{R} = \dot{O} + \boldsymbol{w} \wedge \boldsymbol{r}.$$

II. Acceleration.

7. Differentiating the equations $\dot{\boldsymbol{p}} = \boldsymbol{w} \wedge \boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w} \wedge \boldsymbol{q}$ we find:

$$\begin{aligned}\ddot{\boldsymbol{p}} &= \dot{\boldsymbol{w}} \wedge \boldsymbol{p} + \boldsymbol{w} \wedge \dot{\boldsymbol{p}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{p} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{p}), \\ \ddot{\boldsymbol{q}} &= \dot{\boldsymbol{w}} \wedge \boldsymbol{q} + \boldsymbol{w} \wedge \dot{\boldsymbol{q}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{q} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q}).\end{aligned} \tag{5}$$

These equations are not linear in $\boldsymbol{w}$. Eliminating $\dot{\boldsymbol{w}}$ we have:

$$\begin{aligned}\ddot{\boldsymbol{p}} \times \boldsymbol{p} &= \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{p}) \times \boldsymbol{p} = (\boldsymbol{w} \times \boldsymbol{p})^2 - \boldsymbol{w}^2\boldsymbol{p}^2, \\ \ddot{\boldsymbol{q}} \times \boldsymbol{q} &= \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q}) \times \boldsymbol{q} = (\boldsymbol{w} \times \boldsymbol{q})^2 - \boldsymbol{w}^2\boldsymbol{q}^2, \\ \ddot{\boldsymbol{p}} \times \boldsymbol{q} + \ddot{\boldsymbol{q}} \times \boldsymbol{p} &= 2(\boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - \boldsymbol{p} \times \boldsymbol{q} \cdot \boldsymbol{w}^2).\end{aligned} \tag{5'}$$

If in the trihedral formed by $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{w}$ at O (Fig. 1) we denote the face angles $(\boldsymbol{p}, \boldsymbol{q})$, $(\boldsymbol{p}, \boldsymbol{w})$, $(\boldsymbol{q}, \boldsymbol{w})$ by a, b, c and the dihedral angle at $\boldsymbol{w}$ by α, and if we put

for the sake of brevity

$$-\frac{\ddot{\boldsymbol{p}} \times \boldsymbol{p}}{\boldsymbol{p}^2} = B, \qquad -\frac{\ddot{\boldsymbol{q}} \times \boldsymbol{q}}{\boldsymbol{q}^2} = C,$$

$$-\frac{\ddot{\boldsymbol{p}} \times \boldsymbol{q} + \ddot{\boldsymbol{q}} \times \boldsymbol{p}}{2 \bmod \boldsymbol{p} \cdot \bmod \boldsymbol{q}} = A,$$

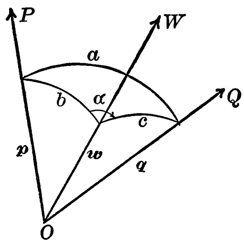


Fig. 1.

the equations become

(6) $\boldsymbol{w}^2 \sin^2 b = B, \quad \boldsymbol{w}^2 \sin^2 c = C, \quad \boldsymbol{w}^2 \sin b \sin c \cos \alpha = A,$

whence

$$\cos \alpha = \frac{A}{\sqrt{BC}}. \tag{7}$$

When the accelerations of the three points O, P, Q are given, the quantities A, B, C can be found. But the vector $\boldsymbol{w}$ is not uniquely determined: its extremity W (Fig. 1) may lie on either side of the plane OPQ, and the sense of $\boldsymbol{w}$ remains indeterminate. Thus there are four distinct solutions.

Moreover, the accelerations $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$ cannot be arbitrarily prescribed. To obtain a real $\boldsymbol{w}$ it is necessary that $A^2 \leqq BC$, *i. e.*,

$$(\ddot{\boldsymbol{p}} \times \boldsymbol{q} + \ddot{\boldsymbol{q}} \times \boldsymbol{p})^2 \leqq 4\ddot{\boldsymbol{p}} \times \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} \times \boldsymbol{q}.$$

8. Suppose, in particular, that the points O, P, Q are such that $\boldsymbol{p}$ and $\boldsymbol{q}$ are rectangular unit vectors and put

$$\boldsymbol{w} = w_1\boldsymbol{p} + w_2\boldsymbol{q} + w_3\boldsymbol{p} \wedge \boldsymbol{q},$$

so that w_1, w_2, w_3 are the rectangular coordinates of $\boldsymbol{w}$ with respect to $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{p} \wedge \boldsymbol{q}$. If, moreover, p_1, p_2, p_3 and q_1, q_2, q_3 are the coordinates of $\ddot{\boldsymbol{p}}$ and $\ddot{\boldsymbol{q}}$ with respect to the same reference set, the equations (5′) become

$$w_2^2 + w_3^2 = -p_1, \quad w_3^2 + w_1^2 = -q_2, \quad 2w_1w_2 = p_2 + q_1. \tag{8}$$

These equations give

$$\begin{aligned} w_1^2 &= \tfrac{1}{2}[p_1 - q_2 + \sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2}], \\ w_2^2 &= -\tfrac{1}{2}[p_1 - q_2 - \sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2}], \end{aligned} \tag{9}$$

where the sign of the square root has been selected so as to make w_1 and w_2 real. For w_3 we find:

$$w_3^2 = -p_1 - w_2^2 = -q_2 - w_1^2 = -\tfrac{1}{2}[p_1 + q_2 + \sqrt{(p_1 - q_2)^2 + (p_2 + q_1)^2}].$$

Hence w_3 is real only if

$$p_1q_2 \geqq \left(\frac{p_2 + q_1}{2}\right)^2,$$

which is the condition at the end of Art. 7.

9. *To express $\dot{\boldsymbol{w}}$ in terms of $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$, and $\boldsymbol{w}$* we may either differentiate (3) or observe that, referring $\dot{\boldsymbol{w}}$ to $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{w}$, we must have:

$$\boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \dot{\boldsymbol{w}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \boldsymbol{p} + \boldsymbol{p} \wedge \dot{\boldsymbol{w}} \times \boldsymbol{w} \cdot \boldsymbol{q} + \boldsymbol{p} \wedge \boldsymbol{q} \times \dot{\boldsymbol{w}} \cdot \boldsymbol{w}$$

Now by (5)

$$\dot{\boldsymbol{w}} \wedge \boldsymbol{q} \times \boldsymbol{w} = \ddot{\boldsymbol{q}} \times \boldsymbol{w}, \qquad \boldsymbol{p} \wedge \dot{\boldsymbol{w}} \times \boldsymbol{w} = -\ddot{\boldsymbol{p}} \times \boldsymbol{w},$$

$$\boldsymbol{p} \wedge \boldsymbol{q} \times \dot{\boldsymbol{w}} = \boldsymbol{p} \times \boldsymbol{q} \wedge \dot{\boldsymbol{w}} = -\boldsymbol{p} \times \ddot{\boldsymbol{q}} + \boldsymbol{p} \times \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{q})$$

$$= -\boldsymbol{p} \times \ddot{\boldsymbol{q}} + \boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - w^2 \cdot \boldsymbol{p} \times \boldsymbol{q};$$

hence

$$\begin{aligned} \boldsymbol{p} \wedge \boldsymbol{q} \times \boldsymbol{w} \cdot \dot{\boldsymbol{w}} &= \boldsymbol{w} \times \ddot{\boldsymbol{q}} \cdot \boldsymbol{p} - \boldsymbol{w} \times \ddot{\boldsymbol{p}} \cdot \boldsymbol{q} \\ &\quad + (\boldsymbol{w} \times \boldsymbol{p} \cdot \boldsymbol{w} \times \boldsymbol{q} - w^2 \cdot \boldsymbol{p} \times \boldsymbol{q} - \boldsymbol{p} \times \ddot{\boldsymbol{q}})\boldsymbol{w}. \end{aligned} \tag{10}$$

The vector $\dot{\boldsymbol{w}}$ may be called the *angular acceleration.*

10. *Acceleration of any point R.* Putting, as in Art. 4,

$$R - O = \boldsymbol{r} = a\boldsymbol{p} + b\boldsymbol{q} + c\boldsymbol{p} \wedge \boldsymbol{q},$$

we have

$$\ddot{\boldsymbol{r}} = a\ddot{\boldsymbol{p}} + b\ddot{\boldsymbol{q}} + c(\ddot{\boldsymbol{p}} \wedge \boldsymbol{q} + \boldsymbol{p} \wedge \ddot{\boldsymbol{q}} + 2\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}}).$$

It may be observed that the acceleration of every point of the plane OPQ is given by

$$\ddot{\boldsymbol{r}} = a\ddot{\boldsymbol{p}} + b\ddot{\boldsymbol{q}};$$

i. e., it is determined completely by the accelerations $\ddot{O}$, $\ddot{P}$, $\ddot{Q}$, of any three non-collinear points of this plane. But the acceleration of any point R not in this plane depends in addition, on the vector $\dot{\boldsymbol{p}} \wedge \dot{\boldsymbol{q}} = \dot{\boldsymbol{p}} \times \boldsymbol{q} \cdot \boldsymbol{w}$.

On the other hand, the relation (4) gives

$$\ddot{\boldsymbol{r}} = \dot{\boldsymbol{w}} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}); \tag{11}$$

i. e., the acceleration $\ddot{R}$ of every point R is determined by the vectors $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$ and the acceleration $\ddot{O}$ of any one point O.

III. Center of Acceleration.

11. To see whether a point R of zero acceleration (or a *center of acceleration*) exists, let us first assume that $\boldsymbol{w} \times \dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \neq 0$, *i. e.*, $(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 \neq 0$, so that we can put

$$R - O = \boldsymbol{r} = a\boldsymbol{w} + b\dot{\boldsymbol{w}} + c\boldsymbol{w} \wedge \dot{\boldsymbol{w}}.$$

We then have to determine a, b, c so as to make $\ddot{R} = 0$, *i. e.*, by (11),

$$\ddot{O} + \dot{\boldsymbol{w}} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = 0.$$

Substituting for $\boldsymbol{r}$ its value we obtain the condition:

$$\ddot{O} + a\dot{\boldsymbol{w}} \wedge \boldsymbol{w} + c\dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) + b\boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) - cw^2 \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0.$$

Multiplying by $\times\, \boldsymbol{w}$, $\times\, \dot{\boldsymbol{w}}$, $\times\, (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})$ we find for a, b, c the conditions:

$$\ddot{O} \times \boldsymbol{w} + c\dot{\boldsymbol{w}} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \times \boldsymbol{w} = 0,$$

$$\ddot{O} \times \dot{\boldsymbol{w}} + b\boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \times \dot{\boldsymbol{w}} = 0,$$

$$\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} - a(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 - c\boldsymbol{w}^2 \cdot (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 = 0,$$

whence

$$a = \frac{\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w}^2}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}, \qquad b = \frac{\ddot{O} \times \dot{\boldsymbol{w}}}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}, \qquad c = -\frac{\ddot{O} \times \boldsymbol{w}}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}.$$

The acceleration center is therefore given by the equation

$$(12) \quad (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2 \boldsymbol{r} = \ddot{O} \times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \boldsymbol{w}^2 \cdot \boldsymbol{w}) \cdot \boldsymbol{w} + \ddot{O} \times \dot{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}.$$

This equation shows that whenever $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} \neq 0$ there exists one and only one acceleration center; if in particular $\ddot{O} = 0$, O is the center.

12. It remains to discuss the cases when $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. We first assume $\ddot{O} \neq 0$.

(*a*) Suppose $\ddot{O} \neq 0$, $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. We then have $\dot{\boldsymbol{w}} = k\boldsymbol{w}$, where $k \neq 0$, and the condition for R to have zero acceleration becomes

$$\ddot{O} + k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = 0.$$

Multiplying by $\boldsymbol{w} \times$ we find $\boldsymbol{w} \times \ddot{O} = 0$ as a first necessary condition for the existence of an acceleration center in this case.

It follows that $\boldsymbol{w} \wedge \ddot{O} \neq 0$ so that we can put $\boldsymbol{r} = a\boldsymbol{w} + b\ddot{O} + c\boldsymbol{w} \wedge \ddot{O}$; substituting this value we find if $\boldsymbol{w} \times \ddot{O} = 0$:

$$[1 - (kc + b)\boldsymbol{w}^2]\ddot{O} + (bk - c\boldsymbol{w}^2)\boldsymbol{w} \wedge \ddot{O} = 0,$$

whence

$$(kc + b)\boldsymbol{w}^2 = 1, \qquad kb = c\boldsymbol{w}^2,$$

i. e.,

$$b = \frac{1}{k^2 + \boldsymbol{w}^2}, \qquad c = \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)},$$

while a remains indeterminate. Thus, *if $\ddot{O} \neq 0$, $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, while $\dot{\boldsymbol{w}}$ is parallel to $\boldsymbol{w}$, there is no point of zero acceleration unless $\ddot{O}$ is normal to $\boldsymbol{w}$; if $\dot{\boldsymbol{w}} \times \ddot{O} = 0$, every point of the line*

$$R = O + a\boldsymbol{w} + \frac{1}{k^2 + \boldsymbol{w}^2}\ddot{O} + \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)}\boldsymbol{w} \wedge \ddot{O},$$

and no other point, has zero acceleration.

(*b*) *Under the same conditions, except that $\dot{\boldsymbol{w}} = 0$, there is no acceleration center if $\boldsymbol{w} \times \ddot{O} \neq 0$; if $\boldsymbol{w} \times \ddot{O} = 0$, the points of the line*

$$R = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\ddot{O}$$

have zero acceleration.

(*c*) If $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$ we have $\ddot{R} = \ddot{O} + \dot{\boldsymbol{w}} \wedge \boldsymbol{r}$; hence a first condition is $\dot{\boldsymbol{w}} \times \ddot{O} = 0$. We can therefore put $\boldsymbol{r} = a\dot{\boldsymbol{w}} + b\ddot{O} + c\dot{\boldsymbol{w}} \wedge \ddot{O}$ so that

$$\ddot{R} = \ddot{O} + b\dot{\boldsymbol{w}} \wedge \ddot{O} - c\dot{\boldsymbol{w}}^2 \cdot \ddot{O} = 0;$$

hence

$$b = 0, \qquad c = \frac{1}{\dot{\boldsymbol{w}}^2},$$

while a is arbitrary. Thus, *if* $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$ *there is no acceleration center if* $\dot{\boldsymbol{w}} \times \ddot{O} \neq 0$; *if* $\dot{\boldsymbol{w}}$ *is normal to* $\ddot{O}$ *all points of the line*

$$R = O + a\dot{\boldsymbol{w}} + \frac{1}{\dot{\boldsymbol{w}}^2}\dot{\boldsymbol{w}} \wedge \ddot{O}$$

have zero acceleration.

(*d*) *If* $\ddot{O} \neq 0$, $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} = 0$ *there is evidently no point of zero acceleration.*

13. Next suppose $\ddot{O} = 0$ and $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. Then we have the following cases:

(*e*) $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$. As $\dot{\boldsymbol{w}} = k\boldsymbol{w}$, where $k \neq 0$, the condition for R to have zero acceleration is

$$0 = k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \boldsymbol{r}) = k\boldsymbol{w} \wedge \boldsymbol{r} + \boldsymbol{w} \times \boldsymbol{r} \cdot \boldsymbol{w} - \boldsymbol{w}^2 \cdot \boldsymbol{r}.$$

Hence in this case *every point of the line through* O *parallel to* $\boldsymbol{w}$ *has zero acceleration.*

(*f*) $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} = 0$. The result is evidently the same.

(*g*) $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} \neq 0$. As $\ddot{R} = \dot{\boldsymbol{w}} \wedge \boldsymbol{r}$, *every point of the line through* O *parallel to* $\dot{\boldsymbol{w}}$ *has zero acceleration.*

(*h*) $\boldsymbol{w} = 0$, $\dot{\boldsymbol{w}} = 0$. As $\ddot{R} = 0$, *every point has zero acceleration.*

IV. Applications.

14. *Example I: Projectile whose axis of spin is permanent.* Let O be the centroid; as $\dot{\boldsymbol{w}} = 0$ we have case (*b*) of Art. 12. Hence there is no point of zero acceleration unless $\boldsymbol{w} \times \ddot{O} = 0$.

Suppose this condition satisfied. Let O_0 be the initial position of the centroid; $\boldsymbol{j}_0$, $\boldsymbol{k}_0$ unit vectors, $\boldsymbol{j}_0$ horizontal and at right angles to $\boldsymbol{w}$, $\boldsymbol{k}_0$ vertical downwards. Then, if v_1, v_2, v_3 are the coördinates of the velocity of O along $\boldsymbol{w}, \boldsymbol{j}_0, \boldsymbol{k}_0$, the position of the centroid at the time t is $O = O_0 + v_1t\boldsymbol{w} + v_2t\boldsymbol{j}_0 + (v_3t + \frac{1}{2}gt^2)\boldsymbol{k}_0$; hence

$$R = O_0 + v_1t\boldsymbol{w} + v_2t\boldsymbol{j}_0 + (v_3t + \tfrac{1}{2}gt^2)\boldsymbol{k}_0 + a\boldsymbol{w} + \frac{g}{\boldsymbol{w}^2}\boldsymbol{k}_0.$$

This is, for a constant t, the line whose points have zero accelerations. For t as a variable parameter, the same equation represents the ruled surface whose generators are those lines of space which in the course of time come to have this property. Thus, the locus in space of all those points that have zero acceleration is a parabolic cylinder.

To obtain the locus, in the body, of these lines, let $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors fixed

in the body and initially coinciding with $\boldsymbol{j}_0$, $\boldsymbol{k}_0$. Then we have for any point S of the body:

$$S = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}(\boldsymbol{g} \times \boldsymbol{j} \cdot \boldsymbol{j} + \boldsymbol{g} \times \boldsymbol{k} \cdot \boldsymbol{k})$$

$$= O + a\boldsymbol{w} + \frac{g}{\boldsymbol{w}^2}\sin wt \cdot \boldsymbol{j} + \frac{g}{\boldsymbol{w}^2}\cos wt \cdot \boldsymbol{k},$$

where $g = \text{mod}\,\boldsymbol{g}$, $w = \text{mod}\,\boldsymbol{w}$. The surface is a circular cylinder.

15. *Example II: Homogeneous circular disk rolling down an inclined plane, starting from rest.*

Let O be the position of the centroid at the time t. We have $\boldsymbol{w} \neq 0$, $\dot{\boldsymbol{w}} \neq 0$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$; *i. e.*, we have case (*a*) of Art. 12, with $\boldsymbol{w} \times \ddot{O} = 0$. Hence, if O_0 is the initial position of the centroid, the points of zero acceleration are given by the equation

$$R = O_0 + (O - O_0) + a\boldsymbol{w} + \frac{1}{k^2 + \boldsymbol{w}^2}\ddot{O} + \frac{k}{\boldsymbol{w}^2(k^2 + \boldsymbol{w}^2)}\boldsymbol{w} \wedge \ddot{O}.$$

If we put

$$\ddot{O} = a_1\boldsymbol{u}_0, \quad O - O_0 = \tfrac{1}{2}a_1t^2\boldsymbol{u}_0, \quad \boldsymbol{w} = a_2t\boldsymbol{w}_0, \quad k = \frac{\text{mod}\,\dot{\boldsymbol{w}}}{\text{mod}\,\boldsymbol{w}} = \frac{1}{t}, \quad \boldsymbol{w}_0 \wedge \boldsymbol{u}_0 = \boldsymbol{v}_0,$$

the equation becomes

$$R = O_0 + \tfrac{1}{2}a_1t^2\boldsymbol{u}_0 + aa_2t\boldsymbol{w}_0 + \frac{t^2}{1 + a_2{}^2t^4}a_1\boldsymbol{u}_0 + \frac{a_1}{a_2(1 + a_2{}^2t^4)}\boldsymbol{v}_0$$

$$= O_0 + aa_2t\boldsymbol{w}_0 + \frac{3 + a_2t^4}{2(1 + a_2{}^2t^4)}a_1t^2\boldsymbol{u}_0 + \frac{a_1}{a_2(1 + a_2{}^2t^4)}\boldsymbol{v}_0.$$

This is the locus, in space, of the points of zero acceleration. To find the locus in the body let $\boldsymbol{u}$, $\boldsymbol{v}$ be unit vectors fixed in the body and initially coinciding with $\boldsymbol{u}_0$, $\boldsymbol{v}_0$. The angle made by $\boldsymbol{u}$ with $\boldsymbol{u}_0$ is $\frac{1}{2}a_2t^2$, and the equation of the locus is

$$S = O + aa_2t\boldsymbol{w}_0 + \frac{t^2}{1 + a_2t^4}(a_1\boldsymbol{u}_0 \times \boldsymbol{u} \cdot \boldsymbol{u} + a_1\boldsymbol{u}_0 \times \boldsymbol{v} \cdot \boldsymbol{v})$$
$$+ \frac{a_1}{a_2(1 + a_2{}^2t^4)}(\boldsymbol{v}_0 \times \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v}_0 \times \boldsymbol{v} \cdot \boldsymbol{v}),$$

where

$$\boldsymbol{u}_0 \times \boldsymbol{u} = \cos\tfrac{1}{2}a_2t^2, \qquad \boldsymbol{u}_0 \times \boldsymbol{v} = -\sin\tfrac{1}{2}a_2t^2, \qquad \boldsymbol{v}_0 \times \boldsymbol{u} = \sin\tfrac{1}{2}a_2t^2,$$
$$\boldsymbol{v}_0 \times \boldsymbol{v} = \cos\tfrac{1}{2}a_2t^2.$$

16. *Example III: Projectile whose momental ellipsoid at the centroid is an ellipsoid of revolution.*

This is the general case so that we have to use equation (12), viz.,

$$R = O + \frac{1}{(\boldsymbol{w} \wedge \dot{\boldsymbol{w}})^2}[(\ddot{O} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} + \boldsymbol{w}^2 \cdot \ddot{O} \times \boldsymbol{w})\boldsymbol{w} + \ddot{O} \times \dot{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \ddot{O} \times \boldsymbol{w} \cdot \boldsymbol{w} \wedge \dot{\boldsymbol{w}}].$$

Taking for O the centroid (Fig. 2) we have $\ddot{O} = \boldsymbol{g}$, $\ddot{O} \times \dot{\boldsymbol{w}} = 0$, and $\boldsymbol{w} \times \dot{\boldsymbol{w}} = 0$; hence, denoting the moduli of $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$, $\boldsymbol{g}$ by w, $\dot{w}$, g, and the constant angle made by $\boldsymbol{w}$ with $\boldsymbol{g}$ by θ, we have:

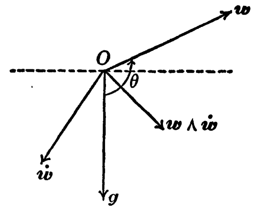


FIG. 2.

$$R = O + \frac{1}{\boldsymbol{w}^2\dot{\boldsymbol{w}}^2}\left[(gw\dot{w}\sin\theta + gw^3\cos\theta)\boldsymbol{w} - gw\cos\theta\cdot\boldsymbol{w}\wedge\dot{\boldsymbol{w}}\right].$$

This shows that the body locus is a circle while the space locus is a transcendental curve.

CENTRAL AXIS.

17. At any given instant t, the locus of those points of the body whose velocity is parallel to $\boldsymbol{w}$ is called the *instantaneous screw axis* of the motion, or the *central axis* of the velocity field. Assuming $\boldsymbol{w} \neq 0$, *i. e.*, assuming that the instantaneous motion is not a translation, it is readily shown that this locus is a straight line parallel to $\boldsymbol{w}$.

For, if the velocity $\dot{R} = \dot{O} + \boldsymbol{w} \wedge \boldsymbol{r}$ of any point R of the body is to be parallel to $\boldsymbol{w}$ we must have $\dot{R} \wedge \boldsymbol{w} = 0$, *i. e.*,

$$\dot{O} \wedge \boldsymbol{w} + (\boldsymbol{w} \wedge \boldsymbol{r}) \wedge \boldsymbol{w} = 0.$$

Putting $\boldsymbol{r} = a\boldsymbol{w} + b\dot{O} + c\boldsymbol{w} \wedge \dot{O}$ we find

$$\dot{O} \wedge \boldsymbol{w} + [\boldsymbol{w} \wedge (b\dot{O} + c\boldsymbol{w} \wedge \dot{O})] \wedge \boldsymbol{w} = 0,$$

i. e.,

$$\dot{O} \wedge \boldsymbol{w} + b(\boldsymbol{w} \wedge \dot{O}) \wedge \boldsymbol{w} + c\boldsymbol{w}^2 \cdot \boldsymbol{w} \wedge \dot{O} = 0.$$

Hence $b = 0$, $c = 1/\boldsymbol{w}^2$, while a remains arbitrary. The equation of the central axis at the time t is therefore

$$R = O + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}. \tag{13}$$

Every value of a gives a point R of the central axis; if R_0 corresponds to $a = 0$, we have $R - R_0 = a\boldsymbol{w}$; *i. e.*, the central axis is parallel to $\boldsymbol{w}$.

18. If in (13) the point O and the vectors $\boldsymbol{w}$ and $\dot{O}$ are given as functions of the time t, so that R becomes a function of the parameters a and t, this equation (13) represents the ruled surface formed by those lines of the body which in the course of time become central axes.

To obtain the equation of the ruled surface formed by those lines of space

which in the course of time become central axes, let O_1 be a point fixed in space and $\dot{O}$ the velocity of that point of the body which at the time t coincides with O_1. Then the required equation is

$$R = O_1 + a\boldsymbol{w} + \frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}. \tag{14}$$

19. By (14), the velocity of any point R of the central axis is

$$\dot{R} = a\dot{\boldsymbol{w}} + \frac{d}{dt}\left(\frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}\right).$$

The ruled surface (14) is developable, *i. e.*, the motion of the central axis is a pure rotation, if the velocity $\dot{R}$ is parallel to the plane of the vectors $\boldsymbol{w}$ and $\dot{\boldsymbol{w}}$, *i. e.*, if $\dot{R} \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0$. The condition for the surface (14) to be developable is therefore

$$\left(\frac{d}{dt}\frac{1}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O}\right) \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0,$$

i. e.,

$$\left(-\frac{2\boldsymbol{w} \times \dot{\boldsymbol{w}}}{\boldsymbol{w}^2}\boldsymbol{w} \wedge \dot{O} + \dot{\boldsymbol{w}} \wedge \dot{O} + \boldsymbol{w} \wedge \ddot{O}\right) \times \boldsymbol{w} \wedge \dot{\boldsymbol{w}} = 0. \tag{15}$$

20. This equation contains $\ddot{O}$ only in the combination $\ddot{O} \times (\boldsymbol{w} \wedge \dot{\boldsymbol{w}}) \wedge \boldsymbol{w}$; *i. e.*, if $\ddot{O}$ be resolved along the rectangular vectors $\boldsymbol{w}$, $\boldsymbol{w} \wedge \dot{\boldsymbol{w}}$, and $\boldsymbol{u} = \boldsymbol{w} \wedge (\boldsymbol{w} \wedge \dot{\boldsymbol{w}})$, the equation contains only the component of $\ddot{O}$ along $\boldsymbol{u}$.

This fact has an interesting dynamical interpretation. Let O be the centroid, and let $\boldsymbol{R}$, $\boldsymbol{H}$ be the vectors of the resultant force and couple for O. Then our result means that, if the surface of the screw axes is to be developable, $\boldsymbol{R}$ must have a fixed projection on $\boldsymbol{u}$; in other words, all values of $\boldsymbol{R}$ that give a developable surface are such that, when drawn from O, their extremities lie in a plane parallel to $\boldsymbol{w}$ and $\boldsymbol{w} \wedge \dot{\boldsymbol{w}}$.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated and provided with an introduction and notes by PHILIP E. B. JOURDAIN. The Open Court Publishing Company, Chicago, 1915. ix+211 pages.

The main purpose of this little volume, the first in the Open Court Series of Classics of Science and Philosophy, is to make accessible in English the two final papers of Georg Cantor on the theory of transfinite numbers, which embody the culmination of his researches on this subject, and which appeared in 1895 and 1897 in the *Mathematische Annalen* (Vols. 46 and 49). In order to give a proper setting to these two papers, the translator precedes them by an introduction of about 80 pages, in which he sketches briefly the progress of ideas which lead up to

these researches and results. A brief note collecting some of the chief developments which have been made in this theory since that time, brings the volume to a close.

In the introduction, we find some brief indications of the genesis of the modern conception of function, out of the consideration of the problem of the vibrating string and the theory of Fourier series. This is followed by a sketch of Weierstrass's theory of irrational numbers, in so far as Weierstrass probably exerted a most potent influence on Cantor. The main section of the introduction, however, is devoted to a résumé of the chief results obtained by Cantor in his researches preceding the papers translated. Beginning with his investigations in the theory of Fourier series, we are shown how Cantor was led to a closer study of point sets, on the one hand in the direction of their " Maechtigkeit " or power, and on the other to the consideration of their derivatives of various orders, the latter culminating in the notion of transfinite ordinal numbers.

The desirability of such an historical introduction is of course apparent. Aside from its value in preparing one for the papers which form the central portion of the book, it is always an interesting matter to trace the development of an idea, to go behind the scenes, so to speak, and see how a bit of investigation gradually assumes a final form. It is therefore very much to be regretted that our author in his attempts to be brief is not always perfectly clear. For instance his exposition of Weierstrass's theory of irrational numbers, on pages 18 and 19, would appeal to one who is not thoroughly conversant with the theory in question as a hopeless conglomeration of aggregates, sums, numerical quantities, and other elements. A simple illustration would probably have served the purpose much better than this attempt at generality without definiteness. Later on, in Section VII, page 52 and following, in the résumé of Cantor's 1883 Grundlagen, much would have been gained in clearness if the author had followed Cantor's original article a little more closely. In particular, the meaning of the word "Anzahl," which is translated enumeral, and seems to lead to confusion in the mind of the author, would have been perfectly clear at the outset as referring particularly to arrangement, rather than to " Maechtigkeit " or power. As it is, only after it has been used repeatedly and frequently do we really get any inkling as to the meaning to be assigned to the term enumeral. Of other sections which might well be improved, we might note the discussion of DuBois Reymond's work on point sets, on page 34; and the proof that the totality of real numbers is not enumerable on page 39.

To give the contents of the papers translated is hardly within the province of this review. Briefly, they contain the developments of the notion of cardinal number, that is, the power of a set, and the laws governing the usual algebraic processes as applied to them. A similar treatment is given for the ordinal numbers. The second of the two papers concerns itself mainly with well-ordered sets. The translation, considered as a whole, is fairly well done. Occasional passages to be sure are reminiscent of the German idiom, and perhaps too literal. For instance, the statement of Theorem H on page 147 is ambiguous as a result of sticking too closely to the German text.

In its general makeup and appearance, we think the book commendable. As a personal opinion, we feel that it would have tended towards clarity if the translator had followed Cantor in italicizing the theorems. Further, we find the printing of the page numbers of the original papers in the text in bold-faced type somewhat disconcerting. They might well have been relegated to the margin and a smaller size of type used. Of misprints, which are liable to cause ambiguity, we note: on page 147, line 14, D ought to be replaced by G; and in line 21, F ought to be replaced by f.

As a contribution to the history of transfinite numbers, and in making the main two papers on the foundations of these numbers accessible in English, we think this little volume is worthy of consideration. As an introduction to the subject, however, it cannot be unqualifiedly recommended.

T. H. HILDEBRANDT.

THE UNIVERSITY OF MICHIGAN.

Solid Geometry. By SOPHIA FOSTER RICHARDSON. Ginn and Co., Boston, 1914. v + 209 pages.

In applying to Professor Richardson's solid geometry the severe test of classroom use, it has been evident that the book possesses many excellent qualities. The use of a single letter to denote a line or a plane and the names skew lines, pencil of lines, pencil of planes, bundle of lines, bundle of planes have tended to make the work concise and the class demonstrations easy to follow. The first chapter contains more theorems than is usual in books on solid geometry. It provides a good drill in the more simple propositions that fix the concepts of perpendicularity and parallelism in space. A feature of interest to the student is the illustration of the principle of duality in several theorems differing only in the interchange of the words line and plane. Throughout the book there is an abundance of valuable and interesting exercises. However in many of the demonstrations a part of the work might well have been left for the student to supply.

The typographical errors are few, and in no instances misleading. The treatment of the incommensurable cases based on the Dedekind-Cantor theory of irrational numbers, presented in the appendix, is doubtless rigorous, but its advantages over the more ordinary methods have not been evident, since otherwise the work could be made much shorter and as rigorous as can be appreciated by first year college students. The propositions leading up to the determination of the surface and the volume of a sphere seem unnecessarily long and so involved that it becomes an arduous task even to quote them.

The formula for the volume of any parallelopiped is developed in two theorems, thus eliminating the celebrated "devil's coffin." This ruthless vandalism is regretted perhaps more by the expectant freshman, who has heard of its fame, than by the instructor, who has become calloused to its fearful and wonderful construction.

Taking everything into consideration it would seem that Professor Richardson

has written a book in advance of many text-books in solid geometry, and one adapted to students somewhat more mature than those of high school grade.

LENNIE P. COPELAND.

WELLESLEY COLLEGE,
WELLESLEY, MASS.

L'Opera "De Corporibus Regularibus" di Pietro Franceschi detto della Francesca usurpata da Fra Luca Pacioli (con dodici tavole). Memoria di G. Mancini. Reale Accademia dei Lincei (anno CCCXII, 1915), Serie Quinta, Volume XIV, Fascicolo VII^B. Pp. 437–580; reprint, pp. 1–144.

During the renaissance there were two noted Italian artists who were also mathematicians. Both were in personal touch with the mathematical oracle of that day, Luca Pacioli. The two artists were Leonardo da Vinci and Pier della Francesca. Recently the mathematical achievements of both of these artists have been subjected to re-examination.[1] That Pacioli used a posthumous manuscript of Pier della Francesca without giving him due credit was affirmed by early writers on Pier della Francesca,[2] but denied by two later biographers of Pacioli.[3] This question was re-opened by G. Pittarelli[4] at the International Mathematical Congress held in Rome in 1908, who asserted that parts of Pacioli's *Divina proportione* (1509) were taken from a Vatican manuscript written by Pier della Francesca. This matter is re-investigated in the monograph under review, in which the Pier della Francesca manuscript, "De Corporibus Regularibus," is published, with extensive comments. While Pacioli's guilt seems now definitely established, it appears also from the history of the time that usurpations of this character were not uncommon. Thus, Pacioli's great compatriots Tartaglia and Cardan are both open to this charge. Gerolamo Mancini's monograph establishes Pier della Francesca's place in the history of the regular solids and also in the advancement of the theory of perspective. From the drawings it appears that Pier della Francesca was a skilled draftsman. His exposition of problems on areas is almost wholly rhetorical. Mancini's biographical and critical parts of the monograph constitute a valuable contribution to the history of early Italian mathematics.

FLORIAN CAJORI.

COLORADO COLLEGE,
COLORADO SPRINGS, COLO.

[1] See P. Duhem, *Études sur Léonard de Vinci*, Paris, 1906.

[2] Vasari, *Vita di Pier della Francesca*, Firenze, 1550, 1568. Egnatio Danti, *Commentari alle due Regole di prospettiva di I. Barozzi*, about 1575.

[3] *Elogio di fra Luca Pacioli, Scritti inediti del P. D. Cossali, pubblicati da B. Boncompagni.* Roma, 1857. H. Staigmüller, *Lucas Paciuolo, eine biographische Skizze* (Zeitschr. f. Math. u, Phys., Bd. XXXIV, 1889).

[4] G. Pittarelli in *Atti del IV. Congresso dei matematici*, tom. III, Roma, 1909.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

471. Proposed by E. T. BELL, University of Washington.

If there is an infinite number of positive integers r for which the equation $\sum_{i=1}^{n} a_i^r = \sum_{i=1}^{m} b_i^r$ holds, where the a_i and b_i are given positive integers, prove that $m = n$, and that in some order the a_i are identical with the b_i.

472. Proposed by E. T. BELL, University of Washington.

If a_i and b_j $(i = 1, \cdots, n;\ j = 1, \cdots, m)$ denote positive integers, and if $\sum_{i=1}^{n} a_i^r = \sum_{j=1}^{m} b_j^r$ for all odd positive integral values of r, prove that $m = n$, and that in some order the a_i are identical with the b_j.

GEOMETRY.

503. Proposed by J. W. CLAWSON, Ursinus College, Penn.

If two points A and B invert with respect to a third point O as center of inversion into A' and B', the middle point of the segment AB inverts into the point other than where the circle of Apollonius (the locus of a point P moving so that $A'P/PB' = A'O/OB'$) cuts the circle $OA'B'$.

504. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a given line. Find the envelop of the side of the pedal triangle opposite the moving vertex.

CALCULUS.

419. Proposed by C. C. YEN, Tangshan, North China.

Find the entire area of the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

420. Proposed by W. J. GREENSTREET, Stroud, England.

The join of the center of curvature of a curve to the origin is at an angle α to the initial line. Prove that with the usual notation,

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2 + \left(\frac{d^2p}{d\psi^2}\right)^2\right] = \frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

MECHANICS.

336. Proposed by C. N. SCHMALL, New York City.

An inclined plane, length l, makes an angle $\phi (< \pi/4)$ with the horizontal plane through its foot. From its foot, a body is projected upward along the plane, with a velocity equal to that of a falling body at the height h, so as to pass over the top and strike the horizontal plane at the *maximum distance* x from the foot of the inclined plane. Show by the methods of the calculus that $x = h/\sin\phi\cos\phi$, and that the corresponding value of l is $2h\cot 2\phi/\cos\phi$.

337. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Assuming that a train may be accelerated by the application of a force equal to 1/40 of its gross weight and be braked with a force equal to 1/10 of its gross weight, show that the least time in which it may run from one to another of two stopping stations 5,000 feet apart is 2 minutes and 5 seconds. Also find the greatest speed during the run to be 54-6/11 miles per hour.

NUMBER THEORY.

255. Proposed by FRANK IRWIN, University of California.

Given any arithmetical progression whose first term a and common difference d are relatively prime integers, and any finite set of positive integers $m_1, m_2, \cdots$ also relatively prime to d, it is required to determine an integer n such that the multiples of $m_1, m_2, \cdots$ may occupy the same positions in the series of natural numbers beginning with n as they do in the arithmetical progression. This is to say that if the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the progression are divisible by m_1, so also will be the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the series $n, n + 1, n + 2, \cdots$, etc. Show that n may be determined as the solution of a congruence $An + B \equiv 0 \pmod{C}$, whose coefficients, A, B, are constants independent of the number and value of the m's.

256. Proposed by FRANK IRWIN, University of California.

Let p be an odd prime, and let the notation $1/k$ stand for the solution of $kx \equiv 1 \pmod{p}$. Then show that if the sum of the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{(p-1)/2}$$

be congruent to zero (mod p)—should that be possible—the same is true for the sum of their products two at a time, as well as four at a time.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A solution of 450 was received from G. Y. SOSNOW and one of 454 from J. J. GINSBURG which have not been acknowledged.

458. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that n terms of the series $1 + 3 + 4 + 6 + 7 + 9 + 10 + \cdots$ is $\frac{1}{4}(n + 1)(3n - 1)$ when n is odd, and $\frac{n}{2}\left(\frac{3n}{2} + 1\right)$ when n is even.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

This series is a combination of two arithmetical progressions $3 + 6 + 9 + \cdots$ and $1 + 4 + 7 + 10 + \cdots$.

When n is even, the first becomes

$$3 + 6 + 9 + \cdots + \frac{3}{2}n = \frac{n}{4}\left(\frac{3}{2}n + 3\right)$$

and the second,

$$1 + 4 + 7 + 10 + \cdots + \left(\frac{3}{2}n - 2\right) = \frac{n}{4}\left(\frac{3}{2}n - 1\right).$$

The sum of these two is

$$\frac{n}{4}\left(\frac{3}{2}n + 3\right) + \frac{n}{4}\left(\frac{3}{2}n - 1\right) = \frac{n}{2}\left(\frac{3}{2}n + 1\right).$$

When n is odd, the first becomes

$$3 + 6 + 9 + \cdots + \frac{3}{2}(n - 1) = \frac{n-1}{4}\left(\frac{3n}{2} + \frac{3}{2}\right)$$

and the second,

$$1 + 4 + 7 + 10 + \cdots + \left(\frac{3n}{2} - \frac{1}{2}\right) = \frac{n+1}{4}\left(\frac{3n}{2} + \frac{1}{2}\right).$$

The sum of these two is

$$\frac{n-1}{4}\left(\frac{3(n+1)}{2}\right) + \frac{n+1}{4}\left(\frac{3n+1}{2}\right) = \frac{1}{4}\left(\frac{3n^2}{2} - \frac{3}{2} + \frac{3n^2}{2} + 2n + \frac{1}{2}\right)$$

$$= \frac{1}{4}(3n^2 + 2n - 1) = \frac{1}{4}(n + 1)(3n - 1).$$

Also solved by L. A. H. WARREN, W. C. EELLS, GEORGE PAASWELL, E. H. WORTHINGTON, HORACE OLSON, G. L. WAGAR, H. N. CARLETON, E. J. OGLESBY, H. C. FEEMSTER, BENJ. SINITSSCY, W. J. THOME, C. C. YEN, R. M. MATHEWS, E. F. CANADY, N. P. PANDYA, J. J. GINSBURG, O. S. ADAMS, A. M. HARDING, and PAUL CAPRON.

GEOMETRY.

485. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the surface generated by the orthogonal projection of a given line upon a variable plane turning about a fixed axis.

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

Take the fixed axis as X-axis, and the common perpendicular of this line and the given line as Z-axis. Call the angle θ made by the line we are projecting and the X-axis, so that the given line has the direction cosines $\cos\theta$, $\sin\theta$, 0; let the coördinates of the point where this line intersects the Z-axis be $(0, 0, a)$. Then the equation of the variable plane may be written in the form

(A) $y - \lambda z = 0$, where λ is a parameter.

The equation of any plane through the given line may be written in the form

(B) $\sin\theta x - \cos\theta y + k(z - a) = 0$, where k is an arbitrary constant.

If (B) is perpendicular to (A), k must equal $-\cos\theta/\lambda$. Substituting this value for k in (B), (A) and (B) give the required equation in parameter form. Eliminating λ, we obtain the explicit equation

$$\tan\theta \cdot xy = y^2 + z^2 - az.$$

Rotating the axes through an angle $\theta/2$ about the Z-axis, and changing the origin to the point $(0, 0, a/2)$, the equation finally becomes

$$x^2\left(\frac{1-\cos\theta}{2\cos\theta}\right) - y^2\left(\frac{1+\cos\theta}{2\cos\theta}\right) - z^2 = -\frac{a^2}{4},$$

which is the equation of a hyperboloid of one sheet.

II. SOLUTION BY THE PROPOSER.

The orthogonal projection u of a given line p upon a plane β passing through a given axis q, may be obtained as the intersection of β with the plane α perpendicular to β and passing through p. When the plane β turns about q, the projecting plane α turns about p; to each position of β corresponds one and only one position of α, and vice versa. The line u is, therefore, the intersection of two corresponding planes α, β of two projective pencils (p), (q).

I. The lines p, q are coplanar. *The line u generates a cone of second degree (C), of which p, q are elements.*

The tangent planes to (C) along p and q are the planes perpendicular to the plane (pq) and passing through the lines p and q respectively. The line of intersection of these tangent planes is perpendicular to (pq) and is the polar ray of this plane with respect to (C). Hence: *The plane of the two given lines is a plane of symmetry of the cone.*

The orthogonal projection P' of any point P of p upon the plane β lies on the line u and in the plane π through P perpendicular to q. When β varies, P' describes, in the plane π, a circle having for diameter the segment joining P to the point of intersection of π with q. This circle is the curve of intersection of (C) with the plane π. Similarly for the orthogonal projection Q' of any point Q of q upon α. Hence: *The planes perpendicular to the given lines are the planes of the circular sections of the cone.*

This cone is sometimes called the *Orthogonal* cone (Theodor Reye, *Geometrie der Lage*, part I, p. 119, fifth edition).

If the given lines p, q are parallel, the cone becomes a cylinder of revolution.

II. The lines p, q are skew. *The line u generates a hyperboloid of one sheet (H), of which p, q are two rays of the same system.*

The planes perpendicular to the given lines are the planes of circular sections of the hyperboloid. Same proof as for the cone above.

If the two pencils of planes (p), (q) be transported parallel to themselves so as to have their axes p, q pass through the center of (H), they will generate the asymptotic cone of the surface. Hence: *The asymptotic cone of the hyperboloid is orthogonal.*

This hyperboloid is sometimes called the *Orthogonal* hyperboloid (Salmon, *Analytic Geometry of Three Dimensions*, Vol. I, p. 116, fifth edition).

REMARK.—From the method of construction of the line u it follows that the cone (C) (or the hyperboloid) is also the surface generated by the orthogonal projection of the line q upon a variable plane turning about the axis p.

The reader may consider the special cases when (1) the given lines are perpendicular to each other; (2) one or both of the given lines are at infinity.

Also solved by H. C. FEEMSTER, G. W. HARTWELL, and FRANK H. LOUD.

486. Proposed by ARON INGVALE, Brooklyn, N. Y.

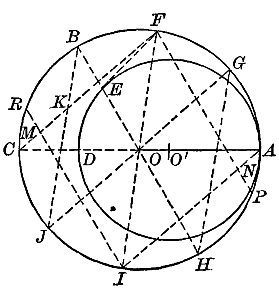
Does the following construction trisect an angle? With the vertex, O, of the given angle as center and with a radius R, describe a circle intersecting the sides of the given angle in A and B. With a radius $\frac{3}{4}R$, and center O' on OA, describe a circle tangent to the other circle at A and cutting the other side of the angle at E. At E draw a tangent to the last circle and produce it to meet the first circle at F. Draw FO. Then is angle BOF one-third of the angle BOA?

REMARK. Though the construction does not, of course, lead to the trisection of an angle in general, yet as a first approximation it is very good. This fact together with the fact that the construction is very simple, and that the proposer's demonstration that it does trisect the angle is very illusive, are the reasons for giving the problem a place in the MONTHLY.

EDITORS.

SOLUTION BY THE PROPOSER.

Let OG bisect $\angle AOF$. Extend FO and GO till they meet the original circle at I and J respectively. Also extend BO to H. Draw FC and AI, which for obvious reasons are parallel to each other and to GJ.



Then draw FP and IR each parallel to BH and prove $\angle BJG = \angle FIA$ as follows:

$\triangle BOJ$ is isosceles and since $CF \parallel JG$, therefore $\triangle KEB$ is also isosceles. But $\triangle KEB$ is similar to $\triangle FNI$ because their sides are respectively parallel.

Hence $\triangle FNI$ is isosceles and also $\triangle IMF$ to which it is congruent.

Hence $\angle CFI = \angle IFP = \angle BJG = \angle FIA$, from which follows $\angle BOF = \angle GOF$. But since $\angle GOF$ was constructed equal to $\angle AOG$, it follows that $\angle AOB$ is trisected.

Note.—The fallacy in this proof has been pointed out to Mr. Ingvale and he admits his error, but it is left as an exercise to others to show that the method applies to a right angle or a straight angle but fails in general. EDITORS.

CALCULUS.

403. Proposed by C. N. SCHMALL, New York City.

A paraboloid of revolution generated by the curve $x^2 = 4ay$, contains a quantity of water such that if a sphere of radius r be dropped to the bottom, it will just be covered by the water. Show that if the volume of water used in this experiment is to be a *minimum*, then we must have $a = r/6$.

Solution by H. S. Uhler, Yale University.

Since the surfaces are figures of revolution we may confine our attention to a parabola with its axis vertical and a co-planar circle whose center lies in this axis. Let the coördinates of the center of the circle be $(0, h)$. The equation of the circle is

$$x^2 + (y - h)^2 = r^2.$$

Elimination of x between this equation and $x^2 = 4ay$ gives

$$y^2 - 2(h - 2a)y - (r^2 - h^2) = 0.$$

In order that the sphere may touch the paraboloid the last equation must have equal roots, the necessary and sufficient condition being

$$4(h - 2a)^2 + 4(r^2 - h^2) = 0, \quad \text{or} \quad h = (r^2 + 4a^2)/4a.$$

The volume of the paraboloid up to the level $h + r$ is given by

$$\pi \int_0^{h+r} x^2 dy = 4\pi a \int_0^{h+r} y dy = 2\pi a (h + r)^2 = \frac{\pi (r + 2a)^4}{8a}.$$

In general, therefore, the volume of the water is expressed by

$$v \equiv \frac{\pi(r + 2a)^4}{8a} - \frac{4\pi r^3}{3}.$$

Two cases now present themselves, namely, (i) a is kept constant while r varies, or (ii) r is fixed in value while a changes.

Case (i)

$$\frac{\partial v}{\partial r} = \frac{\pi[r(r - a)^2 + 11a^2r + 8a^3]}{2a}.$$

Since this expression cannot vanish for any positive value of r the necessary condition for maxima and minima is not fulfilled, as is also apparent from general considerations.

Case (ii)

$$\frac{\partial v}{\partial a} = \frac{\pi(r + 2a)^3(6a - r)}{8a^2},$$

and

$$\frac{\partial^2 v}{\partial a^2} = \frac{\pi}{4a^3}[3a(r + 2a)^3 + (6a - r)(a - r)(r + 2a)^2].$$

When $6a - r = 0$, the first and second derivatives become 0 and $+ 64\pi r$, respectively. Consequently, when $a = r/6$ a minimum of volume is attained. It should be remarked, however, that $h = 5r/3$ and $h - r = 2r/3$, so that the lowest point of the sphere is two-thirds of the radius above the vertex of the paraboloid; in other words, the sphere has not literally been "dropped to the bottom."

Also solved by A. W. Smith, F. A. Pottle, Horace Olson, J. W. Clawson, C. A. Nickle, O. S. Adams, Paul Capron, J. A. Bullard, Geo. W. Hartwell, J. A. Caparo, A. G. Rau, H. C. Feemster, and G. Paaswell.

404. Proposed by B. J. BROWN, Victor, Colorado.

Solve the differential equation, $(x^2 - y^2)(1 + dy/dx) = 2xy(1 - dy/dx)$.

Solution by H. L. Agard, Williams College.

The equation may be written,

$$(x^2 - 2xy)dx + (2xy - y^2)dy + x^2dy - y^2dx = 0,$$

which, after adding and subtracting x^2dx and y^2dy, becomes,

$$2x(x - y)dx + 2y(x - y)dy - (x^2 + y^2)(dx - dy) = 0.$$

Introducing the integrating factor $(x - y)^{-2}$, we have,

$$\frac{(x - y)(2xdx + 2ydy) - (x^2 + y^2)(dx - dy)}{(x - y)^2} = 0,$$

which, upon integration, becomes

$$\frac{x^2 + y^2}{x - y} = c, \qquad \text{or} \qquad x^2 + y^2 - c(x - y) = 0.$$

Also solved by A. W. SMITH, NORMAN ANNING, J. W. CLAWSON, J. A. BULLARD, G. PAASWELL, O. S. ADAMS, ELIJAH SWIFT, FREDERICK WOOD, HORACE OLSON, C. A. BARNHART, L. M. COFFIN, G. W. HARTWELL, J. D. BOND, A. G. RAU, C. A. HUTCHINSON, CLARIBEL KENDALL, C. S. ATCHINSON, J. W. CROMWELL, C. P. SOUSLEY, J. A. CAPARO, and PAUL CAPRON.

405. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the greatest quadrilateral which can be formed with the four given sides a, b, c, and d taken in order.

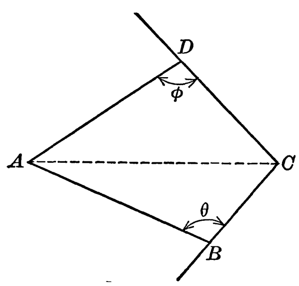
SOLUTION BY A. M. HARDING, University of Arkansas.

In the quadrilateral $ABCD$, if $AB = a$, $BC = b$, $CD = c$, $AD = d$, we have

$$\overline{AC}^2 = a^2 + b^2 - 2ab \cos \theta = c^2 + d^2 - 2cd \cos \phi.$$

Hence,

$$ab \cos \theta - cd \cos \phi = \frac{a^2 + b^2 - c^2 - d^2}{2}. \tag{1}$$



Also, area $= \frac{1}{2}ab \sin \theta + \frac{1}{2} cd \sin \varphi$. For a maximum or minimum we must have

$$ab \cos \theta d\theta + cd \cos \varphi d\varphi = 0. \tag{2}$$

From (1) we obtain, by differentiation,

$$- ab \sin \theta d\theta + cd \sin \varphi d\varphi = 0. \tag{3}$$

It follows from (2) and (3) that

$$\tan \phi = - \tan \theta; \qquad i.\ e., \qquad \phi + \theta = 180°. \tag{4}$$

Therefore,

$$\cos \theta = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

Hence, the given quadrilateral will be a maximum when it can be inscribed in a circle.

Note.—It is evident from the nature of the problem that it is necessary to consider only *convex* quadrilaterals, and that (4) gives a maximum and not a minimum.

406. Proposed by C. N. SCHMALL, New York City.

Given $f(x + h) + f(x - h) = f(x) \cdot f(h)$, determine by Taylor's theorem or otherwise the nature of the function f.

SOLUTION BY W. M. CARRUTH, Hamilton College, New York.

Given

$$f(x + h) + f(x - h) = f(x) \cdot f(h). \tag{1}$$

Put $h = 0$ in this equation. Then $2 \cdot f(x) = f(0) \cdot f(x)$. Hence, either

$$f(x) = 0, \tag{2}$$

which is a trivial solution of the problem and will be ignored until the end of the discussion, or

$$f(0) = 2. \tag{3}$$

Changing the sign of h in (1) does not alter the left-hand member; hence.

$$f(x)\cdot f(h) = f(x)\cdot f(-h),$$

or $f(h) = f(-h)$. Differentiating this, $f'(h) = -f'(-h)$; whence

$$f'(0) = 0. \tag{4}$$

Differentiate (1) twice with respect to h. Then, $f''(x+h) + f''(x-h) = f(x)\cdot f''(h)$. Putting $h = 0$ in this equation, we get

$$2\cdot f''(x) = f''(0)\cdot f(x). \tag{5}$$

Replace $f''(0)$, which is a constant, by $\pm 2c^2$; and put y for $f(x)$, and d^2y/dx^2 for $f''(x)$. Although the prime was used to represent differentiation with respect to h, (5) is an identity and the change here to x is legitimate. (5) may now be written

$$\frac{d^2y}{dx^2} \mp c^2y = 0. \tag{6}$$

Using the upper sign in (6), the solution may be written in the form $y = A\cdot\cosh(cx + B)$, where A and B are constants to be determined. From (3) and (4) above,

$$A\cdot\cosh B = 2, \quad \text{and} \quad Ac\cdot\sinh B = 0.$$

Therefore, if $c \neq 0$, we have $B = 0$ and $A = 2$ (the values $B = n\pi i$, $A = (-1)^n\cdot 2$ are no more general than these). Hence, one solution of (1), which is easily verified, is

$$f(x) = 2\cdot\cosh cx. \tag{7}$$

(The fact that (7), and (8) below, may be verified on substitution in (1) proves that the constant c is an arbitrary constant.)

Using the lower sign in (6), the solution may be written in the form

$$y = A\cdot\cos(cx + B).$$

As before,

$$A\cdot\cos B = 2, \quad \text{and} \quad Ac\cdot\sin B = 0:$$

whence

$$B = 0, \quad \text{and} \quad A = 2.$$

Another solution of (1) is, therefore,

$$f(x) = 2\cdot\cos cx. \tag{8}$$

If $c = 0$ in (6), we have the solution $y = Ax + B$. Here, from (3) and (4) as before, $A = 0$, and $B = 2$. But the solution $f(x) = 2$ is included in both (7) and (8), if $c = 0$ is permitted there.

Collecting the results that have been given in (2), (7), and (8), we may say that the function $f(x)$ defined by equation (1) must be equivalent to one of the forms

$$0, \quad 2\cdot\cosh cx, \quad 2\cdot\cos cx,$$

where c is any real number. If imaginary values of c are permitted, the last two forms of $f(x)$ are equivalent.

If a solution by Maclaurin's theorem rather than by differential equations is desired, the work following equation (5) above may take this form:

Replace $f''(0)$, which is a constant, by $\pm 2c^2$, and consider the primes to mean differentiation with respect to x. Then (5) may be written

$$f''(x) = \pm c^2f(x).$$

Successive differentiation of this equation leads to the general result

$$f^{(n+2)}(x) = \pm c^2f^{(n)}(x),$$

or, if $x = 0$ be substituted,

$$f^{(n+2)}(0) = \pm c^2f^{(n)}(0). \tag{6'}$$

From (4) and (6′), all the odd ordered derivatives of $f(x)$ are zero when $x = 0$. By successive applications of (6′) we have (since $f''(0) = \pm 2c^2$)

$$f^{\mathrm{IV}}(0) = 2c^4, \quad f^{\mathrm{VI}}(0) = \pm 2c^6, \cdots, f^{(4n)}(0) = 2c^{4n}, \quad f^{(4n+2)}(0) = \pm 2c^{4n+2}.$$

Hence, by Maclaurin's theorem,

$$f(x) = 2 \pm 2c^2x^2/2! + 2c^4x^4/4! \pm 2c^6x^6/6! + \cdots.$$

This gives either

$$f(x) = 2\cdot\cosh cx, \quad \text{or} \quad f(x) = 2\cdot\cos cx,$$

to which solutions $f(x) = 0$ should be added as before.

Also solved by H. C. FEEMSTER, OSCAR S. ADAMS, PAUL CAPRON, and the PROPOSER.

MECHANICS.

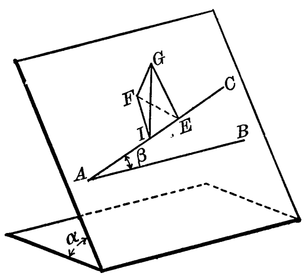
319. Proposed by LAENAS G. WELD, Pullman, Illinois.

A hexagonal pencil lies upon the inclined top of a drawing table and is on the point of either rolling or sliding. Find the angle between its direction and the horizontal edge of the table, the coefficient of friction being μ.

SOLUTION BY H. S. UHLER, Yale University.

The condition for being on the verge of sliding is expressed by $\mu = \tan \alpha$, where α denotes the angle between the table top and the horizontal and is sometimes called the "limiting angle of repose." The proof of this relation seems superfluous in this place because it is given in practically all text-books which are devoted in part or entirely to elementary mechanics.

In order that the pencil may be on the point of rolling, it is obviously necessary and sufficient that the vertical line through the center of gravity intersect the lateral edge which is at the lowest level. In the diagram let $\overline{AB}$ and $\overline{AC}$ indicate respectively a horizontal line on the table top and the lowest lateral edge of the hexagonal prism. G marks the center of gravity which is assumed to lie in the geometric axis of the pencil. $\overline{GI}$ denotes the vertical through G, that is,



the line of action of the weight of the pencil which intersects the edge $\overline{AC}$ in the point I. $\overline{GE}$ and $\overline{GF}$ are perpendiculars dropped from G upon the edge $\overline{AC}$ and the table top, respectively. In other words, the plane EFG contains a right section of the prism passing through the center of gravity. Without quoting the well-known theorems of elementary geometry we see at once that $\angle EFI = \beta$, $\angle EGF = 30°$, $\angle FGI = \alpha$, $\angle EFG = \angle FEI = \angle GFI = 90°$. Consequently, $\overline{FG}/\overline{EF} = \cot 30° = \sqrt{3}$, $\overline{EF}/\overline{FI} = \cos\beta$, and $\overline{FI}/\overline{FG} = \tan\alpha$. Multiplying these three equations together we obtain $1 = \sqrt{3}\cos\beta\tan\alpha$, but $\tan\alpha = \mu$, hence

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{3}\mu}\right).$$

Remark.—Since the greatest value of a cosine is unity the least value of μ is equal to $1/\sqrt{3} \doteq 0.57735$ corresponding to $\alpha = 30°$ and $\beta = 0°$, as it should be.

Also solved by W. J. THOME, M. R. BOWERMAN, and GEORGE PAASWELL.

320. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A heavy uniform chain of length l is hung over a rough horizontal cylinder of radius r. Show that one end of the chain will be $[2\mu r/(\mu^2+1)](e^{\mu\pi}+1)+l(e^{\pi\mu}-1)$ units lower than the other, just when the chain begins to move, the coefficient of friction being μ.

Solution by William Hoover, Columbus, Ohio.

Let x = the longest part hanging down, and $l-(\pi r+x)$ = the shortest, θ being the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right; and T, T' the tensions at any points in those parts of the string to which x and $l-(\pi r+x)$ belong. Then if C be a constant, we have

$$T = Ce^{\mu\theta}+\frac{r}{\mu^2+1}\{2\mu\sin\theta+(1-\mu^2)\cos\theta\}, \tag{1}$$

by theory, μ and θ being both negative, the string being supposed to move in the direction of x.

When $\theta=-\pi/2$, $\mu=-\mu$, and $T=x$, a unit of length of string weighing a unit, we have

$$C=\left(x-\frac{2\mu r}{\mu^2+1}\right)e^{-(1/2)\pi\mu}, \tag{2}$$

and (1) then becomes

$$T=\left(x-\frac{2\mu r}{\mu^2+1}\right)e^{[\theta-(\pi/2)]\mu}+\frac{r}{\mu^2+1}\{-2\mu\sin\theta+(1-\mu^2)\cos\theta\}. \tag{3}$$

When $\theta=0$,

$$T=\left(x-\frac{2\mu r}{\mu^2+1}\right)e^{-(1/2)\pi\mu}+\frac{r(1-\mu^2)}{\mu^2+1}, \tag{4}$$

which is the tension at the vertex.

Again, $T'=l-(\pi r+x)$, when $\theta=\pi/2$, $\mu=-\mu$; then

$$Ce^{-(1/2)\pi\mu}-\frac{2\mu r}{\mu^2+1}=l-(\pi r+x),$$

or

$$C=\left\{l-(\pi r+x)+\frac{2\mu r}{\mu^2+1}\right\}e^{(1/2)\pi\mu}. \tag{5}$$

Hence

$$T'=\left\{l-\pi r-x+\frac{2\mu r}{\mu^2+1}\right\}e^{\mu[\theta+(\pi/2)]}+\frac{r}{\mu^2+1}\{-2\mu\sin\theta+(1-\mu^2)\cos\theta\}. \tag{6}$$

When $\theta=0$,

$$T'=\left\{l-\pi r-x+\frac{2\mu r}{\mu^2+1}\right\}e^{(1/2)\pi\mu}+\frac{r(1-\mu^2)}{\mu^2+1}. \tag{7}$$

For equilibrium, $T=T'$, and then

$$\left(l-\pi r-x+\frac{2\mu r}{\mu^2+1}\right)e^{(1/2)\pi\mu}=\left(x-\frac{2\mu r}{\mu^2+1}\right)e^{-(1/2)\pi\mu}, \tag{8}$$

giving

$$x=\frac{l-\pi r}{1+e^{-\pi\mu}}+\frac{2\mu r}{\mu^2+1}, \tag{9}$$

and

$$l-(\pi r+x)=\frac{(l-\pi r)e^{-\pi\mu}}{1+e^{-\pi\mu}}-\frac{2\mu r}{\mu^2+1}. \tag{10}$$

(9) and (10) give the two vertical parts of the string, their difference being

$$\frac{(e^{\pi\mu}-1)(l+\pi r)}{1+e^{-\pi\mu}}+\frac{4\mu r}{\mu^2+1},$$

not agreeing with the required difference in the statement of the problem.

Also solved with same result by Paul Capron, Ralph E. Root, and Elijah Swift.

NUMBER THEORY.

217. (May, 1914.) Proposed by E. T. BELL, University of Washington.

(i) If r is a prime greater than 2, and $p \equiv 2^a r + 1$ is prime, the only solution, when n is greater than 2, of $x^n - y^n = p$, is $n = 3$, $x = 2$, $y = 1$.

(ii) The only primes that are simultaneously of the forms $4k + 1$ and $3^m - 2^m$ are 1 and 5.

(iii) Generalize (ii).

SOLUTION BY FRANK IRWIN, University of California.

(i) Since $x^n - y^n$ is divisible by $x - y$, we must, if it is to be a prime, have $x - y = 1$, or $x = y + 1$.

Again, n must be a prime; for $x^{st} - y^{st}$ is divisible by $x^s - y^s$, and cannot, therefore, be a prime.

Now $2^a r = p - 1 = x^n - y^n - 1 = (y + 1)^n - y^n - 1 = ny^{n-1} + \binom{n}{2} y^{n-2} + \cdots + ny;$ so that $2^a r$ is divisible by ny, since n, being a prime, divides each of the binomial coefficients. Consequently we must have $n = r$ (n is greater than 2 by hypothesis) and y a power of 2.

But by writing $2^a r$ as $x^n - (x - 1)^n - 1$, we may show, just as above, that x also is a power of 2.

Now as $x = y + 1$, these results are consistent with each other only if $y = 1$, $x = 2$.

In this last case we have $2^a r + 1 = 2^n - 1^n = 2^r - 1$, since $n = r$. Hence, $2^a(2^{r-a} - r) = 2$, and, therefore, $a = 1$ and $2^{r-a} - r = 1$, that is, $2^{r-1} = r + 1$; so that $r = n = 3$.

(ii) $3^m - 2^m = (2 + 1)^m - 2^m = m2^{m-1} + \binom{m}{2} 2^{m-2} + \cdots + m2 + 1$, and can be of the form $4k + 1$ only if m is even (unless $m = 1$, when $3^m - 2^m = 1$).

But if m is even, $= 2n$, say, $3^m - 2^m = 9^n - 4^n$ and is therefore divisible by $9 - 4$ or 5, the other factor being greater than 1, unless $n = 1$, and is, therefore, prime in this last case only.

(iii) A parallel argument leads to the result that the only primes that are simultaneously of the forms $a^2k + 1$ and $(a + 1)^m - a^m (a > 1)$ are 1 and, possibly, $(a + 1)^a - a^a$. This latter number may or may not be prime as the cases $a = 3, 4$ show.

Also solved by MARY E. CARTER.

221. (September, 1914.) Proposed by T. E. MASON, Bloomington, Indiana.

Find a number x such that the sum of the divisors of x is a perfect square. [CARMICHAEL, *Theory of Numbers*, p. 17.]

SOLUTION BY E. B. ESCOTT, Kansas City, Mo.

(1) Let x be prime. Then $x + 1 = r^2$, or $x = (r + 1)(r - 1)$. This is possible only if $r = 2$; otherwise, x is not prime. Hence, $x = 3$ satisfies the condition.

(2) Let $x = abc\cdots$, where $a, b, c, \cdots$ are different primes. Denoting the sum of the divisors of x by Σx,

$$\Sigma x = (a + 1)(b + 1)(c + 1)\cdots.$$

Forming a table of factors of $a + 1$, it is very easy to pick out products of two or more which shall be square, *e. g.*,

x	Σx	x	Σx
2.11	6^2	7.97	28^2
2.47	12^2	.	.
11.47	24^2	.	.
5.23	12^2	.	.
5.53	18^2	2. 5.7	12^2
23.53	36^2	5. 7.11	24^2
7.17	12^2	5. 7.47	48^2
7.31	16^2	2.13.41	42^2
7.71	24^2	5.67.101	204^2
		.	.
		.	.
		.	.

(3) Corresponding to a factor a^2 in the number, we have in the sum of the divisors $1 + a + a^2$, etc. By forming a table of $1 + a + a^2$, $1 + a + a^2 + a^3$, $\cdots$ we can find examples where the number contains square factors, cube factors, $\cdots$.

EXAMPLES:

x	Σx	x	Σx
$7^2.11^2.5.13$	798^2	$3^3.7.19$	80^2
$41^3.83$	2436^2	$2^2.5.41$	42^2
$41^3.5.13$	2436^2	.	.
$3^3.89$	60^2	.	.
		.	.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kans.

REPLIES.

31. What are the actual courses now offered in colleges and universities in this country for the preparation of teachers (1) for secondary schools, (2) for colleges? The discussion may well lead to the consideration also of what courses *should be offered* for the preparation of teachers of mathematics (1) for secondary schools, and (2) for colleges.

REPLY BY U. G. MITCHELL, University of Kansas.

In the American Report, Committee No. V, of the International Commission on the Teaching of Mathematics, published in 1911, we find (pp. 5–6) these statements:[1]

"Twenty years ago no professional training of university grade existed in this country to prepare teachers of mathematics for secondary schools. At that time the young teacher's sole preparation for his work was the taking of as many academic courses as possible, plus, in some instances, a course on the history of education or some lectures on pedagogy. . . . About 15 years ago we find conditions throughout the country beginning to change in this respect. At least five different educational institutions had by this time (1895) established courses on the teaching of algebra and geometry, which, together with a course on general pedagogy, formed a certain professional training for high school teaching in mathematics. Up to 1900 only four other colleges are known to this committee to have added courses in the pedagogy of secondary mathematics to their programs.

"The past 10 years have shown far greater interest in pedagogical matters and a much more rapid growth in courses of this kind. At present (1910) about 25 other colleges in addition to those above mentioned, have developed such courses."

The writer has just completed an examination of the most recent catalogs from a selected list of 100 colleges and universities to see how many of them are now offering courses especially designed for students preparing to teach mathematics. The result shows that the change referred to above is continuing. Fifty-three of the 100 institutions are offering one or more courses of the following four different classes:

(1) *Courses especially designed for teachers of secondary mathematics.* Such courses were listed by 40 of the 100 institutions and were variously designated as "Teachers' Course," "Mathematical Methods," "History and Teaching of Mathematics," "Teachers' Course in Algebra and Geometry," etc. In five cases "reviews of secondary mathematics with special attention to methods"

[1] *Bulletin, 1911, No. 12, U. S. Bureau of Education* (Washington, D. C.).

constituted a part or all of the course. In a considerable number of cases it was stated that courses in analytic geometry and calculus were prerequisites. The credit given for such courses varied from 1 to 6 hours.

(2) *Courses in practice teaching.* In 6 of the 100 institutions students were offered opportunity to do actual teaching under the direction of supervisors in training schools.

(3) *Courses in the history of mathematics.* Such courses were offered in 24 of the 100 institutions and, with a single exception, were given either two or three hours credit. Two institutions offered more than one such course. Nine of the institutions which gave no separate course in history of mathematics stated that a part of the time of the teachers' course was devoted to history. Hence, some formal teaching in the history of mathematics was offered in at least 33 of the 100 institutions.

(4) *Courses in the foundations of mathematics.* Such courses were offered in 8 of the 100 institutions. They were variously designated as "Fundamental Concepts of Mathematics," "Fundamental Theorems of Algebra and Geometry," "Foundations of Geometry," etc., and were generally three-hour courses.

The writer failed to find any courses which were said to be designed especially to prepare teachers of collegiate mathematics. It is quite possible, however, that some of the history and foundation courses have been established primarily for students who expect to become professors of mathematics. In this respect, the condition is practically the same as described by the American Report, Committee X, of the International Commission on the Teaching of Mathematics in the following language[1] (italics mine):

"There is a universal feeling that courses in the pedagogy of mathematics are of very small advantage to the future college teacher. Those professors who are willing to see such courses introduced specify either that they should be in addition to all the present courses in mathematics and not a substitute for any purely mathematical subject, or that they should be introduced only to satisfy the imperative demands of the schools.

"A wiser view seems to be that which argues that the college teacher can well afford to spend some time in learning the best methods of teaching, whatever his subject; and that a course in the pedagogy of mathematics will be of very great value to the teacher of mathematics when the graduate school or university shall have developed a course suited to his needs. *There appears to be sufficient material at hand, and it is strange that no American institution has solved the problem of a course in mathematical pedagogy in such a manner as to appeal to the professor of mathematics.*"

As to what courses should be offered for the preparation of teachers of secondary mathematics, a considerable number of American institutions are now offering and, in large part, requiring, the following standard suggested as a minimum by Professor J. W. A. Young at the Cambridge meeting[2] (1912) of the International Commission:

(*a*) Trigonometry, college algebra, analytic geometry.
(*b*) Surveying, or descriptive geometry, or elementary astronomy.
(*c*) The differential and integral calculus with applications to geometry, mechanics and physics.
(*d*) Modern geometry.

[1] *Bulletin, 1911, No. 7, U. S. Bureau of Education* (Washington, D. C.), pp. 23–4.
[2] See *L'Enseignement Mathématique* for Nov. 15, 1912, p. 483.

(*e*) The elements of analytic mechanics.
(*f*) The elements of theoretic and laboratory physics.
(*g*) Algebra from a modern standpoint.
(*h*) One or more courses introductory to important fields of modern mathematics.
(*i*) One or more courses on the history of mathematics.
(*j*) One or more courses on the teaching of mathematics.

At the university with which the writer is connected, a student majoring in mathematics is required to take before graduation at least 30 hours of work in pure mathematics courses. Besides the usual courses in college algebra, trigonometry, analytical geometry and calculus (15 hours), a two-hour course in higher algebra, a three-hour course in modern geometry, a two-hour course in history of mathematics and a three-hour course in analytic mechanics or advanced calculus, are specifically required. The remaining 5 or more hours are elective. In order to be certificated for teaching mathematics in the secondary schools in Kansas the student must also take a three-hour teachers' course in mathematics, a semester's work in practice teaching in the training school, a three-hour course in general psychology, a three-hour course in educational psychology, a three-hour course in the history of education and 6 hours of elective work in the school of education. While these requirements are probably less than some others, they doubtless do not differ greatly from those of a number of middle-western universities.

The writer believes that courses in education possess a distinct value for the prospective teacher of mathematics. They give him some conception of the human side of his work. More than one young professor, splendidly equipped mathematically, has failed as a teacher because he spent his energy in trying to teach mathematics *to* students and never acquired the point of view of teaching students *by means of* mathematics.

Reply by R. C. Archibald, Brown University.

Within the past decade many of those charged with directing the education of youth in the United States have had their outlook immensely extended, and their ideas radically changed by careful study and consideration of methods employed in other countries. They have discovered that the standards to which teachers in practically all the secondary schools of France, Germany, Italy and many other countries must attain are far higher than those demanded in even the best secondary schools of the United States. As far as mathematics is concerned this has been set forth with great explicitness, during the past few years through the publications inspired by the International Commission on the Teaching of Mathematics.

As a result, thoughtful inquirers have reached the conclusion that the efficiency and general well-being of this country demand that some radical changes be made in the method of conducting secondary education; moreover, that as a beginning of necessary reform, newly appointed teachers should be college graduates who are informed as to the problems and methods in secondary education and who, while they have learned to appreciate varied forms of scholarship and culture, have yet been specially equipped to teach one or two nearly related subjects.

It is with deep appreciation of the important work to be done in promoting the cause of secondary education, and therefore of higher education, that the department of mathematics in Brown University lays emphasis, by virtue of special courses and personal effort, on the methods employed in the preparation of teachers of mathematics. The annual increase in the number of students taking this work in the department seems to testify to the fact that a real need is being met.

It is considered important that prospective teachers of mathematics should have a thorough scientific reconsideration in college of the principles of secondary school mathematics, that the true inwardness of all operations, and a clear understanding of the foundations of the various subjects, should be acquired. This knowledge, together with adequate facility in presenting it, presupposes that the student has been carried far beyond the subject matter actually to be taught at a later day in the secondary school. But it is only with such preparation that a teacher of mathematics can claim to be competent to lay the foundations of a subject which is doubtless of more far-reaching importance than any other.

The eleven semester courses (each three hours per week) required for recommended teachers are: algebra, geometry, plane trigonometry, analytical geometry, and differential and integral calculus.

The two-semester *teachers' course in algebra* constitutes an introduction to some of the concepts of modern analysis. Among the topics treated are, the number system, with special reference to irrational numbers, limits, infinite series, fundamental operations, determinants and proofs of the fundamental theorem of algebra.

In the two-semester *teachers' course in elementary geometry* the student is taught: methods for attacking Euclidean problems; discussions of famous problems; the existence of transcendental numbers and the proof of the transcendence of π; means of rigorous discussion of the more delicate and difficult parts of the subject such as the systems of axioms; something about (*a*) the history and literature of elementary geometry and (*b*) the most important French, German and Italian texts.

As to the course in *solid and spherical geometry*, the ability to reason accurately from fundamental axioms, and to solve originals, is demanded.

The work in *analytical geometry* occupies one and one-half semesters and that in *calculus* two and one-half semesters.

In addition to those mentioned above it is possible to take six or eight other year-courses, partaking somewhat of a graduate character, and some such courses are usually elected by our embryonic secondary school teachers. For example, during the year 1915–16 Goursat's *Mathematical Analysis*, volume 1, was a text in one course; in another, Pierpont's *Functions of a Complex Variable*, and in a third, Papelier's *Coordonnées tangentielles*. In no one of these courses were there less than ten students, the majority of whom were preparing to be secondary school teachers. For the year 1916–17, the first volume of Weber's

Algebra is a text in one course and in another, in differential geometry, the texts are the last part of volume 1 of Goursat's *Mathematical Analysis*, Gauss's memoirs on *Curved Surfaces* and selected chapters from Humbert's *Cours d'Analyse*.

Another advantage in connection with our preparation of teachers is the meetings of the Mathematical Club. The club is a recent organization which aims to create an atmosphere in which to establish friendly intercourse among students and teachers, and to supply an opportunity for the presentation, by advanced students and professors, of papers of general interest.

Practically all students recommended as teachers of mathematics to secondary schools have had two or more courses in education besides work in science. Many students return for graduate work leading to the degree of Master of Arts. Their work is mainly in the department of education and in the schools of Providence and the surrounding cities. Under suitable restrictions tuition fees are met through scholarships awarded by the Board of Education of the state of Rhode Island. Such students (men and women) are not only well equipped scholastically, but have also acquired valuable insight and experience with reference to methods of imparting knowledge.

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

The distinguished mathematical physicist, Professor DON JOSÉ ECHEGARAY, of the University of Madrid, died on Sept. 15, aged eighty-three years.

Mr. F. S. NOWLAN, of Columbia University, has been appointed instructor in mathematics at the Carnegie School of Technology, Pittsburgh.

At Pennsylvania State College, Dr. J. E. ROWE has been promoted from assistant professor to associate professor of mathematics.

Mr. E. J. OGLESBY, instructor in mathematics in the University of Virginia, has been appointed professor of mathematics at the College of William and Mary.

Mr. ALBERT H. HOLMES, a contributor to the MONTHLY in the department of Problems and Solutions, and a charter member of the Association, died at Brunswick, Maine, Sept. 10, 1916, at the age of sixty-five years.

Dr. J. A. BULLARD, Mr. C. E. NORWOOD, and Mr. J. J. TANZOLA have been appointed instructors in mathematics at the U. S. Naval Academy, Annapolis, Maryland.

At Dartmouth College, Dr. R. D. BEETLE and Dr. F. M. MORGAN have been made assistant professors of mathematics, and Dr. C. H. FORSYTH, of the University of Michigan, has been appointed instructor in mathematics.

Dr. H. M. SHEFFER, of the College of the City of New York, has been appointed lecturer on philosophy at Harvard University, offering the courses in logic formerly given by the late Professor Royce.

Mr. J. W. LASLEY, JR., returns to the faculty of the University of North Carolina after a year's leave of absence as fellow by courtesy in mathematics at Johns Hopkins University.

The Mathematical Club of the University of North Carolina was organized on October 19th. The officers are: WILLIAM CAIN, president, ARCHIBALD HENDERSON, vice-president, and J. W. LASLEY, Jr., secretary-treasurer. The club proposes to engage in mathematical investigation, especially along historical and pedagogical lines.

Miss EDITH M. COON, who for the past two years has held a position as instructor in the department of mathematics at Mount Holyoke College, has gone to Madras, India, as vice-principal of the Christian College for Women that has been recently founded there under English and American auspices. Miss Coon will teach mathematics and physics.

The October number of *The Monist* is devoted to a commemoration of the scientific and philosophical work of GOTTFRIED WILHELM LEIBNITZ and its influence on modern thought. This material in celebration of the Leibnitz bicentenary has been collected and edited by Mr. P. E. B. JOURDAIN, the Cambridge scholar, who is especially interested in the fields of mathematics, physics, and logic.

A review of the Napier Tercentenary Memorial Volume by Professor L. C. Karpinski was published in *Science* for September 22, 1916. This volume contains thirty papers, touching a variety of interests related to logarithms, by representatives of the British Empire, France, Germany, Italy, Denmark, Turkey, and the United States. The papers by representatives of this country have already been noted in these columns in earlier issues.

At Sheffield Scientific School, Yale University, the following changes have been announced: Dr. H. F. MACNEISH, instructor in mathematics, has resigned to accept an appointment in the De Witt Clinton High School, New York City; Dr. D. D. LEIB has gone to the Connecticut College for Women, New London; Dr. P. R. RIDER has become instructor in mathematics in Washington University, St. Louis, Missouri; and Dr. D. F. BARROW, of the University of Texas, and Mr. J. K. WHITTEMORE have been appointed instructors in mathematics.

During the past year, the Mathematical Club of Rutgers College held regular monthly meetings at which the following papers were presented: "Tangencies of circles," Dr. J. A. Ingham; "The occurrence of conic sections in the

orbits of the heavenly bodies," Professor W. B. Stone; "Life insurance; the life tables," Professor E. Brasefield; "The use of the auxiliary angle," Professor Richard Morris; "Summation of certain trigonometric series," Mr. C. P. Osborne; "Logarithms; exposition of Napier's principles," Mr. Ed. S. Ingham; "Ptolemy's theorem and its relation to trigonometry," Mr. Edwin Florance; "The triangle and its circles," Mr. L. B. Gittleman.

It is with regret that we record the death of Dr. L. L. CONANT, professor of mathematics in the Worcester Polytechnic Institute, Worcester, Massachusetts. He was killed by an auto truck in front of his home on Oct. 11, 1916. Professor Conant was a graduate of Dartmouth College, class of 1879, A.M., 1887, and Ph.D. of Syracuse University, 1893. He engaged in public school work from 1879 to 1887, at which time he became professor of mathematics at the Dakota School of Mines, remaining in this position for three years. After one year's study at Clark University, he accepted the professorship of mathematics at Worcester in 1891, in which position he continued until his death. He was interested in all phases of education, serving not only in his official capacity at Worcester, but as a member of the Massachusetts State Board of Education from 1909–1914. Professor Conant was a member of the American Mathematical Society, the London Mathematical Society, the Mathematical Association of America, and other scientific societies. He was the author of "The Number Concept," "Exercises in Plane Geometry," and "Plane and Spherical Trigonometry with Tables." Professor Conant made a provisional bequest of $10,000 to the American Mathematical Society, the income to be offered once in five years as a prize for original work in pure mathematics.

The sixty-third annual meeting of the Indiana State Teachers' Association was held at Indianapolis on Oct. 26–28, attended by more than 10,000 teachers of the state representing all grades of school work from the kindergarten to the college. The general meetings were held in five sections, and the sectional meetings relating to special subjects and interests met in thirty different sections. The mathematics section composed of high school and college teachers of mathematics met during the forenoon of Oct. 26 in Cleb Mills Hall. Some three hundred teachers of mathematics were present. The following program was presented: "The hurdle system of teaching algebra," by Mr. M. A. DALMAN, of manual training high school, Indianapolis; "Efficiency tests in mathematics," by Professor D. A. ROTHROCK, of Indiana University; "Some school-room problems," by President R. J. ALEY, of the University of Maine; and a round table "Discussion of recent tendencies in opposition to mathematics in the secondary schools," participated in by Professor F. H. HODGE, of Franklin College, Professor E. N. JOHNSON, of Butler College, Professor R. B. STONE, of Purdue University, and by others. In this discussion Professor HODGE reviewed the criticisms of the high-school course in mathematics by Commissioner David Snedden, of Massachusetts, and Dr. ABRAHAM FLEXNER, of New York; Professor JOHNSON presented the sentiment of a representative group of "Who's

Who" men and women, showing some 90 per cent. of those expressing themselves as being strongly in favor of retaining mathematics in the curriculum of the high school.

The Mathematics Section of the Indiana State Teachers' Association dates back to 1891 at which time the College Association of Mathematics Teachers, which had been formed a few years earlier, took on the broader field of secondary as well as college mathematics. The programs are arranged so as to include topics of interest to mathematics teachers of any grade. At the close of the sectional meeting a call was made for teachers of college grade to meet and make preliminary organization with reference to forming an Indiana branch of the Mathematical Association of America. The preliminary organization was effected by the election of Professor S. C. DAVISSON, of Indiana University, chairman, Professor W. O. MENDENHALL, of Earlham College, secretary-treasurer, and Professor F. H. HODGE, of Franklin College, as member of the executive committee. This is the fifth section of the Association to be organized. The others are Kansas, Ohio, Missouri, and Iowa.

The *Scientific American Supplement* for August 5, 1916, contains an article by Professor R. E. MORITZ, of the University of Washington, describing the "Clyco-Harmonograph," an instrument for drawing in ink or pencil some sixty-three distinct species of mathematical curves, among them such known curves as the conchoids, the nephroids, the foliates, and harmonic curves. The method commonly employed in the construction of such curves is to plot the curve by points determined by their equations. This process is laborious, and accumulates inaccuracies. The Moritz instrument eliminates these errors, and constructs the curves with the greatest ease and precision. The article in the "Scientific American Supplement" shows a photograph of the instrument and reproductions of a number of harmonic curves constructed by it.

Recent papers read before the ASSOCIATION and the SOCIETY indicate that renewed interest is apparent in all phases of mathematical history. Hence, no apology is needed for the publication of notes such as the following:

In *Nature*, December 3, 1914, p. 363, Professor CAJORI showed that the cross $\times$ as a symbol of multiplication, which is said in histories to occur first in William Oughtred's *Clavis mathematicae* (1631), is given in form of the letter x and X in Edward Wright's translation of John Napier's *Mirifici logarithmorum canonis descriptio*, second edition, London, 1618, where we read, page 4: "The note of addition is (+), of subtracting (−), of multiplying ($\times$)." This is taken from a part of the book headed "Appendix to the Logarithmes," the authorship of which is not given but is believed now most probably to be attributed to William Oughtred.

In 1902 Professor W. W. BEMAN pointed out (*L'Intermédiaire des mathématiciens*, T. 9, Paris, p. 229, question 2424) that the colon (:) occurs as the symbol for geometric ratio at the end of the tables in Oughtred's *Trigonometria* of 1657. Professor CAJORI has found that the colon was so used by the astronomer Vincent

Wing in 1651, 1655 and 1656 and by a Suffolk schoolmaster with the initials "R. B." in 1655. For further details see *Nature,* Dec. 31, 1914, p. 477.

The first designation of the sides of a triangle by the same letters, respectively, as the angles opposite, one group of letters being capitals A, B, C, and the other group small letters a, b, c, has been attributed to Leonhard Euler (*Histoire de l'acadêmie de Berlin, annêe,* 1753, p. 231), but Professor CAJORI finds that it occurs in a pamphlet containing trigonometric formulas published by Richard Rawlinson of Queen's College, Oxford, sometime between 1655 and 1668. Additional information on this is given in *Nature,* Feb. 11, 1915, pp. 642 and 643.

Recent discussions on the teaching of mathematics to students of engineering recall some earlier references to this topic. For instance, in the December, 1914, number of the *Bulletin* of the American Mathematical Society Mr. GEORGE PAASWELL, C.E., calls attention to the great need of applying fuller mathematical analysis to the problems of the applied science professions, and speaks of the appalling gaps in analysis which the engineer must bridge with assumptions far from rigorous or satisfactory; he appeals to producing mathematicians to turn their attention to bettering this state of affairs.

In the April 1915 number of the *Bulletin* Professor C. N. HASKINS makes reply to two leading points of criticism: (1) While the curricula of the schools of applied science are not sufficiently intensive or extensive to enable their graduates to meet these outstanding problems with a great wealth of mathematical power, the required mathematical courses represent very nearly the maximum of what can be effectively assimilated and used by the average student in such schools. Mathematically able students might however *elect* broader and deeper courses of a suitable sort. (2) As to the criticism that an engineer despairs of being able to read modern mathematical treatises unless in lines already familiar to him, it must be recognized that this "has its exact counterpart in the mathematician's despair of keeping up with modern engineering thought and practice." In analogy with courses leading to the degree of Doctor of Public Health, Professor Haskins suggests the establishing in a few advanced institutions of graduate courses wherein competent men may devote themselves to research in the problems of engineering and may at the same time prepare themselves for effective work in the mathematical problems of engineering.

SECOND ANNUAL MEETING OF THE ASSOCIATION.

The second annual meeting of the MATHEMATICAL ASSOCIATION OF AMERICA will be held at Columbia University in New York City on Thursday, Friday and Saturday, December 28, 29, 30, 1916. The meeting will open with a joint session on Thursday afternoon of the Association with the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science; at which time Professor

E. W. Brown, of Yale University, will deliver his retiring address as president of the American Mathematical Society, and Professor A. O. Leuschner will deliver his retiring address as vice-president of Section A of the American Association.

On Friday morning Professor Florian Cajori, of Colorado College, will give an address before the Association on the History of Fluxions; and Professor M. W. Haskell, of the University of California, will speak on University Courses in Mathematics intended for Teachers of Secondary Mathematics. The latter paper will be discussed by Professor J. W. Young of Dartmouth College and Professor Edward Kasner of Columbia University.

On Friday afternoon will be the meeting of Institutional Delegates devoted to a discussion of the subject of Mathematical Libraries for Colleges, including a report of the Library Committee by the Chairman, Professor W. B. Ford of the University of Michigan; and a paper by Dr. T. H. Gronwall of New York City on "A Nucleus for a Mathematical Library."

On Saturday morning Professor E. B. Wilson, of the Massachusetts Institute of Technology, will give an address on The Mathematics of Aerodynamics, and the subject will be discussed by Professor A. G. Webster of Clark University, and Professor E. W. Brown of Yale University.

On Thursday evening there will be a joint dinner of the Mathematical Association of America with the American Mathematical Society, and Section A of the American Association for the Advancement of Science.

On Friday between twelve and two o'clock there will be an exhibition of Portraits and Medals of Mathematicians from the collection of David Eugene Smith.

The annual business meeting, with election of officers and other important matters, will occur on Friday afternoon at half past three o'clock. All meetings of the ASSOCIATION will be held in room 301 of Hamilton Hall, Columbia University. The headquarters of both the Association and the Society will be at the Murray Hill Hotel. Members should make reservations at the earliest possible date.

The Program Committee, Professor David Eugene Smith, Chairman, stands as announced in the November issue. The Committee on Arrangements, Professor Thomas S. Fiske, Chairman, has been enlarged by the addition of Professor Joseph Bowden, of Adelphi College, Professor C. O. Gunther, of Stevens Institute, and Professor J. B. Chittenden, of the Brooklyn Polytechnic Institute.

The Committee on Libraries, whose preliminary report will be given at this meeting, was announced in the November MONTHLY, but through an oversight of the editors the names of two members were omitted. The full committee is as follows: Professor W. B. Ford of the University of Michigan, Chairman, Professor Florian Cajori of Colorado College, Professor E. S. Crawley of the University of Pennsylvania, Professor S. Lefschetz of the University of Kansas, Professor W. R. Longley of Yale University, and Professor R. E. Root of the United States Naval Academy.

ERRATA NOTED IN VOLUME XXIII.

Page 53, in the figure the point M. should be symmetrical with the point k.
Page 54, line 3, R before the integral sign should be R^2.
Page 54, line 9, the denominator of the last term should be "mR" instead of "$\sqrt{R^2 - b^2}$."
Page 54, line 11, the first parenthesis should precede "tan."
Page 160, first line of note 2, for "collectiònis" read "collectio."
Page 161, line 14, add "or internal" after "external."
Page 161, line 15, add "or external" after "internal."
Page 161, line 3 of footnote 1, for "Elucid" read "Euclid."
Page 304, line 16, for "radical axis" read "given line."
Page 329, line 18, for "$u = 1$" read "$m = 1$."
Page 334, line 18, for "$(n - 1)$th term" read "the term a_{n-1}."
Page 353, Example 228, for "Hermon C." read "Herman R."
Page 354, line 14, change $y = 8$ to $y = 5$.
Line 2 up, for "Ox" read "OY."
In the diagram change B' to B.
Page 360, line 22, for "first" read "second."

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